Interacting Stochastic Processes

Problem sheet 2

1. Convex combinations of measures (mixtures)
Let ν_0 and ν_1 be the product measures on $X = \{0,1\}^\Lambda$ with densities 0 and 1, respectively. Is it true that for each $\rho \in (0,1)$ the product measure ν_ρ is given by

$$\nu_\rho = \rho\nu_1 + (1-\rho)\nu_0 \quad ?$$

2. On $X = \{0,1\}^\mathbb{Z}$ let ν_ρ be the blocking measure with density

$$\rho(x) = \frac{(p/q)^x}{1+(p/q)^x} \quad \text{with} \quad p > q\,.$$

Use the first Borel Cantelli Lemma to prove that almost surely under ν_ρ

$$\sum_{x\leq 0} \eta(x) < \infty \quad \text{and} \quad \sum_{x\geq 0} \big(1-\eta(x)\big) < \infty\,.$$

(You can find the BC Lemma on Wikipedia or in any book on probability theory.)

3. CP-invariance
On $X = \{0,1\}^\Lambda$ with $\Lambda = \mathbb{Z}$ consider a transformation $\tau : X \to X$ which flips occupation numbers and inverts space at the same time.

(a) Write down a formula for τ. Is τ a bijection on X?

(b) Show that τ is a (simple) symmetry for the ASEP on X (also called CP-invariance).
Is this also true on the finite lattice $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$?

(c) In general: Do symmetries commute, i.e. does $T_1, T_2 \in \mathcal{S}$ imply that $T_1T_2 = T_2T_1$?

4. Exam question from last year.
Consider the ASEP on the finite lattice $\Lambda_L = \{1,\ldots,L\}$ with closed boundary conditions, i.e. for $p,q > 0$ the boundary rates are given by

$$\begin{aligned} c(1,2,\eta) &= p\eta(1)\big(1-\eta(2)\big)\,, \quad c(2,1,\eta) = q\eta(2)\big(1-\eta(1)\big)\,,\\ c(L{-}1,L,\eta) &= p\eta(L{-}1)\big(1-\eta(L)\big)\,, \quad c(L,L{-}1,\eta) = q\eta(L)\big(1-\eta(L{-}1)\big)\,. \end{aligned}$$

(a) Write down the generator for this process.

(b) Find a family of stationary product measures ν_ρ and give the corresponding density profile $\rho(x)$.
You may use any result from the course, provided that it is clearly stated.
(Hint: Consider first a simple random walker on Λ_L.)

(c) Sketch the density profiles $\rho(x)$ you found in (b) for $p > q$, $p < q$ and $p = q$.
Are the measures you found in (b) reversible? (Justify your answer.)

(d) Identify at least one simple symmetry τ for the process.
Justify your answer by showing that the jump rates are invariant under τ.

5. Show that a bounded linear operator $T : C(X) \to C(X)$ is a symmetry if and only if $\mathcal{L}T = T\mathcal{L}$.
(Hint: Look at the proof of Prop. 1.8 on stationarity iff $\mu(\mathcal{L}f) = 0$.)

6. Are there dynamical phase transitions for ZRPs analogous to the ones for the ASEP? Justify your answer.

7. Consider the totally asymmetric simple exclusion process (TASEP) $(\eta_t : t \geq 0)$ on the finite periodic lattice $\tilde{\Lambda}_L = \mathbb{Z}/L\mathbb{Z}$ with jump rates $p = 1$ and $q = 0$. Suppose that one of the particles is special, i.e. it jumps with rate $\alpha > 0$.

(a) Explain how a TASEP configuration η can be mapped onto a configuration ζ of a zero-range process (ZRP) with a special site in the origin.
Define the ZRP $(\zeta_t : t \geq 0)$ corresponding to the TASEP $(\eta_t : t \geq 0)$ by giving the lattice Λ, the state space and the generator.

(b) Compute the grand-canonical stationary product measures ν_ϕ of the ZRP and give the marginals on site 0 and any other site $x \neq 0$.

(c) Compute the stationary density $\rho_x(\phi) = \nu_\phi(\eta(x))$ as a function of the fugacity for $x = 0$ and $x \neq 0$.

(d) The total density is given by

$$\rho(\phi) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \rho_x(\phi) .$$

Use this to derive the current-density relation $j(\rho)$, i.e. the stationary current as a function of ρ. Do not attempt to solve the quadratic equation yourself but use something like MATLAB or MAPLE, then plot the function $j(\rho)$ for $L = 100$ and $\alpha = 0.5$ and 2.

(e) Explain why the total density ρ in the ZRP is related to the TASEP density $\tilde{\rho}$ via

$$\rho = \frac{1 - \tilde{\rho}}{\tilde{\rho}} ,$$

and the stationary currents are related via $\tilde{j} = \tilde{\rho}\, j$.
Use this to plot the current-density relation $\tilde{j}(\tilde{\rho})$ of the TASEP for $L = 100$ and $\alpha = 0.5$ and 2. Interpret your results.

8. (a) Consider a general ZRP and define $p_x(\phi) = \log z_x(\phi)$ (called pressure in statistical mechanics). Show that you can write $\rho_x(\phi) = \phi \frac{\partial}{\partial \phi} p_x(\phi)$. Use this to show that $\rho_x(\phi)$ is monotone increasing, which is equivalent to $p_x(\phi)$ being convex. To do this, compute the derivative of $\rho_x(\phi)$ and write it as the variance of $\eta(x)$ (which is positive).

(b) Consider now a ZRP on $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ with $g_x(k) = \alpha k$ and translation invariant jump probabilities $p(x, y) = p(x - y)$. Show that the stationary measure is homogeneous and Poisson and compute $\rho(\phi)$ and the current.

9. Suppose the stationary current of a lattice gas on $\Lambda = \mathbb{Z}$ is given by $j(\rho) = -\rho \log \rho$ and its density obeys the macroscopic conservation law

$$\frac{\partial}{\partial t}\rho(x, t) + \frac{\partial}{\partial t} j\big(\rho(x, t)\big) = 0 .$$

(a) Starting with a step initial measure

$$\mu = \nu_{\rho_l, \rho_r} \quad \text{product measure with} \quad \nu_{\rho_l, \rho_r}(1_x) = \begin{cases} \rho_l & , \; x \leq 10000 \\ \rho_r & , \; x > 10000 \end{cases} ,$$

what density $\mu_t(1_0)$ do you expect in the origin as $t \to \infty$?

(b) Summarize your results in a phase diagram for the corresponding dynamic phase transition.

(c) What density $\mu_t(1_{[t]})$ do you expect to see at site $[t]$ as $t \to \infty$?