Interacting Stochastic Processes

Problem sheet 1

1. Let $W_1, W_2, \ldots$ be a sequence of independent exponential random variables $W_i \sim Exp(\lambda_i)$.

(a) Show that $\mathbb{E}(W_i) = 1/\lambda_i$ and that

$$\min\{W_1, \ldots, W_n\} \sim Exp\Big(\sum_{i=1}^{n} \lambda_i\Big) .$$

(b) The sum of iid exponentials with $\lambda_i = \lambda$ is Γ-distributed, i.e.

$$\sum_{i=1}^{n} W_i \sim \Gamma(n, \lambda) \quad \text{with pdf} \quad \frac{\lambda^n w^{n-1}}{(n-1)!}\, e^{-\lambda w} .$$

2. The Poisson process $(N_t : t \geq 0)$ with rate $\lambda > 0$ is a Markov chain with $X = \mathbb{N} = \{0, 1, \ldots\}$, $N_0 = 0$ and rates $c(n, m) = \lambda\, \delta_{n+1,m}$.

(a) Show that $N_t \sim Poi(\lambda t)$ has a Poisson distribution for all $t > 0$.

(b) Show that $(N_t : t \geq 0) \sim PP(\lambda)$ if and only if it has stationary, independent increments, i.e.

$$N_{t+s} - N_s \sim N_t - N_0 \quad \text{and} \quad N_{t+s} - N_s \quad \text{independent of} \quad (N_u : u \leq s) ,$$

and for each t, $N_t \sim Poi(\lambda t)$.

(c) Show that for independent Poisson variables $Y_1, Y_2, \ldots$ with $Y_i \sim Poi(\lambda_i)$ we have $\mathbb{E}(Y_i) = Var(Y_i) = \lambda_i$ and

$$\sum_{i=1}^{n} Y_i \sim Poi\Big(\sum_{i=1}^{n} \lambda_i\Big) .$$

3. The single server queue (M/M/1)
Let $(\eta_t : t \geq 0)$ be a continuous time Markov chain with state space $\mathbb{N} = \{0, 1, \ldots\}$ and jump rates

$$c(\eta, \eta + 1) = \alpha , \quad c(\eta, \eta - 1) = \beta(1 - \delta_{0,\eta}) .$$

η_t can be interpreted as the number of customers at time t, arriving at rate $\alpha > 0$ and being served at rate $\beta > 0$.

(a) Write down the master equation for this process.

(b) Show that for $\alpha > \beta$ the process is transient, i.e. $\eta_t \to \infty$ $a.s.$ as $t \to \infty$.
Hint: Compare to an asymmetric random walk and use the strong law of large numbers.

(c) Show that for $\alpha < \beta$ the process is positive recurrent by giving its stationary distribution μ. Is the distribution reversible?

(d) What do you think happens for $\alpha = \beta$?

(e) Let $A \sim PP(\alpha)$ be the arrival process of customers. Show that for $\alpha < \beta$ the departure process D is also Poisson $D \sim PP(\alpha)$ given that the process is stationary (this is called **Burke's theorem**).
Hint: There is an elegant proof using reversibility. Alternatively, condition on the value of η_t and show that

$$\mathbb{P}\big(\text{at least one departure in } [t, t + \Delta t)\big) = 1 - e^{-\alpha \Delta t} .$$

What do you think happens for $\alpha \geq \beta$?

4. Give a graphical construction for the linear voter model on $\Lambda = \mathbb{Z}$ with nearest neighbour interaction $p(x,y) = \delta_{x,y+1} + \delta_{x,y-1}$. Look at the sample path in reversed time. How does it look like?

5. Resolve the following 'paradox':
A single continuous-time random walker on $\mathbb{Z}$ does not have a stationary distribution, but an IPS of many random walkers has!

6. A generic algorithm to simulate continuous-time IPS is called **random sequential update**.
Consider the TASEP with $p = 1$, $q = 0$ on $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ (periodic boundary conditions).
Claim: To simulate (or construct a sample path of) the process do the following:

- Pick a site $x \in \Lambda_L$ uniformly at random;
- update your time counter $t \mapsto t + \Delta t$ by $\Delta t \sim Exp(L)$ (independently each time);
- if $\eta(x) = 1$ and $\eta(x+1) = 0$ move the particle, i.e. put $\eta(x) = 0$, $\eta(x+1) = 1$ ($+$ to be understood modulo L for periodic boundary conditions);

then start over again.

(a) Show that the number of timesteps k it takes for a given particle to attempt a jump is a geometric random variable $k \sim Geo(1/L)$ with mean L.

(b) Show that the waiting time until a jump attempt $t = \sum_{i=1}^{k} \Delta t_i$ is exponential $t \sim Exp(1)$, where the Δt_i are iid realizations of Δt for each time step.

(c) For large system size L, time increments are often replaced by their mean for simplicity, i.e. $\Delta t = 1/L$. Show that in this case the waiting time $t = k * \Delta t$ is still exponential in the limit $L \to \infty$.

(d) How does this algorithm have to be modified to simulate the ASEP with $p, q > 0$?

7. (Hard) part of the first exam question 2008/09:
We have seen that the condition on the jump rates

$$\sup_{y\in\Lambda} \sum_{x\in\Lambda} \sup_{\eta\in X} c(x,y,\eta) < \infty$$

implies (but is not equivalent to) $\mathcal{L}f$ being a convergent sum for all cylinder functions f.

(a) Which of the following conditions

$$(i)\ \sup_{y\in\Lambda} \sup_{x\in\Lambda} \sup_{\eta\in X} c(x,y,\eta) < \infty \qquad (ii)\ \sum_{y\in\Lambda} \sum_{x\in\Lambda} \sup_{\eta\in X} c(x,y,\eta) < \infty$$

has the same implication, i.e. $\mathcal{L}f$ converges for all cylinder functions f.
Justify your answer by using the above result or by giving an example such that $\mathcal{L}f$ diverges.

(b) Suppose that the rates are translation invariant and of finite range $R > 0$, i.e. for all $x, y \in \mathbb{Z}$

$$c(x,y,\eta) = c(0, y-x, \tau_{-x}\eta)\ , \quad c(x,y,\eta) = 0 \text{ if } |x-y| > R$$

and $c(x,y,\eta^z) = c(x,y,\eta)$ for all $z \in \mathbb{Z}$ such that $|x-z|, |y-z| > R$.
Show that $\mathcal{L}f$ converges for all cylinder functions f.