Complex Geometry

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Contents

Chapter 1

Course Description

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Lecture time/room:

Monday 1pm - 2pm	MA_B3.02 (except week 10), A1.01 (week 10)
Wednesday 9am - 11am	MA_B1.01

Reference books:

- P. Griffiths, J. Harris: Principles of Algebraic Geometry, Wiley, 1978.
- D. Huybrechts: Complex geometry: An Introduction, Universitext, Springer, 2005.
- K. Kodaira: Complex manifolds and deformation of complex structures, Springer, 1986.
- R.O. Wells: Differential Analysis on Complex Manifolds, Springer-Verlag, 1980.
- C. Voisin: Hodge Theory and Complex Algebraic Geometry I/II, Cambridge University Press, 2002.
- W. Ballmann: Lectures on Kähler manifolds, ESI Lectures in Mathematics and Physics, European Mathematical Society, 2006.
- K. Fritzsche, H. Grauert: From Holomorphic Functions To Complex Manifolds, GTM 213, Springer-Verlag, 2002.
- S.S. Chern: Complex manifolds without potential theory, Springer-Verlag, 1979.

I have also observed Dominic Joyce's TCC module "Kähler Geometry" will be taught in this term:

https://people.maths.ox.ac.uk/∼joyce/KahlerGeom2020/index.html

Prerequisites: Familiarity with topics covered in MA3H5 Manifolds, MA3B8 Complex Analysis, MA3H6 Algebraic Topology.

MA475 Riemann Surfaces, MA4C0 Differential Geometry, MA4A5 Algebraic Geometry, MA4J7 Cohomology and Poincare duality would be certainly very helpful.

Contents: The primary goal of this Module is to present some fundamental techniques from several complex variables, Hermitian differential geometry (and partial differential equations, potential theory, functional analysis), to study the geometry of complex, and in particular, Kähler manifolds. Hodge theory will be one important major topic of this course. Some possible topics:

- Basics/definitions concerning complex manifolds, vector bundles and sheaf theory
- Some selected topics from several complex variables: the Cauchy integral, the Cauchy-Riemann equations, Hartogs's principle, plurisubharmonic functions, domains of holomorphy, holomorphic convexity, Riemann extension theorem, Hörmander's L2 estimates . . .
- Hermitian differential geometry, curvature of Hermitian holomorphic vector bundles, Chern classes
- Some elliptic operator theory, Kähler manifolds, Hodge decomposition, Kodaira embedding
- Outlook on the topology of varieties, Morse theory, Lefschetz pencils, variation of Hodge structures, Clemens-Schmid exact sequences, *etc.*

Chapter 2

Structures

2.1 Complex manifolds

We recall the definitions of differentiable manifolds and adapt them to complex manifolds.

A topological manifold is a second countable Hausdorff space M equipped with a covering by open sets U_α, which are homeomorphic, via local charts ϕ_α, to open sets of $\mathbb{R}^n$. Such an n is necessarily independent of α when M is connected, and is then called the dimension of M.

Definition 2.1.1. *A C^k differentiable manifold is a topological manifold equipped with a system of local charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that the open sets U_α cover M and the change of chart morphisms $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are differentiable of class C^k.*

A C^k differential function on such a manifold is a function f such that for each U_α, $f \circ \phi_\alpha^{-1}$ is differentiable of class C^k.

Let $f : M \to N$ be a continuous function between C^k differentiable manifolds M and N of dimensions m and n. We call f C^k differentiable if for any charts (U, ϕ) and (V, ψ) from the atlases on M and N, the map $\psi \circ f \circ \phi^{-1} : \phi(U) \cap f^{-1}(V) \to \psi(V)$ is a C^k map.

An atlas on M is a collection of local charts indexed by some set A which are pairwise compatible and $M = \cup_{\alpha \in A} U_\alpha$. An atlas is called maximal if every chart compatible with all the charts of the atlas is already in the atlas. Every atlas extends to a unique maximal atlas.

In the above definition, $k \in \mathbb{Z}^+ \cup \{\infty\}$. It is a theorem of Whitney that any C^k-differentiable manifold with $k \geq 1$ its maximal atlas contains a C^∞ atlas on the same underlying set. In particular, it could have a unique compatible structure of C^∞-differentiable (*i.e.* smooth) manifold. So later, we might misleadingly call all these C^k manifolds smooth manifolds. In fact, any C^k–structure is smoothable to a real analytic (C^ω) structure.

A smooth map $f : N \to M$ between smooth manifolds is an embedding if it is a diffeomorphism onto its image. We refer to the image of such a map as a submanifold of M. As an equivalent definition, A k-dimensional submanifold of M is a subset $S \subset M$ such that for every point $p \in S$ there exists a chart (U, ϕ) of M containing p such that $\phi(S \cap U)$ is the intersection of a k-dimensional plane with $\phi(U)$.

If M is a smooth manifold of dimension $2n$, we can define when it is a complex manifold.

Definition 2.1.2. *A complex manifold M is a smooth manifold admitting an open cover $\{U_\alpha\}$ and local charts $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are holomorphic. The complex dimension of M is n.*

A holomorphic function on a complex manifold is a complex valued function f such that for each U_α, $f \circ \phi_\alpha^{-1}$ is holomorphic.

Let $f : M \to N$ be a continuous function between complex manifolds M and N of dimensions m and n. We call f holomorphic if for any charts (U, ϕ) and (V, ψ) from the atlases on M and N, the map $\psi \circ f \circ \phi^{-1} : \phi(U) \cap \phi \circ f^{-1}(V) \to \psi(V)$ is a holomorphic map.

The set (ring) of all holomorphic functions on will be denoted by $\mathcal{O}(M)$. If M is compact, then any global holomorphic function f must be constant by maximal principle (restrict it on a neighborhood of a maximal point of $|f|$). That is $\mathcal{O}(M) = \mathbb{C}$. In general, we can ask when we have sufficiently many holomorphic function, such that a collection of them $(f_1, \cdots, f_N)$ could provide a proper embedding from M to $\mathbb{C}^N$. This is a strong restriction on the manifold, and any of such is called a Stein manifold.

Let M be a complex manifold of complex dimension n and let $N \subset M$ be a smooth submanifold of real dimension $2k$. Then N is a complex submanifold if there exists a holomorphic atlas $\{(U_\alpha, \phi_\alpha)\}$ of M such that $\phi_\alpha : U_\alpha \cap N \cong \phi_\alpha(U_\alpha) \cap \mathbb{C}^k$. A generalization of complex submanifold is analytic subvariety. An analytic subvariety of M is a closed subset $N \subset M$ such that for any point $x \in N$ there exists an open neighborhood $x \in U \subset M$ such that $N \cap U$ is the zero set of finitely many holomorphic functions in $\mathcal{O}(U)$.

Being a complex submanifold or subvariety is a very restrictive condition. For example, a real line in a complex plane cannot be. But on contrary, any closed subset of $\mathbb{R}^2$ could be the zero set of a smooth function by Whitney extension theorem.

2.1.1 Examples of Complex manifolds

1. $\mathbb{C}^n$ is a complex manifold of dimension n, A complex domain $\Omega \subset \mathbb{C}^n$ is a complex n-manifold. For example, $GL(n, \mathbb{C})$ is a complex manifold.

2. The complex projective space $\mathbb{C}P^n$ is the quotient space of $\mathbb{C}^{n+1} \backslash \{0\}$ by the equivalent relation:

$$(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n), \lambda \in \mathbb{C}^*.$$

We denote the equivalence class by the homogeneous coordinate $[z_0 : \cdots : z_n]$.

We define $U_0, \cdots, U_n$ by $U_i = \{[z_0 : \cdots : z_n] | z_i \neq 0\}$. U_i is the set of complex lines in $\mathbb{C}^{n+1}$ which pass through the origin and do not belong to the hyperplane $z_i = 0$. Define $\phi_j : U_j \to \mathbb{C}^n$ by

$$\phi_j[z_0 : \cdots : z_n] = (\frac{z_0}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j}).$$

The transition function

$$\phi_{01}(u_1, \cdots, u_n) = \phi_1([1 : u_1 : \cdots : u_n]) = (\frac{1}{u_1}, \frac{u_2}{u_1}, \cdots, \frac{u_n}{u_1})$$

is holomorphic on $\phi_0(U_0 \cap U_1)$, making $\mathbb{C}P^n$ a complex manifold of dimension n.

Two ways of generalization. First, $\mathbb{C}P^n$ parametrizes lines or hyperplanes in $\mathbb{C}^{n+1}$. We can also parametrize complex subspaces of dimension k in $\mathbb{C}^n$, called Grassmannian $Gr_{\mathbb{C}}(k, n)$. It has dimension $k(n-k)$. More generally, we can paramatrize all Flags as sequences of vector spaces: $0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{C}^n$. This is called flag varieties. Even more generally, we have complex Lie Groups modulo any parabolic subgroups.

Definition 2.1.3. *A complex manifold M is projective if M is a compact complex submanifold in $\mathbb{C}P^n$.*

By Chow's theorem, a projective complex manifold (or variety) is the zero set of finitely many homogeneous polynomials.

All the above generalizations of $\mathbb{C}P^n$ are projective. For Grassmannian, a famous embedding is the Plücker embedding:

$$\iota : Gr(k, n) \to \mathbb{P}(\Lambda^k \mathbb{C}^n),$$

defined by $\iota(W) = [w_1 \wedge \cdots \wedge w_k]$ for any basis $\{w_1, \cdots, w_k\}$ of a k-dimensional subspace W.

Exercise: What is the defining equation of $Gr(2, 4)$?

3. Analytic or biholomorphic automorphisms form a group with respect to composition, $Aut(M)$. Let G be a subgroup of it. G is called properly discontinuous if for any pair of compact subsets $K_1, K_2 \subset M$, the set $\{g \in G | gK_1 \cap K_2 \neq \emptyset\}$ is finite.

Proposition 2.1.4. *If G is properly discontinuous and has no fixed point, then the quotient space M/G is a complex manifold.*

Conversely, if $\pi : M \to N$ is a topological (unramified) covering and N is a complex manifold, then M has a complex structure (such that π is holomorphic).

Examples include complex Tori. Let $\Lambda = \mathbb{Z}^{2n} \subset \mathbb{C}^n$ is a discrete lattice then $\mathbb{C}^n/\Lambda$ is a complex torus. When $n = 1$, they are always projective. When $n > 1$, it is not always true. For example, the complex manifold obtained from Λ generated by $(\sqrt{-5}, \sqrt{-7}), (\sqrt{-2}, \sqrt{-3}), (1, 0), (0, 1)$ is not projective, and even does not have any non-trivial subvariety. In general, we have Riemann relations to charaterize projective complex tori, or Abelian varieties.

Another example is Hopf manifolds. Consider $\mathbb{C}^n\backslash\{0\}, n > 1$, again. We define a biholomorphic automorphism ϕ by $\phi(z) = \alpha z$ where $\alpha \in \mathbb{C}$ is a constant with $0 < |\alpha| < 1$. Then the group $\langle\phi\rangle = \mathbb{Z}$ acts freely and properly discontinuously on $\mathbb{C}^n\backslash\{0\}$. Then the complex structure of $\mathbb{C}^n\backslash\{0\}$ descends to the quotient M^n_α, being a compact complex manifold. It is called a Hopf manifold. The underlying smooth manifold is $S^1 \times S^{2n-1}$. In general, we could choose α as the multivector $(\alpha_1, \cdots, \alpha_n)$.

Another important family is provided by quotient of unit disc $D^n = \{z|||z|| < 1\} \subset \mathbb{C}^n$. We can also view is as an open subset in $\mathbb{C}P^n$. If we introduce the Hermitian product on $\mathbb{C}^{n+1}$ given by the diagonal matrix $\mathrm{diag}(1, -1, \cdots, -1)$. Then D^n is the open subset in $\mathbb{C}P^n$ with $\langle z, z\rangle > 0$, which is invariant under $\mathrm{SU}(\langle,\rangle) = \mathrm{SU}(1, n)$. A ball quotient is a quotient of D^n by any discrete group $\Gamma \subset \mathrm{SU}(1, n)$. An example is a Riemann surface of genus $g > 1$, which is the quotient of D^1 by a Fuchsian group (discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$). $n = 1$ and $n > 1$ are quite different as the lattice in first is not local rigid, which corresponds to continuous moduli of Riemann surface, while the latter is rigid.

4. It is a natural question to ask which smooth manifold can be endowed with a complex structure. Certainly, it has to be even dimensional and orientable (*i.e.* the Jacobian determinants of the transition functions are positive) as the transition matrix has positive determinant (exercise). But the restriction on the topology is actually more subtle and does not know a general criterion.

Even we restrict on spheres. It is known that S^{2n} does not have complex structure for $n \neq 1, 3$ (Borel-Serre 1953). As $S^2 \cong \mathbb{C}P^1$, it has a standard complex structure. But it is a long standing question whether S^6 has one or not.

It is also an important problem to know the restrictions on being projective manifolds (or Kähler manifolds). In the class, we will discuss Hodge theory, which will give nontrivial restrictions.

2.2 Vector bundles and the tangent bundle

We first define tangent space which is a fiber of the tangent bundle. To define the tangent space, we recall the notion of germ. A germ is an equivalence class of pairs (U, f_U), where U is an open neighborhood of x

and $f_U : U \to \mathbb{R}$ is a smooth function. The pairs (U, f_U) and (V, f_V) are equivalent if f_U and f_V are equal on some neighborhood $W \subset U \cap V$ of x. Let $\mathcal{O}_x$ be the vector space of germs at x.

A local derivation of $\mathcal{O}_x$ is a linear map $X : \mathcal{O}_x \to \mathbb{R}$ such that

$$X(fg) = f(x)X(g) + g(x)X(f).$$

Any such local derivative is called a tangent vector.

Let (U, ϕ) be a local chart with coordinate functions $x_1, \cdots, x_n$ and $a_1, \cdots, a_n \in \mathbb{R}$, then

$$Xf = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}|_x := \sum a_i \frac{\partial f \circ \phi^{-1}}{\partial r_i}|_{\phi(x)}$$

where r_i are standard coordinate functions on $\mathbb{R}^n$, is a local derivation. In fact, every local derivation is of this form.

Definition 2.2.1. *The tangent space T_xM is the set of such local derivatives. It is an n-dimensional real vector space if M is an n-dimensional manifold.*

Follows from the above discussion, $\{\partial_{x_1}, \cdots, \partial_{x_n}\}$ is a basis of T_xM.

Intuitively, a tangent vector is an equivalence class of paths through x: two paths are equivalent if they are tangent at x. By a path we mean a smooth map $u : (-\epsilon, \epsilon) \to M$ such that $u(0) = x$ for some $\epsilon > 0$. Given a function, we can use the path to define a local derivation $Xf = \frac{d}{dt} f(u(t))|_{t=0} = \sum a_i \frac{\partial f}{\partial x_i}(x)$. We sometimes write this $X \in T_xM$ as $u'(0)$.

By a vector field X on M, we mean a rule that assigns to each point $x \in M$ an element $X_x \in T_xM$ and the assignment $x \mapsto X_x$ is smooth.

Proposition 2.2.2. *There is a one-to-one correspondence between vector fields on a smooth manifold M and derivations of $C^\infty(M)$.*

This implies vector fields can be restricted and patched, hence a sheaf.

Now consider two vector fields X, Y. Locally, $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum b_i \frac{\partial}{\partial x_i}$ where a_i, b_i are smooth functions. We have $[X, Y] = \sum_{i,j} (a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}) \frac{\partial}{\partial x_i}$.

Given a smooth map $\theta : M \to N$, one can describe the map $d_x\theta : T_xM \to T_{\theta(x)}N$ (sometimes, we also write it as θ_*) by saying that it sends $v \in T_xM$ to $(\theta \circ \gamma)'(0)$ for any curve γ in M with $\gamma'(0) = v$. Or more directly, $d_x\theta(X_x)(f) = X_x(f \circ \theta)$. Two vector fields X, Y on M and N respectively are said to be θ-related if $d_x\theta(X_x) = Y_{\theta(x)}$ for all $x \in M$.

The dual space of the tangent space is called the cotangent space of M at x, and is denoted by T_x^*M. In the special case $N = \mathbb{R}$ above, we have $X_x(f) = d_xf(X_x)$. In other words, d_xf is a cotangent vector at x. In local charts, the dual basis of $\{\partial_{x_1}, \cdots, \partial_{x_n}\}$ in T_x^*M is $\{dx_1, \cdots, dx_n\}$. And we have $df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n$ at x.

For complex manifolds, we can write the complex coordinate functions $z_1 = x_1 + iy_1, \cdots, z_n = x_n + iy_n$. If we let $\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - \sqrt{-1}\partial_{y_i})$ and $\partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} + \sqrt{-1}\partial_{y_i})$, then the complexified tangent space $T_xM \otimes \mathbb{C}$ is spanned by $\{\partial_{z_1}, \cdots, \partial_{z_n}, \partial_{\bar{z}_1}, \cdots, \partial_{\bar{z}_n}\}$. We have $T_xM \otimes \mathbb{C} = T_x^{1,0}M \oplus T_x^{0,1}M$ where the holomorphic tangent space $T_x^{1,0}M$ is spanned by $\{\partial_{z_1}, \cdots, \partial_{z_n}\}$ and the anti-holomorphic tangent space $T_x^{0,1}M$ is spanned by $\{\partial_{\bar{z}_1}, \cdots, \partial_{\bar{z}_n}\}$. We also have the dual spaces spanned by $dz_1, \cdots, dz_n$ and $d\bar{z}_1, \cdots, d\bar{z}_n$ respectively. In particular, any holomorphic function f has $df = \frac{\partial f}{\partial z_1}dz_1 + \cdots + \frac{\partial f}{\partial z_n}dz_n$ at x.

The implicit function theorem implies the following.

Theorem 2.2.3. *If $p \in N$ is a regular value of smooth map $f : M \to N$ between manifolds, then $f^{-1}(p)$ is a manifold of dimension $\dim M - \dim N$. Moreover, $\ker d_q f = T_q(f^{-1}(p))$ for any $q \in f^{-1}(p)$.*

Here a regular value means for all the preimage $q \in f^{-1}(p)$, the tangent map $d_q f$ is surjective (or of rank $\dim M - \dim N$). The theorem also has the complex version where f is a holomorphic map between complex manifolds and we look at points p such that at any $q \in f^{-1}(p)$ has the complex version of Jacobian $\mathcal{J}(f) = (\frac{\partial f_i}{\partial z_j})$ is of full rank. Then $f^{-1}(p)$ is a complex manifold of dimension $\dim M - \dim N$.

Apply to projective manifold, we have the following construction. If $P(z_0, \cdots, z_n)$ is a non-zero homogeneous complex polynomial, and for all the point in $\mathbb{C}P^n$ with $P(z_0, \cdots, z_n) = 0$, we have $\frac{\partial P}{\partial z_i} \neq 0$ for some i. Then

$$M = \{[z_0, \cdots, z_n] \in \mathbb{C}P^n : P(z_0, \cdots, z_n) = 0\}$$

is a projective manifold of dimension $n - 1$.

Exercise: Generalize it to higher codimensions.

Example 2.2.4. *Plane curves.*

In fact, all these tangent spaces could be bundled together to form a manifold TM, called the tangent bundle. We recall the definitions for vector bundle.

A real (resp. complex) topological vector bundle of rank m over a topological space X is a topological space E equipped with a map $\pi : E \to M$ such that for an open cover $\{U_\alpha\}$ of M, we have local trivialization homeomorphisms

$$\tau_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^m \quad (\text{resp. } U_\alpha \times \mathbb{C}^m)$$

such that

1. $pr_1 \circ \tau_\alpha = \pi$.

2. The transition functions

$$\tau_{\alpha\beta} = \tau_\beta \circ \tau_\alpha^{-1} : \tau_\alpha(\pi^{-1}(U_\alpha \cap U_\beta)) \to \tau_\beta(\pi^{-1}(U_\alpha \cap U_\beta))$$

are $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) on each fiber $u \times \mathbb{R}^m$ (resp. $u \times \mathbb{C}^m$).

Such a transformation between $U_\alpha \cap U_\beta \times \mathbb{R}^m \subset U_\alpha \times \mathbb{R}^m$ and $U_\alpha \cap U_\beta \times \mathbb{R}^m \subset U_\beta \times \mathbb{R}^m$ must respect pr_1, and is thus described by a continuous function from $U_\alpha \cap U_\beta$ to $\mathrm{GL}(m, \mathbb{R})$ or $\mathrm{GL}(m, \mathbb{C})$. These are called transition matrices.

Definition 2.2.5. *If M is a smooth manifold, a smooth vector bundle E is a vector bundle with given local trivializations whose transition matrices are smooth.*

A smooth complex vector bundle $\pi : E \to M$ over a complex manifold M is holomorphic if E is a complex manifold such that $\pi_E : E \to M$ is holomorphic.

Equivalently, a complex vector bundle E is a holomorphic vector bundle if and only if both E and M are complex manifolds such that for any $x \in M$ there exists $x \in U$ in M and a trivialization $\tau_U : E_U \to U \times \mathbb{C}^k$ that is a biholomorphic map of complex manifolds. This is equivalent to the transition functions for E are holomorphic. If there is a global trivialization, we call it trivial bundle (in smooth or holomorphic sense). Notice there are smoothly trivial but holomorphically non-trivial vector bundles.

We can also use transition functions to define a vector bundle. Given an open cover $\{U_\alpha\}$ of M and smooth (resp. holomorphic) maps $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(m, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ satisfying $\tau_{\alpha\alpha} = I$ and $\tau_{\alpha\beta}\tau_{\beta\gamma}\tau_{\gamma\alpha} = I$ on $U_\alpha \cap U_\beta \cap U_\gamma$. It is not hard to check there is a unique (smooth or holomorphic) vector bundle $E \to M$ with transition functions $\{\tau_{\alpha\beta}\}$ and thus these two definitions are equivalent.

For two holomorphic (resp. smooth) vector bundles E and F over M, a holomorphic (resp. smooth) vector bundle homomorphism between E and F is a holomorphic (resp. smooth) map $\phi : E \to F$ with $\pi_E = \pi_F \circ \phi$ such that the induced map $\phi(x) : E_x \to F_x$ is linear with $\mathrm{rk}(\phi(x))$ is independent of $x \in M$. Two vector bundles E and F are holomorphically (resp. smoothly) isomorphic if there is a biholomorphic (resp. bi-smooth) vector bundle homomorphism.

Use the transition function viewpoint, two vector bundles $\{\tau_{\alpha\beta}\}$ and $\{\tau'_{\alpha\beta}\}$ are holomorphically (resp. smoothly) equivalent if there are nonvanishing matrix valued holomorphic (resp. smooth) functions $h_\alpha(x)$ on U_α such that $\tau'_{\alpha\beta} = h_\alpha \tau_{\alpha\beta} h_\beta^{-1}$.

A section of a vector bundle $E \to M$ is a map $s : M \to E$ such that $\pi \circ s = Id_M$. This section is said to be continuous, smooth, or holomorphic if s is so respectively. We will denote by $\Gamma(U, E)$ the set of all holomorphic sections of E over U.

We can also construct vector bundles out of what we have. Suppose F and G are vector bundles with rank r and s and transition functions $\{f_{\alpha\beta}\}$ and $\{g_{\alpha\beta}\}$. The following constructions work in any of the category: continuous, smooth, holomorphic, $\cdots$.

1. Whitney sum $F \oplus G$ has rank $r+s$ and is given by transition functions

$$h_{\alpha\beta} = \left(\begin{array}{cc} f_{\alpha\beta} & 0 \\ 0 & g_{\alpha\beta} \end{array} \right).$$

2. Tensor product $F \otimes G$ has rank rs and the transition functions are given by

$$h^{ik,jl}_{\alpha\beta} = f^{ij}_{\alpha\beta} g^{kl}_{\alpha\beta}.$$

3. Dual bundle F^* has rank r and transition function ${}^t f^{-1}_{\alpha\beta}$.

4. Exterior product $\Lambda^q F$ has rank $\binom{r}{q}$ and transition functions

$$h^{IJ}_{\alpha\beta} = \sum_{\pi} \epsilon_\pi \, f^{i_1 j_{\pi(1)}}_{\alpha\beta} \cdots f^{i_q j_{\pi(q)}}_{\alpha\beta}$$

where $I = (i_1, \cdots, i_q), 1 \leq i_1 < \cdots < i_q \leq r$ and $J = (j_1, \cdots, j_q), 1 \leq j_1 < \cdots < j_q \leq r$ and the sum is taken from all the permutations π and $\{1, \cdots, q\}$.

5. Symmetric product $S^q F$, leave as exercise.

6. If the original bundle is complex, we have complex conjugate $\bar{F}$. It is defined by transition functions $\{\bar{f}_{\alpha\beta}\}$.

7. A subset $F \subset E$ of a vector bundle $\pi : E \to M$ such that $\pi|_F : F \to M$ is a vector bundle and $F \subset \pi^{-1}(x)$ is a vector subspace of $\pi^{-1}(x)$ is said to be a subbundle of the vector bundle E. We can then choose local trivializations such that the transition function $g_{\alpha\beta}$ of E to be

$$g_{\alpha\beta} = \left(\begin{array}{cc} h_{\alpha\beta} & k_{\alpha\beta} \\ 0 & q_{\alpha\beta} \end{array} \right),$$

where F has transition functions $h_{\alpha\beta}$. And the quotient bundle E/F whose fibers are E_x/F_x has transition functions $q_{\alpha\beta}$. We have short exact sequence $0 \to F \to E \to E/F \to 0$. This always splits in smooth category, *i.e.* $E = F \oplus E/F$ and local trivialization could be chosen such that $k_{\alpha\beta} = 0$. But this is not true in holomorphic category.

8. Let $f : N \to M$ be a holomorphic (resp. smooth) map between complex (resp. smooth) manifolds and let E be a holomorphic (smooth) vector bundle on M given by $\{U_\alpha\}$ and $\tau_{\alpha\beta}$. Then the pull back f^*E is a holomorphic (resp. smooth) vector bundle over N that is given by $\{f^{-1}(U_\alpha)\}$ and $\tau_{\alpha\beta} \circ f$. For any $y \in N$, there is a canonical isomorphism $(f^*E)_y \cong E_{f(y)}$.

Come back to the tangent bundle TM of a smooth manifold M (of dimension n). Let M have the atlas with local charts $(U_\alpha, \phi_\alpha = (x_{\alpha 1}, \cdots, x_{\alpha n}))$

with α in an index set A. Then the local trivialization of TM over U_α is given in terms of the basis $\{\partial_{x_{\alpha 1}}, \cdots, \partial_{x_{\alpha n}}\}$. That is,

$$(x, \sum_{i=1}^{n} a_k \partial_{x_{\alpha i}}|_x) \mapsto (x, (a_1, \cdots, a_n)).$$

The transition maps between $U_\alpha \cap U_\beta \times \mathbb{R}^n \subset U_\alpha \times \mathbb{R}^n$ and $U_\alpha \cap U_\beta \times \mathbb{R}^n \subset U_\beta \times \mathbb{R}^n$ are given by $(u, v) \to (u, d\phi_{\alpha\beta}(v))$. Here $d\phi_{\alpha\beta}$ is the Jacobian matrix $(\frac{\partial x_{\alpha i}}{\partial x_{\beta j}})_{n\times n}$. Precisely,

$$\tau_\beta \circ \tau_\alpha^{-1}(x, \mathbf{a}) = (x, \sum_{i=1}^{n} a_i \partial_{x_{\alpha i}}|_x) = (x, \sum_{i=1}^{n} a_i \frac{\partial x_{\alpha 1}}{\partial x_{\beta i}}, \cdots, \sum_{i=1}^{n} a_i \frac{\partial x_{\alpha n}}{\partial x_{\beta i}}).$$

A section of the tangent bundle is a vector field.

A smooth manifold is orientable if and only if we can have an atlas such that the transition matrices are in $\mathrm{GL}^+(n, \mathbb{R})$.

Similarly, the cotangent spaces also form a vector bundle, called the cotangent bundle T^*M. It is the dual bundle of TM. A smooth section of the cotangent bundle is a 1-form (or sometimes called a covector). Given each $x \in M$, we can form the pth exterior power of the cotangent space. This is also a vector bundle, $\Lambda^p T^*M$. A section of it is called a p-form.

This construction also applies to complex setting. First, $T^{1,0}M$ or $T^*_{1,0}M$ is a holomorphic bundle. Thus, all of its exterior products $\Lambda^{p,0}M = \wedge^p T^*_{1,0}M$ are so. In particular, if $\dim_{\mathbb{C}} M = n$, then $\Lambda^{n,0}M$ is a holomorphic line bundle, which is called the canonical bundle $\mathcal{K}_M$.

2.2.1 Holomorphic vector bundles

We start to talk about holomorphic line bundles and divisors. A divisor D on M is a locally finite formal linear combination $D = \sum a_i V_i$ where V_i are irreducible analytic hypersurfaces of M and $a_i \in \mathbb{Z}$. A divisor is called effective if $a_i \geq 0$ for all i. The set $Div(M)$ of divisors is a group under addition in the obvious way. There is a basic correspondence between divisors and holomorphic line bundles. First, for a meromorphic section s of a holomorphic line bundle (*i.e.* a locally defined holomorphic function with values in $\mathbb{C}P^1$), we can associate a divisor $(s) := \sum_V ord_V(s)V$ by a weighted sum of its zeros and poles, where V are irreducible hypersurfaces. Let f_α be local defining functions of D over some open cover $\{U_\alpha\}$ of M. Then the functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ are holomorphic and nonzero in $U_\alpha \cap U_\beta$ with $g_{\alpha\beta} g_{\beta\alpha} = 1$, and in $U_\alpha \cap U_\beta \cap U_\gamma$ we have $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. With these identities for $\{g_{\alpha\beta}\}$, we can construct a line bundle L by taking the union $\cup_\alpha U_\alpha \times \mathbb{C}$ with points $(x, \lambda) \in U_\beta \times \mathbb{C}$ and $(x, g_{\alpha\beta}(x)\lambda) \in U_\alpha \times \mathbb{C}$ identified. The line bundle given by the transition functions $\{g_{\alpha\beta}\}$ is called the associated line bundle of D and denoted by L_D (or $\mathcal{O}(D)$). It is easy to check that it is well defined. Denote by $Pic(M)$ the set of isomorphism classes of holomorphic line bundles. In sheaf theory language, $Pic(M) \equiv H^1(M, \mathcal{O}^*)$.

Recall that collection of transition functions define the same line bundle if and only if there exists non-vanishing holomorphic functions f_α on U_α such that $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$. Tensor product makes $Pic(M)$ into an abelian group, called the Picard group of M ($L \otimes L^* = End(L)$ is a trivial line bundle because identity $L \to L$ gives a nowhere zero section). The kernel of the homomorphism $Div(M) \to Pic(M)$ is those divisors (f) where f is a meromorphic section of the trivial bundle, *i.e.* a meromorphic function on M. In fact, if D is given by f_α and L_D is trivial, then there exists $h_\alpha \in \mathcal{O}^*(U_\alpha)$ such that $\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$. Then $f = f_\alpha h_\alpha^{-1}$ is a global meromorphic function on M with divisor D. Two divisors are called linearly equivalent if $D \sim D'$, *i.e.* when $D - D' = div(f)$. Thus the group homomorphism factors through an injection $Div(M)/\sim \to Pic(M)$. This homomorphism need not to be surjective, although it is true when M is a projective manifold.

There is a holomorphic line bundle which is the "universal" line bundle. This is a tautological bundle over $\mathbb{C}P^n$, denoted $\mathcal{O}(-1)$: the fibre of $\mathcal{O}(-1)$ over z is precisely the line $L_z \subset \mathbb{C}^{n+1}$ corresponding to z, or formally, the pairs $(l, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1}$ with $z \in l$. Over open subsets $U_i \subset \mathbb{C}P^n$, a canonical trivialization is given by $\tau_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}, (l, w) \mapsto (l, w_i)$. The transition map $\tau_{ij}(l) : \mathbb{C} \to \mathbb{C}$ are given by $w \mapsto \frac{z_j}{z_i} w$ where $l = [z_0 : \cdots : z_n]$.

The dual of $\mathcal{O}(-1)$, *i.e.* the bundle whose fiber corresponds to linear functional over the line, is denoted by $\mathcal{O}(1)$. More generally, we have $\mathcal{O}(k)$ as the k^{th} tensor power of the $\mathcal{O}(1)$ for $k > 0$. When $k < 0$, $\mathcal{O}(k) = \mathcal{O}(-k)^*$ and $\mathcal{O}(0)$ is the trivial bundle.

The global linear coordinates $z_0, \cdots, z_n$ on $\mathbb{C}^{n+1}$ define natural sections of $\mathcal{O}(1)$. For $k > 0$, consider a homogeneous degree k polynomial. It gives a map $\mathbb{C}^{n+1} \to \mathbb{C}$, restricting to L_x we get a degree k map $L_x \to \mathbb{C}$. It is a linear map $L_x^k \to \mathbb{C}$ and so a section of $\mathcal{O}(k)$. On the other hand, these are all holomorphic sections of $\mathcal{O}(k)$. We only need to show that the effective divisor $D = (s)$ of a holomorphic section s is the zero set of a homogeneous degree k polynomial. Let s_F be a section as above, then $\frac{s}{s_F}$ is a meromorphic function on $\mathbb{C}P^n$. Pull it back to $\mathbb{C}^{n+1} \backslash \{0\}$ and multiply it with F, we get a holomorphic function G. It could be extended to $\mathbb{C}^{n+1}$ by Hartogs' theorem.

Theorem 2.2.6 (Hartogs). *A holomorphic function on the complement of a point in an open set $U \subset \mathbb{C}^n, n \geq 1$ extends to a holomorphic function in all of U.*

Proof. We look at each slice $z_i = const$, $i = 1, \cdots, n-1$. Set

$$F(z_1, \cdots, z_{n-1}, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{|w_n|=r} \frac{f(z_1, \cdots, z_{n-1}, w_n) dw_n}{w_n - z_n}.$$

F is define on U. It is clearly holomorphic in z_n, and since $\frac{\partial}{\partial \bar{z}_i} = 0, i = 1, \cdots, n-1$, F is holomorphic in $z_1, \cdots, z_{n-1}$ as well. Moreover, $F = f$ on

$U \setminus \{p\}$ first on slices with at least one $z_i \neq 0$ by Cauchy's formula and then by continuity on $z_1 = \cdots = z_{n-1} = 0$. $\square$

Moreover, we have $G(\lambda X) = \lambda^k G(X)$. By restricting on a line in $\mathbb{C}^n$, *i.e.* look at $\iota^* G$ for $\iota : t \mapsto (\mu_0 t, \cdots, \mu_n t)$, we know $\iota^* G$ is either identically zero or has a zero of order k at $t = 0$ and pole of order k at $t = \infty$. So the power series expansion for G around the origin contains only terms with degree k. Such a holomorphic function G is a homogeneous degree k polynomial. Hence $\dim H^0(\mathbb{C}P^n, \mathcal{O}(k)) = \frac{(k+n)!}{k!n!}$.

Exercise: What is the holomorphic tangent bundle of $\mathbb{C}P^1$ in terms of the notation $\mathcal{O}(k)$?

The above description of $\mathcal{O}(1)$ also leads to the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{C}P^n} \to \mathcal{O}(1)^{\oplus n+1} \to T^{1,0}\mathbb{C}P^n \to 0.$$

The first map is $1 \mapsto (z_0, \cdots, z_n)$. For the second map, let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$. We look at U_0 and let $z_0, \cdots, z_n$ be coordinates for $\mathbb{C}^{n+1}$ and $Z_i = \frac{z_i}{z_0}$ be coordinates on $U_0 \subset \mathbb{C}P^n$. Then we have

$$\pi_* \frac{\partial}{\partial z_i} = \frac{1}{z_0} \cdot \frac{\partial}{\partial Z_i}, i = 1, \cdots, n, \quad \pi_* \frac{\partial}{\partial z_0} = -\sum \frac{z_i}{z_0^2} \frac{\partial}{\partial Z_i}.$$

Hence for any linear functions $l(z)$ on $\mathbb{C}^{n+1}$, the vector field $v(z) = l(z)\frac{\partial}{\partial z_i}$ descends to $\mathbb{C}P^n$. That is $\pi_* v(z) = \pi_* v(\lambda z)$. Moreover, $T^{1,0}\mathbb{C}P^n$ is spanned by $\{\pi_*(z_i \frac{\partial}{\partial z_j})\}_{i,j=0,\cdots,n}$ with the single relation provided by the Euler vector field $E = \sum_{i=0}^n z_i \partial_{z_i}$: $\pi_* E = 0$. This is true since $\sum_{i=0}^n z_i \partial_{z_i} f = d \cdot f$ on homogeneous function f of degree d and also functions on $\mathbb{C}P^n$ corresponds to homogeneous functions of degree 0. With these understood, the second map is $(\sigma_0, \cdots, \sigma_n) \mapsto \pi_*(\sigma_i(z)\frac{\partial}{\partial z_i})$, where σ_i are sections of $\mathcal{O}(1)$.

Thus the kernel of the second map is multiples of $(z_0, z_1, \cdots, z_n)$ and the exact sequence holds.

We remark that the above description of sections of $\mathcal{O}(k)$ actually implies Chow's theorem:

Theorem 2.2.7. *Any analytic subvariety of projective space is algebraic.*

The above constructions could be generalized to Grassmannian $Gr_{\mathbb{C}}(k, n)$. We have a tautological holomorphic vector bundle S of rank k whose fiber over $p \in Gr_{\mathbb{C}}(k, n)$ is just the vector space corresponding to p. So S is a subbundle of the trivial bundle of rank n. Its quotient bundle is called the universal quotient bundle of $Gr_{\mathbb{C}}(k, n)$.

We can classify holomorphic vector bundles over S^2.

Theorem 2.2.8 (Grothendieck-Birkhoff)**.** *Any holomorphic vector bundle E on S^2 is isomorphic to a sum $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$. The ordered sequence $a_1 \geq \cdots \geq a_n$ is uniquely determined.*

Proof. Write $S^2 = U_0 \cup U_\infty$ where $U_0 = S^2 - \{\infty\}$ and $U_\infty = S^2 - \{0\}$. Then $\mathcal{O}(k)$ could be determined from the gluing of $U_0 \times \mathbb{C}$ and $U_\infty \times \mathbb{C}$ by the map

$$(z, \lambda) \mapsto (z, z^k \lambda).$$

For our holomorphic bundle E, its restriction to U_0 and U_∞ are trivial as both are Stein. So E is obtained by gluing $U_0 \times \mathbb{C}^n$ to $U_\infty \times \mathbb{C}^n$ by a holomorphic function $\tau : U_0 \cap U_\infty \to \mathrm{GL}(n, \mathbb{C})$. By Birkhoff's factorization theorem, we can factor τ as $\tau_- \cdot D \cdot \tau_+$, where τ_+ and τ_- can be extended to and are holomorphic in U_0 and U_∞ respectively and $D = z^{\mathbf{a}}$ is diagonal. If we change coordinates in $U_0 \times \mathbb{C}^n$ by τ_+ and in $U_\infty \times \mathbb{C}^n$ by τ_-^{-1}, we will have transition matrix given by a diagonal one with elements $z^{a_1}, \cdots, z^{a_n}$, in other words a direct sum $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$. □

The original version of Birkhoff factorization says that any (smooth, holomorphic, *etc.*) function $f : S^1 \to GL(n, \mathbb{C})$ can be factored as $f_- \cdot D \cdot f_+$ where f_+ is the boundary of a map from $\{|z| \leq 1\}$ to $GL(n, \mathbb{C})$, f_- is the boundary of a map from $\{|z| \geq 1\}$ to $GL(n, \mathbb{C})$, and D is a diagonal matrix $z^{\mathbf{a}}$. This result could be applied in our setting by let $f = \tau|_{|z|=1}$ and by the unique continuation of holomorphic functions.

2.3 Almost complex structure and integrability

When we have a complex manifold, there is a $\mathbb{R}$-linear operator $J : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ defined by $J\partial_{z_j} = i\partial_{z_j}$ and $J\partial_{\bar{z}_j} = -i\partial_{\bar{z}_j}$; or on TM by $J\partial_{x_j} = \partial_{y_j}$ and $J\partial_{y_j} = -\partial_{x_j}$. Such an endomorphism could be defined on the tangent bundle of certain smooth manifolds with even dimension. An almost complex structure J is an endomorphism $J : TM \to TM$ of the tangent bundle TM such that $J^2 = -Id$. Almost complex manifolds have even dimension since $\det(J)^2 = \det(J^2) = \det(-Id) = (-1)^m$. Moreover, M has to be orientable. This is because we can choose a base $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ in each T_xM and any such two bases give the same orientation for T_xM. (basically due to the fact $\mathrm{GL}(n, \mathbb{C})$ could be canonically embedded to $\mathrm{GL}^+(2n, \mathbb{R})$ or $U(n) \subset SO(2n)$. More explicitly, the matrix representation of changing a base is

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

which is of positive determinant.) This gives the orientation of TM.

Given almost complex structure J on M, the complexification of the tangent bundle of M decomposes as $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ where J acts on $T^{1,0}M$ as i and on $T^{0,1}M$ as $-i$. For any $X \in TM$, $X + iJX \in T^{1,0}M$ and $X - iJX \in T^{0,1}M$. For cotangent bundle, we have $T^*M \otimes \mathbb{C} = T^*_{1,0}M \oplus T^*_{0,1}M$ where $T^*_{1,0}M$ annihilates $T^{0,1}M$. A $(1,0)$-form is a smooth section of $T^*_{1,0}M$;

similarly for a $(0,1)$-form. The splitting of the cotangent bundle induces a splitting of all exterior powers. Write $\Lambda^{p,q}M = \Lambda^p(T^*_{1,0}M) \otimes \Lambda^q(T^*_{0,1}M)$. Then

$$\Lambda^r T^*M \otimes \mathbb{C} = \oplus_{p+q=r} \Lambda^{p,q}M.$$

A (p,q)-form is a smooth section of the bundle $\Lambda^{p,q}M$. The space of all such sections is denotes $\Omega^{p,q}(M)$.

Apparently, a complex manifold is almost complex. For example, the tangent bundle of an almost complex manifold is merely a complex vector bundle, while the tangent bundle of a complex manifold is a holomorphic vector bundle. An almost complex structure is integrable if it induces a complex structure. For any almost complex structure J, we could associate it with the Nijenhuis tensor

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY].$$

Newlander-Nirenberg theorem shows that an almost complex structure is integrable if and only if the Nijenhuis tensor vanishes.

Exercise: $N_J = 0$ if and only if $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

Exercise: Any almost complex structure on a 2-dimensional manifold is integrable.

The Newlander-Nirenberg Theorem could provide an alternative viewpoint of standard complex structure on Flag manifolds. I learned it from Raoul Bott. As the flag manifolds parametrize (partial) flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

with dimensions $\dim V_i = d_i$ and $0 = d_0 < d_1 < \cdots < d_k = n$. When $k = \dim V$, it is called a complete flag. These could be understood as a homogeneous space for the the general linear group. We discuss it in the complex setting, so $G = \mathrm{GL}(n,\mathbb{C})$. The stablizer of a flag is a nonsingular block upper triangular matrices, where the dimensions of the blocks are $n_i = d_i - d_{i-1}$.

Restricting to $\mathrm{SL}(n,\mathbb{C})$, the flag manifold is $\mathrm{SL}(n,\mathbb{C})/P$ where this notion could be generalize to $G_{\mathbb{C}}/P$ for a complex semisimple Lie group $G_{\mathbb{C}}$ and a parabolic group P. In the complete flag case, P is a Borel subgroup (*i.e.* a maximal Zariski closed and connected solvable algebraic subgroup). In general, each $G_{\mathbb{C}}/P$ is a coadjoint orbit of G on $\mathfrak{g}^*$.

If we choose an inner product on V, then any flag can be split into a direct sum, and so the flag manifold is isomorphic to the homogeneous space $U(n)/U(n_1) \times \cdots \times U(n_k)$. In the complete flag case, this viewpoint generalizes to G/T where G is a compact connected semisimple Lie group and $T \subset G$ a maximal torus.

Example 2.3.1. *Assume that G is a compact connected semisimple Lie group and $T \subset G$ a maximal torus. Let $\mathfrak{g}$ and $\mathfrak{t}$ be Lie algebras of G and*

T respectively. The set of all roots R is by definition constituted of all elements $\alpha \in i\mathfrak{t} \subset \mathfrak{t} \otimes \mathbb{C}$ such that there exists a nonzero $X \in \mathfrak{g} \otimes \mathbb{C}$ with $[H, X] = \langle \alpha, H \rangle X$ for all $H \in \mathfrak{t}$, or equivalently, the characters of the corresponding irreducible representation of T, if we identify the character group $Hom(T, \mathbb{C}^*)$ with $(\mathbb{C}^*)^{\dim T} \subset \mathfrak{t} \otimes \mathbb{C}$. The eigenspaces are denoted $\mathfrak{g}_\alpha$.

The tangent space of G/T at the coset eT decomposes, in terms of Lie algebra, as

$$(\mathfrak{g}/\mathfrak{t}) \otimes \mathbb{C} = \oplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

By left action, such a decomposition induces a decomposition of $T^*(G/T) \otimes \mathbb{C}$. By choosing a regular element $H_0 \in i\mathfrak{t}$, we have the decomposition of $R = R^+ \cup R^-$ to positive and negative roots. Notice $R^+ = -R^-$ as $\alpha \in R$ if and only if $-\alpha \in R$. We have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{2.1}$$

for any positive roots α, β.

In fact, the decomposition $R = R^+ \cup R^-$ is a decomposition $T(G/T) \otimes \mathbb{C} = T^{1,0}(G/T) \oplus T^{0,1}(G/T)$, thus gives rise an almost complex structure on G/T. The relation (2.1) is the integrability condition. This is the well known complex structure on G/T.

We can modify this construction a bit such that R is decomposed as a disjoint union of two sets R_1 and R_2, with the relation $R_1 = -R_2$. But we do not require the condition (2.1). This gives different choices of non-integrable almost complex structures on G/T.

For a general flag manifold $G_{\mathbb{C}}/P$, the holomorphic tangent plane is given by a subset of R^+ which is closed under addition. So the above discussion still applies.

Different from being a complex manifold, being an almost complex manifold is a topological condition. For example, in complex dimension two, the condition is guaranteed by the Wu's theorem. It asserts that almost complex structures on M are classified (up to homotopy) by the integrable lifts c_1 of the Stiefel-Whitney class $w_2(M)$ (which means $c_1 \cdot [A] = [A] \cdot [A]$ mod 2 for any $[A] \in H_2(M, \mathbb{Z})$) that satisfy

$$(c_1^2, [M]) = 3\sigma(M) + 2\chi(M).$$

Dessai further specified it as a purely topological description.

Theorem 2.3.2. *An oriented 4-manifold M admits an almost complex structure if and only if $\chi(M) + \sigma(M) \equiv 0$ mod 4 and one of the following conditions is satisfied:*

1. *The intersection form Q on $H_2(M, \mathbb{Z})$ is indefinite.*
2. *Q is positive definite and $b_1 - b_2 \leq 1$.*

3. Q is negative definite and, in case $b_2 \leq 2$, $4(b_1 - 1) + b_2$ is the sum of b_2 integer squares.

For example, we see that S^4 is not almost complex. It was a challenging problems to produce examples of almost complex but not complex manifolds before Donaldson, although the first such example was found by Yau. Now, we know abundant such examples, the simplest one might be $(2n+1)\mathbb{C}P^2$ with $n \geq 1$ (almost complex because $3 \times (2n+1) + 2 \times (2n+3) = 10n + 9 = 3^2 \times (n+1) + 1^2 \times n$). A bold conjecture of Yau says that in complex dimensions 3 and higher, any almost complex manifold admits a complex structure (of course, it is NOT saying every almost complex structure is integrable). Its special case is S^6 where we have a canonical almost complex structure, but do not know whether there are complex structure or not.

Example 2.3.3. *Let $e_1, e_2, \cdots, e_7$ be the standard basis of $\mathbb{R}^7$ and $e^1, e^2, \cdots, e^7$ be the dual basis. Denote e^{ijk} the wedge product $e^i \wedge e^j \wedge e^k$ and define*

$$\Phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

Then Φ induces a unique bilinear mapping, the cross product: $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ by $(u \times v) \cdot w = \Phi(u, v, w)$, where $\cdot$ is the Euclidean metric on $\mathbb{R}^7$. It follows that $u \times v = -v \times u$ and

$$(u \times v) \cdot u = 0. \tag{2.2}$$

Also, further discussion shows that

$$u \times (u \times v) = (u \cdot v)u - (u \cdot u)v. \tag{2.3}$$

Just as 3-dimensional cross product can be expressed in terms of quaternions, the cross product on $\mathbb{R}^7$ could also be understood by octonions. Identify $\mathbb{R}^7$ with the imaginary octonions, then

$$u \times v = \Im(uv) = \frac{1}{2}(uv - vu).$$

But unlike the 3-dimensional cross product which is invariant under $SO(3)$, the 7-dimensional cross product is invariant under $G_2 \subset SO(7)$.

Let

$$S^6 = \{u \in \mathbb{R}^7, u \cdot u = ||u|| = 1\}.$$

The tangent space at $u \in S^6$ is $T_u S^6 = \{v \in \mathbb{R}^7 | u \cdot v = 0\}$. Let $J_u = u \times \_$ be the cross product operator of u. Then $J_u(T_u S^6) \subset T_u S^6$ and $J_u^2 = -id$ on $T_u S^6$ by (2.2),(2.3). Let $\mathsf{J} = \{J_u, u \in S^6\}$. Then J gives an almost complex structure on S^6 which is the standard almost complex structure we consider. It is known that J is not integrable since the Nijenhuis tensor of J is nowhere-vanishing.

Exercise: Define the complex structure on S^2 in a similar manner using the standard cross product.

If M is a complex manifold, the splitting of the complex cotangent bundle induces a splitting of the exterior derivative: $df = \partial f + \bar{\partial} f$ where $\partial : C^\infty(M, \mathbb{C}) \to C^\infty(T^{1,0}M)$ with $\partial f = (df)^{1,0}$ and similarly for $\bar{\partial}$. In local coordinates $z_j = x_j + iy_j$,

$$\partial f = \sum \frac{\partial f}{\partial z_j} dz_j, \quad \bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

where

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}), \quad \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}).$$

We can naturally extend $\partial, \bar{\partial}$ to $\Omega^{p,q}$. By definition $\partial : \Omega^{p,q} \to \Omega^{p+1,q}, \bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$.

For complex manifold M, $\bar{\partial}^2 = 0$. Hence it defines a cohomology.

Definition 2.3.4. *The (p, q)-Dolbeault cohomology group of the complex manifold M is the vector space*

$$H^{p,q}(M) = \frac{\ker \bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}}{\text{Im } \bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q}}.$$

Recall that the de Rham cohomology of a smooth manifold M could be defined in a similar manner as $d^2 = 0$:

$$H^k_{dR}(M, \mathbb{R}) = \frac{\ker d : \Omega^k \to \Omega^{k+1}}{\text{Im } d : \Omega^{k-1} \to \Omega^k}.$$

The de Rham cohomology is isomorphic to the usual (singular) cohomology $H^k(M, \mathbb{R})$. The Betti numbers are defined as $b_i = \dim H^i(M, \mathbb{R})$.

Notice $H^{p,0}$ is simply the space of holomorphic p-forms, which by definition is the space of holomorphic sections of the holomorphic bundle $\Lambda^{p,0}$.

In general, if M is merely almost complex, we can also define $\partial\alpha$ and $\bar{\partial}\alpha$ by projection to its $(p+1, q)$ and $(p, q+1)$ components if $\alpha \in \Omega^{p,q}$. However, we don't necessarily have $\bar{\partial}^2 = 0$. The latter condition would guarantee the integrability. More precisely, since

$$d\Omega^{0,0} \subset \Omega^{1,0} + \Omega^{0,1}, \quad d\Omega^{1,0} \subset \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}, \quad d\Omega^{0,1} \subset \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2},$$

and $\Omega^{0,0}, \Omega^{1,0}, \Omega^{0,1}$ generate $\oplus\Omega^{p,q}$, we have

$$d\Omega^{p,q} \subset \Omega^{p-1,q+2} + \Omega^{p,q+1} + \Omega^{p+1,q} + \Omega^{p+2,q-1}.$$

Hence, we can write

$$d = \bar{\mu} + \bar{\partial} + \partial + \mu,$$

with the components $\bar{\mu}$ and $\bar{\partial}$ being complex conjugate to μ and ∂, and each component derivation $\bar{\mu}, \bar{\partial}, \partial, \mu$ has bidegrees $(-1,2), (0,1), (1,0), (2,-1)$ respectively. μ (and $\bar{\mu}$) are essentially Nijenhuis tensor in guise. In fact,

$$\bar{\mu} + \mu = -\frac{1}{4}(N_J \otimes id_{\mathbb{C}})^*.$$

As both sides are derivations, it suffices to check this on 1-forms. which follows from Cartan's formula relating exterior differential and Lie bracket.

This is essentially the following. We could view Nijenhuis tensor as measuring how far the Lie bracket of the two $(1,0)$ complex vector fields, *i.e.* i-eigenvectors of J, $X - iJX$ and $Y - iJY$ from being a $(1,0)$ complex vector field as well. This is a Frobenius type condition. There is another interpretation: it measures the $(0,2)$ part of $d\Omega^{1,0}$. This is because if Z and W are complex vector fields of type $(0,1)$, then for $\omega \in \Omega^{1,0}$, we have

$$d\omega(Z,W) = Z(\omega(W)) - W(\omega(Z)) - \omega([Z,W]) = \omega(N_J(\mathrm{Re}\ Z, \mathrm{Re}\ W)^{1,0}).$$

If we take $\omega = df$, the above measures $\bar{\partial}^2 f$. It is easy to see that $\mathrm{i}d\Omega^{1,0} \subset \Omega^{2,0} + \Omega^{1,1}$ would imply $d\Omega^{p,q} \subset \Omega^{p+1,q} + \Omega^{p,q+1}$.

To prove the equality,

$$d\omega(X,Y) = (\iota_X \circ d\omega)(Y) = (L_X\omega)(Y) - (d \circ \iota_X\omega)(Y).$$

Notice $(d \circ \iota_X\omega)(Y) = Y(\omega(X))$, and $(L_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y])$. The latter is because of the general formula $Y(T(\alpha_1, \alpha_2, \cdots, X_1, X_2, \cdots)) = (L_Y T)(\alpha_1, \alpha_2, \cdots, X_1, X_2, \cdots) + T(L_Y\alpha_1, \alpha_2, \cdots) + T(\alpha_1, L_Y\alpha_2), \cdots) + \cdots$.

We can also see the Nijenhuis tensor from the following setting.

We let $\{\theta^1, \cdots, \theta^n\}$ be a local unitary coframe for $T^{1,0}M$. We have a dual base $\{e_1, \cdots, e_n\}$ on the complexified tangent bundle. The metric g can be written as $g = \theta^i \otimes \bar{\theta^i}$. We extend an affine connection ∇ (*i.e.* a linear map $\nabla : \Omega^0(TM) \to \Omega^1(TM)$ satisfying Leibniz rule $\nabla(fX) = df \otimes X + f\nabla X$ for smooth function f and vector field X) linearly to $TM \otimes \mathbb{C}$. It is called almost-Hermitian if $\nabla J = \nabla g = 0$. Assume ∇ is almost Hermitian from now on. Since $J(\nabla e_j) = i\nabla e_j$, we have a matrix of complex valued 1-forms $\{\theta_i^j\}$, called the connection 1-forms, such that

$$\nabla e_i = \theta_i^j e_j.$$

It is a metric connection implies the matrix is skew-Hermitian: $\theta_i^j + \bar{\theta_j^i} = 0$. The torsion Θ is defined as

$$d\theta^i = -\theta_j^i \wedge \theta^j + \Theta^i, \quad i = 1, \cdots, n.$$

The curvature as a skew-Hermitian matrix of 2-forms is defined by

$$d\theta_j^i = -\theta_k^i \wedge \theta_j^k + \Psi_j^i.$$

It is known that there exists a unique almost-Hermitian connection whose torsion has no $(1,1)$ part. Hence, we define $N^i_{\bar{j}\bar{k}}$ by

$$(\Theta^i)^{(0,2)} = N^i_{\bar{j}\bar{k}}\bar{\theta}^j \wedge \bar{\theta}^k.$$

This is another interpretation of Nijenhuis tensor. And then it is easy to see that $\bar{\partial}^2 f = 0$ for any smooth function f if and only if the Nijenhuis tensor is zero. Once we have this, we can formally argue that $d\Omega^{p,q} \in \Omega^{p+1,q} + \Omega^{p,q+1}$.

2.4 Kähler manifolds

Let (M, J) be a complex manifold. A Riemannian metric g on M is called Hermitian if $g(Ju, Jv) = g(u, v)$ for all $u, v \in TM$. This is equivalent to require the 2-form defined ω defined by $\omega(u, v) = g(Ju, v)$ is a $(1,1)$ form. In fact, a Hermitian metric $h = g - i\omega$. The fact g is positive definite implies that ω tames J, *i.e.* $\omega(u, Ju) > 0$ for all $u \neq 0$. Overall, the Hermitian structure is a compatible triple (g, J, ω), any two determines the third.

A Kähler manifold is a Hermitian manifold such that $d\omega = 0$. We have the following equivalent definitions.

Proposition 2.4.1. *Let (M, J, g) be a Hermitian manifold and ∇ be the Levi-Civita connection. The following are equivalent:*

- *(M, J, g) is Kähler.*
- *$\nabla J = 0$.*
- *$\nabla \omega = 0$.*
- *Locally, one can write $\omega = i\bar{\partial}\partial\phi$ for a real valued function ϕ, called a local Kähler potential.*
- *There exist holomorphic coordinates $z_1, \cdots, z_n$ in which the Hermtian metric is Euclidean metric on $\mathbb{C}^n$ to second order: $h = \sum dz_i \otimes d\bar{z}_i + O(|z|^2)$.*

The last is called normal coordinates for Kähler manifold. It implies if our calculation involves only first derivatives of the Kähler structure, then we could check it only for flat metric in $\mathbb{C}^n$. For example, it implies $\nabla J = 0$ because the Christoffel symbols Γ^i_{jk} only depends on the the first derivatives of the metric.

For any compact Kähler manifold, the Kähler class $[\omega] \in H^2(M, \mathbb{R})$ is nontrivial. This is because $\frac{\omega^n}{n!}$ is the volume form. We derive it by noting the $(1,1)$-form associated to the Euclidean metric on $\mathbb{C}^n$ is

$$\omega = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j.$$

2.4.1 Examples.

0. $M = \mathbb{C}^n$. Use notation $z_j = x_j + iy_j$.

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

If g is the Euclidean metric, then its Kähler form is

$$\omega = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j = \sum_j dx_j \wedge dy_j.$$

1. The most important example is the complex projective plane $\mathbb{C}P^n$. Write $||w|| = \sum_{i=0}^n |w_i|^2$. For $\lambda \in \mathbb{C}^*$, $\log||\lambda w||^2 = \log|\lambda|^2 + \log||w||^2$, so

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\log||w||^2$$

is a well-defined, closed $(1,1)$ form on $\mathbb{C}P^n$. This is called the Fubini-Study form.

On U_0, $z_i = \frac{w_i}{w_0}$, $1 \leq i \leq n$. Hence

$$\frac{2\pi}{i}\omega = \sum_{i,j}\left(\frac{dz_i \wedge d\bar{z}_i}{1+|z|^2} - \frac{\bar{z}_i dz_i \wedge z_j d\bar{z}_j}{(1+|z|^2)^2}\right).$$

The eigenvalues of the matrix z^*z are $|z|^2$ and $n-1$ copies of 0. So the metric is positive definite. Recall that $\frac{2}{i}\omega = \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. The Fubini-Study metric is the unique $U(n+1)$-invariant (up to scaling) Riemannian metric on $\mathbb{C}P^n$. Use this viewpoint, we could check it is positive form also by checking at one point, say $[1,0,\cdots,0]$, where it is $\omega = \frac{i}{2\pi}\sum_j dw_j \wedge d\bar{w}_j$.

1'. A similar looking but dramatically different metric is the Bergman metric on the unit ball $D^n = \{z \in \mathbb{C}^n \mid |z|^2 < 1\}$ in $\mathbb{C}^n$. Define

$$\omega = -\frac{i}{2\pi}\partial\bar{\partial}\log(1-|z|^2) = \frac{i}{2\pi}\sum_{i,j}\left(\frac{dz_i \wedge d\bar{z}_i}{1-|z|^2} + \frac{\bar{z}_i dz_i \wedge z_j d\bar{z}_j}{(1-|z|^2)^2}\right).$$

ω is closed and invariant under the action of $\mathrm{SU}(1,n)$. Thus its descends to a Kähler form on any ball quotient.

2. Let (X, J, ω) be Kähler and $Y \subset X$ a complex submanifold. Then Y is Kähler with Kähler form $\omega|_Y$.

3. Any complex submanifold of $\mathbb{C}P^n$ is Kähler. In particular, any projective variety is Kähler.

4. For the general operations, the covering preserve the Kähler structure, but the quotient is not, *e.g.* the Hopf surface.

2.4.2 Blowups

We now define blowups:

1. Blowing-up at a point. Let $B = B_r(0)$ be a ball in $\mathbb{C}^n$, $n \geq 2$. Let $z = (z_1, \cdots, z_n)$ be the standard coordinates of $\mathbb{C}^n$ and $w = [w_1, \cdots, w_n]$ be homogeneous coordinates on $\mathbb{C}P^{n-1}$. The blowing-up of B at 0, denoted by $\tilde{B} \subset B \times \mathbb{C}P^{n-1}$ is the complex n-manifold given by

$$\tilde{B} = \{(z, w) \in B \times \mathbb{C}P^{n-1} | z_i w_j = z_j w_i, \quad \forall 1 \leq i < j \leq n\}.$$

If we cover $\mathbb{C}P^{n-1}$ by $U_1, \cdots, U_n$, then $\tilde{B} \cap (B \times U_i)$ is defined by the $n-1$ equations

$$z_j = z_i \frac{w_j}{w_i}, \quad 1 \leq j \leq n, j \neq i.$$

Let $\pi : \tilde{B} \to B$ be the restriction of the projection map from $B \times \mathbb{C}P^{n-1}$ onto B. In other words, the blowing-up of $0 \in B \subset \mathbb{C}^n$ is the restriction of the projection of the holomorphic line bundle $\mathcal{O}(-1) \subset \mathbb{C}P^{n-1} \times \mathbb{C}^n$ over $\mathbb{C}P^{n-1}$ to the second factor $\mathcal{O}(-1) \to \mathbb{C}^n$ on B.

For any $z \neq 0$, $\pi^{-1}(z)$ is a single point $(z, [z])$. While for the origin $0 \in B$, the inverse image is the complex projective space $E := \pi^{-1}(0) = \{0\} \times \mathbb{C}P^{n-1}$. E is called the exceptional divisor of the blowing up.

In general, if M^n is a complex n-manifold and $p \in M$. Take a neighborhood $p \in U \subset M$ along with $\phi : U \to B$ a coordinate disc around x. $\tilde{M}_x = M - \{x\} \cup_\pi \tilde{B}$ obtained by replacing $B \subset M$ with $\tilde{B}$ is the blow-up of M.

Topologically, $\tilde{M} = M \# \overline{\mathbb{C}P^n}$. To show this, we notice this is a local statement so reduced to the case when $M = \mathbb{C}^n$. As shown above, the blowup of $\mathbb{C}^n$ is biholomorphic to the total space of $\mathcal{O}(-1)$. As $\mathcal{O}(-1)$ can be identified with a complex conjugate of $\mathcal{O}(1)$, we know their total spaces are identical up to an orientation reversing diffeomorphism. Moreover, the total space of $\mathcal{O}(1)$ is biholomorphic to $\mathbb{C}P^n - \{x\}$. So the identification follows. Notice in this identification, all real rays going into the origin of $\mathbb{C}^n$ are transformed into rays going out of x in $\mathbb{C}P^n$.

2. More generally, one can blow up any codimension $k \geq 2$ complex submanifold Z of $\mathbb{C}^n$. Suppose Z is the locus of the equation $x_1 = \cdots = x_k = 0$, then the blow-up of Z is the locus of the equations $x_i y_j = x_j y_i, i, j \in 1, \cdots, k$ in $\mathbb{C}^n \times \mathbb{C}P^{k-1}$. Of course, we can blow up any submanifold of any complex manifold M by applying this construction locally. The restriction $\pi|_E : E \to Z$ could be seen as the projectivization of the normal bundle of Z in M.

3. The inverse operation of a blowup is called blowdown.

The blow-ups (of points and submanifolds) always preserve Kählerness, but not blowing-down along submanifolds. The first claim is very fundamental in Kähler geometry, but it seems there is no proof in any textbook.

We will prove it after introducing geometry of Hermitian bundles. The second statement corresponds to Moishezon manifolds. The easiest example might be when we have two homologous $\mathcal{O}(-1,-1)$ curves and we flop one of them.

Chapter 3

Geometry

3.1 Hermitian Vector Bundles

We now focus on holomorphic vector bundles. Recall a holomorphic vector bundle is a complex vector bundle over a complex manifold X such that the total space E is a complex manifold and the projection map $\pi : E \to X$ is holomorphic. Equivalently, we can say that a holomorphic bundle is uniquely determined by a system of holomorphic transition functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$ are holomorphic. One can define a $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ operator on sections. To define it, we first work in a local trivialization U_α. Here sections are functions $U_\alpha \to \mathbb{C}^k$ and we define it by the ordinary $\bar{\partial}$ operator. Now, the transition functions of different trivializations are holomorphic, and so our $\bar{\partial}$ are identical at the intersections, which means the the local definitions patch together to give an operator defined on global sections. Notice that there is no naturally defined the exterior derivative d on sections of a vector bundle (flat bundle has one?).

Let $\Omega^{0,q}(X, E)$ be the smooth sections of $\Lambda^{0,q} \otimes E$. Then $\bar{\partial}_E^2 = 0$ and the cohomology of the complex $(\Omega^{0,\cdot}(X, E), \bar{\partial}_E)$ is called the the Dolbeault cohomology of X with values in E, denoted by $H^{0,\cdot}(X, E)$. It is canonically isomorphic to $H^q(X, \mathcal{O}_X(E))$ of the sheaf $\mathcal{O}_X(E)$ of holomorphic sections of E over X. We shortly denote $H^q(X, E) := H^q(X, \mathcal{O}_X(E)) = H^{0,q}(X, E)$. Similarly, we can define $H^{p,q}(X, E)$.

A Hermitian metric h is a C^∞ field of Hermitian inner products in the fibers of E.

Definition 3.1.1. *A connection ∇ on a complex vector bundle $E \to M$ is a $\mathbb{R}$-linear map $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying Leibnitz rule*

$$\nabla(fs) = df \otimes s + f\nabla s$$

for smooth function f and $s \in \Omega^0(E)$.

If ∇_1 and ∇_2 are two connections on $E \to M$, then their difference is C^∞-linear. That is $(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 s - \nabla_2 s)$. Hence $\nabla_1 - \nabla_2$

is induced by a 1-form on M with values in $\mathrm{Hom}(E,E) = E \otimes E^*$, *i.e.* $\nabla_1 - \nabla_2 \in \Omega^1(M, End(E)$. In other words, the space of connections on E is an affine space for $\Omega^1(M, End(E)$.

Given a connection ∇, we can define operators

$$\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$$

by Leibniz rule

$$\nabla(\psi \wedge \xi) = d\psi \otimes \xi + (-1)^p \psi \nabla \xi$$

for $\psi \in \Omega^p(E)$ and $\xi \in \Omega^0(E)$. In particular, we have $\nabla^2 : \Omega^0(E) \to \Omega^2(E)$ which is easy to check is linear over Ω^0, *i.e.* $\nabla^2(fs) = f\nabla^2 s$. Hence, it is induced by bundle map $E \to \Lambda^2 T^*M \otimes E$ and thus corresponds to a global section Θ, called curvature, of the bundle

$$\Lambda^2 T^*M \otimes Hom(E,E) = \Lambda^2 T^*M \otimes E^* \otimes E.$$

Then on U_α with the fixed trivialization with frames $e_1, \cdots, e_k$ for E, we can write

$$\nabla_A = d + A^\alpha.$$

Here A^α is a matrix value 1-form. More precisely, if we write $s = \sum s_i e_i$ and $\nabla e_i = \sum A^\alpha_{ij} e_j$ on U_α, we have

$$\nabla s = \sum_j (ds_j + \sum_i s_i A^\alpha_{ij}) e_j.$$

In terms of this basis, we can write

$$\nabla^2 e_i = \sum \Theta_{ij} \otimes e_j.$$

Θ_{ij} is called the curvature matrix and will be denoted by Θ_A or simply Θ later.

If E has a Hermitian metric h, we could require the connection compatible with it, which means $dh(X,Y) = h(\nabla X, Y) + h(X, \nabla Y)$. Equivalently, it means the structure group is the unitary group $U(k)$ and A^α is now a $u(k)$-valued 1-form, *i.e.* $A^\alpha = -(A^\alpha)^*$, if the trivialization is unitary with respect to h. A connection ∇ on E is said to be compatible with the holomorphic structure on E if $\pi^{0,1}(\nabla s) = \bar{\partial}_E s$ for all sections s of E. Here $\bar{\partial}_E$ has $\bar{\partial}_E^2 = 0$ and is naturally extended from $\bar{\partial}$ on M and the holomorphic structure on E as explained in the beginning of this section.

Proposition 3.1.2. *Let E be a Hermitian holomorphic vector bundle. Then there is a unique connection on E compatible with the metric and the holomorphic structure.*

Actually, a unitary connection on a Hermitian complex vector bundle is compatible with a holomorphic structure if and only if it has curvature of type $(1,1)$.

Proof. In a local trivialization, there are two ways to choose frames. If we choose a holomorphic frame $\{e_1, \cdots e_k\}$, then a connection compatible with the holomorphic structure has the form $\nabla_A = d + A^\alpha$ where A^α is a matrix valued $(1,0)$-form. The Hermitian structure h is given by the matrix $h_{ij} = h(e_i, e_j)$. Later we will also use h to represent this matrix. The condition that ∇ is compatible with the Hermitian metric h gives us $A^\alpha h + h(\bar{A}^\alpha)^T = dh$. Comparing the $(1,0)$ part, we have $A^\alpha = \partial h \cdot h^{-1}$. When E is a line bundle, it is $\partial \log h$.

If we work with a local unitary trivialization, then A^α is a $u(k)$-valued 1-form, since $0 = dh(e_i, e_j) = A_{ij} + \bar{A}_{ji}$. However, under this frame, $\bar\partial_E$ behaves as $\bar\partial + B^\alpha$ where B^α is a matrix valued $(0,1)$-form. The matrix B^α is determined by $\bar\partial_E e_i = B^\alpha_{ij} e_j$. Hence $A^\alpha = B^\alpha - (B^\alpha)^*$ is uniquely determined.

For the second fact, one could see it as follows. A connection is compatible with a holomorphic structure if and only if $\Theta_A^{0,2} = \bar\partial_E^2 = 0$. For a unitary connection, the curvature is also skew adjoint. Hence $\Theta_A^{2,0} = -(\Theta_A^{0,2})^* = 0$ and the statement holds. When E is a line bundle, it is $\bar\partial\partial \log h$. Notice $i[Tr\Theta_A] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. $\square$

When $E = TM$ for a complex manifold M, this connection is exactly the Levi-Civita connection with respect to the Kähler metric.

This unique connection is called the Chern connection. Our argument for the Chern connection also holds for a complex vector bundle E with a pseudoholomorphic structure $\bar\partial_E$ over an almost complex manifold M. For a complex vector bundle E, a pseudoholomorphic structure on E is given by a differential operator $\bar\partial_E : \Omega^0(E) \to \Omega^{0,1}(E)$ which satisfies the Leibniz rule $\bar\partial_E(fs) = \bar\partial f \otimes s + f\bar\partial_E s$, where f is a smooth function and s is a section of E. The Koszul-Malgrange theorem says that a pseudoholomorphic structure is induced from a holomorphic structure on E over a complex manifold M if and only if $\bar\partial_E^2 = 0$. This follows from Newlander-Nirenberg theorem as a pseudoholomorphic structure is one-to-one corresponds to a bundle almost complex structure, which is an almost complex structure $\mathcal{J}$ on TE so that

(i) the projection is $(\mathcal{J}, J)$-holomorphic,

(ii) $\mathcal{J}$ induces the standard complex structure on each fiber, i.e. multiplying by i,

(iii) the fiberwise addition $\alpha : E \times_M E \to E$ and the fiberwise multiplication by a complex number $\mu : \mathbb{C} \times E \to E$ are both pseudoholomorphic.

And the integrability condition $\bar\partial_E^2 = 0$ implies this almost complex structure is integrable by Newlander-Nirenberg.

Notice for unitary trivializations, the transition map is no longer holomorphic. So if we simply say trivialization, we mean a holomorphic one. If we change a trivialization, $\{e'_1, \cdots, e'_k\} = \{e_1, \cdots, e_k\}g$, then $h' = g^T h \bar{g}$, and $A^{\alpha g} = gA^\alpha g^{-1} + (dg)g^{-1}$. The last one is because $\nabla e'_i = A^{\alpha g}_{ij} e'_j = A^{\alpha g}_{ij} e_l g_{jl} = A^{\alpha g}_{ij} g_{jl} \otimes e_l$ and $\nabla e'_i = \nabla(e_j g_{ij}) = dg_{ij} \otimes e_j + g_{ij} \nabla e_j = dg_{il} \otimes e_l + g_{ij} A^\alpha_{jl} \otimes e_l$, which means $A^{\alpha g} g = gA^\alpha + dg$.

As $\Theta_A u = \nabla(du + Au) = d(Au) + A \wedge du + A \wedge Au = (dA + A \wedge A)u$, we have $\Theta_A = dA + A \wedge A$. Hence, we have $\Theta_{g(A)} = g\Theta_A g^{-1}$ (Use the fact $dg^{-1} = -g^{-1}(dg)g^{-1}$). For line bundle $d\Theta_A = 0$. In general, we have Bianchi Identity $d\Theta_A = \Theta_A \wedge A - A \wedge \Theta_A$.

Example 3.1.3. *An important example is the tangent bundle of a Riemann surface M. Write the metric in isothermal coordinate $ds^2 = h^2 dz \otimes d\bar{z}$. Let the compatible Kähler form be $\omega = \frac{i}{2} h^2 dz \wedge d\bar{z}$. Then*

$$\Theta = -2\left(\frac{\partial^2}{\partial z \partial \bar{z}} \log h\right) dz \wedge d\bar{z} = -\frac{1}{2} \Delta \log h dz \wedge d\bar{z}.$$

We have $i\Theta = K \cdot \omega$, where $K = \frac{-\Delta \log h}{h^2}$ is the Gaussian curvature.

The Gauss-Bonnet theorem follows from Chern-Weil theory by noticing $\int_M K\omega = 2\pi c_1(TM) = 2\pi\chi(M)$.

For a holomorphic line bundle L with a Hermitian metric, the structure group is $U(1)$. So, in a local holomorphic trivialization, a connection on L is represented by a 1-form A and the curvature is the 2-form $\Theta = dA = \bar{\partial}\partial \log h$. It is closed form representing a de Rham cohomology class in $H^2(X; i\mathbb{R})$. To summarize, we have

Proposition 3.1.4. *Given a Hermitian metric h in a holomorphic line bundle $L \to M$, the curvature Θ_h of the Chern connection is closed and $\frac{i}{2\pi}[\Theta_h] \in H^2(M, \mathbb{R})$ is independent of the choice of h. Hence $[\Theta_h]$ is independent of the metric h.*

This class $\frac{i}{2\pi}[\Theta]$ is defined as the first Chern class $c_1(L)$ of the line bundle.

Proof. If there is another metric $h' = e^f h$. The curvatures are related by $\Theta_{h'} = \Theta_h + \bar{\partial}\partial f$. □

In fact, we can use the same definition of Chern class for any unitary connection for any complex line bundle, not just one compatible with the holomorphic structure. If we have another connection $A' = A + a$ with $a \in \Omega^1(M, End(E))$, which implies $\Theta' = \Theta + da$, so $[\Theta'] = [\Theta]$. We can also define higher Chern classes for vector bundles of higher rank in a similar manner. But now as we have $\Theta_{g(A)} = g\Theta_A g^{-1}$, the right things we should look at are polynomials invariant under conjugation, *e.g.* trace, determinant *etc.*, for matrix $i\Theta_A$. Precisely, for any $k \times k$ matrix P, we can define elementary symmetric functions of eigenvalues $f_i(P)$ by

$$\det(P + tI) = f_k(P) + f_{k-1}(P)t + \cdots + f_1(P)t^{k-1} + f_0(P)t^k.$$

It is a good exercise to check that $f_i(\frac{i}{2\pi}\Theta_A)$ are all closed. Then the Chern forms and Chern classes are defined as $f_i(\frac{i}{2\pi}\Theta_A)$ and their cohomology classes. It is also direct to check that the cohomology classes $f_i(\frac{i}{2\pi}\Theta_A)$ are independent of connection A.

Let M be an almost complex manifold. The first Chern class of M is defined by $c_1(X) = -c_1(K)$ where $\mathcal{K} = \Lambda^n(T^*_{1,0}M)$ is the canonical line bundle.

A holomorphic line bundle L over a complex manifold M is *positive* if there exists a metric with curvature form Θ such that $\frac{i}{2\pi}\Theta$ is a positive $(1,1)$ form. Here, a $(1,1)$ form ω is called positive if $g(u,v) = \omega(u, Jv)$ defines a Hermitian metric. In terms of local holomorphic coordinates $z = (z_1, \cdots, z_n)$, a form ω is positive if $\omega = \frac{i}{2}\sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$ with $(h_{ij}(z))$ a positive definite Hermitian matrix for each z. The positivity of a line bundle is a topological property. More precisely, L is positive if and only if its Chern class $c_1(L)$ may be represented by a positive form. It follows from the following

Proposition 3.1.5. *Let M be Kähler. If ω is any real, closed $(1,1)$ form with $[\omega] = c_1(L)$, then there exists a Hermitian metric on L whose Chern connection has curvature form $\Theta = \frac{2\pi}{i}\omega$. Moreover, if $H_1(M;\mathbb{R}) = 0$, there is a unique such connection up to gauge transformation.*

In particular, we know that a trivial bundle cannot be positive when the base is compact. This implies Stein manifolds are non-compact since strictly plurisubharmonic functions could be viewed as Hermitian metrics on the trivial bundle.

3.2 (Almost) Kähler identities

We assume (M, J) is an almost complex manifold of dimension $2n$. We can still define Hermitian metric $h = g - i\omega$ with respect to J, by requiring $g(X,Y) = g(JX, JY)$. There is a real $(1,1)$ form $\omega(X,Y) := g(JX, Y)$.

For a general Riemmanian manifold (M, g) is compact, we have global L^2 inner product

$$(\alpha, \beta) = \int_M g(\alpha(z), \beta(z)) \frac{\omega^n}{n!}.$$

Here the pointwise inner product $(\alpha(z), \beta(z))$ is extended from that on T^*M by requiring that $(\alpha(z), \beta(z)) = \det(\alpha_i, \beta_j)$ for any decomposable k-vectors $\alpha(z) = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta(z) = \beta_1 \wedge \cdots \wedge \beta_k$.

When we are in the almost complex setting and define L^2 inner product using almost Hermitian metric instead of Riemannian metric, we can define the adjoint operator of $\bar\partial$

$$\bar\partial^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$$

by requiring that

$$(\bar{\partial}^* \alpha, \beta) = (\alpha, \bar{\partial}\beta).$$

Similarly for ∂^*.

One can give an explicit formula for $\bar{\partial}^*$ in terms of the Hodge star

$$* : \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$$

by requiring

$$h(\alpha(z), \beta(z))\frac{\omega^n}{n!} = \alpha(z) \wedge *\bar{\beta}(z).$$

Notice the Hodge star is defined for any almost complex manifold and our convention makes sure it is linear. Locally, if we write $\alpha = \sum_{I,J} \alpha_{I\bar{J}}\phi_I \wedge \bar{\phi}_J$ where $\{\phi_1, \cdots, \phi_n\}$ is a unitary coframe of $T^*_{1,0}M$ such that $h = g - i\omega$ is $\sum_{i=1}^n \phi_i \otimes \bar{\phi}_i$ under this basis, then h on $\Lambda^{p,q}$ is defined by letting $\{\phi_\alpha \wedge \bar{\phi}_\beta\}$ be orthonormal. In particular $h(\phi_\alpha \wedge \bar{\phi}_\beta, \phi_\alpha \wedge \bar{\phi}_\beta) = 2^{p+q}$.

$$*\alpha = \sum_{I,J} \epsilon_{IJ}\alpha_{I\bar{J}}\phi_{J^0} \wedge \bar{\phi}_{I^0},$$

where $I^0 = \{1, \cdots, n\} - I$ and ϵ_{IJ} is the sign of the permutation

$$(1, \cdots, n, 1', \cdots, n') \to (I, J, J^0, I^0).$$

We have $* * \alpha = (-1)^{p+q}\alpha$. The sign $(-1)^{p+q}$ is the sign of changing (I, J, J_0, I_0) to (J_0, I_0, I, J), which is $(-1)^{(p+q)(2n-p-q)}$.

The Hodge star is in fact defined on any Riemanninan manifold, for an orthonormal basis $\{e_1, \cdots, e_n\}$ where the volume form is $e_1 \wedge \cdots \wedge e_n$, it could be computed as

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \epsilon \cdot e_{j_1} \wedge \cdots \wedge e_{j_{n-k}},$$

where $\epsilon = sgn(i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$ and $* * \alpha = (-1)^{k(n-k)}$ for $\alpha \in \Omega^k$.

One can check that $\bar{\partial}^* = - * \partial *$. In fact, we have the following.

Proposition 3.2.1. *Let X be a compact, almost Hermitian manifold with dimension $2n$. The L^2-adjoint of d is the codifferential $d^* = - * d*$ and the adjoints of $\mu, \partial, \bar{\partial}, \bar{\mu}$ are given by*

$$D^* = - * \bar{D} *$$

for any $D \in \{\mu, \partial, \bar{\partial}, \bar{\mu}\}$.

Proof. We only need to check it for d, others follow from projection onto different (p, q) types.

This identity follows from Stokes' theorem for smooth forms. When M is a compact manifold without boundary, and $\eta \in \Omega^k$, $\xi \in \Omega^{k+1}$,

$$0 = \int_M d(\eta \wedge *\bar{\xi}) = \int_M d\eta \wedge *\bar{\xi} + (-1)^k \eta \wedge d * \bar{\xi} = (d\eta, \xi) + (-1)^k (\eta, (-1)^{2n-k} * d * \xi).$$

Since $(d\eta, \xi) = (\eta, d^*\xi)$, we have $d^* = - * d*$.

As both sides can map a (p,q) form to $(p+1, q-2), (p, q-1), (p-1, q), (p-2, q+1)$ forms, we have the other 4 relations for $\mu, \partial, \bar{\partial}, \bar{\mu}$. □

Now we can define different Laplacians:

$$\Delta_D = DD^* + D^*D, \quad D \in \{d, \mu, \partial, \bar{\partial}, \bar{\mu}\}.$$

In particular, we denote $\Delta = \Delta_d$.

There are two more important linear operators on form spaces. The Lefschetz operator is given by

$$L : \Omega^{p,q} \to \Omega^{p+1,q+1}, \quad L(\eta) = \omega \wedge \eta.$$

The dual Lefschetz operator is defined as $\Lambda := *^{-1} L* : \Omega^{p,q} \to \Omega^{p-1,q-1}$. We can check $\Lambda = (-1)^k * L*$ is the formal adjoint of L: *i.e.*, $(L\alpha, \beta) = (\alpha, \Lambda\beta)$. This is because $(L\alpha, \beta) = \int L\alpha \wedge *\bar{\beta} = \int \alpha \wedge \overline{\omega \wedge *\beta} = (\alpha, *^{-1}(\omega \wedge *\beta))$.

If we let $H = \sum_{k=0}^{2n} (k-n)\Pi^k$ where Π^k is for the projection to k-forms, then we have the following relations.

Proposition 3.2.2.

$$[H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \quad [L, \Lambda] = H.$$

Notice this is a pointwise property.

Proof. If $\alpha \in \Omega^k$, then $[H, L](\alpha) = (k+2-n)\omega\wedge\alpha - \omega\wedge((k-n)\alpha) = 2\omega\wedge\alpha$. Similarly, $[H, \Lambda] = (k-2-n)\Lambda\alpha - \Lambda((k-n)\alpha) = -2\Lambda\alpha$.

For the third one, as it is a pointwise relation, we can study it at a given cotangent space and choose the standard coordinate for vector space $T^*_{1,0}M$, $\{e_1, \cdots, e_n\}$, and the standard form $\omega = \frac{i}{2}\sum e_j \wedge \bar{e}_j$. Let $\{e^1, \cdots, e^n\}$ be dual basis in $T^{1,0}_x M$. Apply Lemma 3.2.3 and using the formula $X\rfloor(\beta\wedge\gamma) = X\rfloor\beta\wedge\gamma + (-1)^p\beta\wedge(X\rfloor\gamma)$ for p-form β, we have (notice L, Λ each contributes $\frac{i}{2}$, so cancelled -4 in item 5 of Lemma 3.2.3)

$$L\Lambda\alpha = \sum e_j\wedge\bar{e}_j\wedge(\bar{e}^k\rfloor(e^k\rfloor\alpha)) = -\sum_{j\neq k} e_j\wedge(\bar{e}^k\rfloor(\bar{e}_j\wedge(e^k\rfloor\alpha))) + \sum_j e_j\wedge\bar{e}_j\wedge(\bar{e}^j\rfloor(e^j\rfloor\alpha)).$$

$$\Lambda L\alpha = \sum \bar{e}^k\rfloor(e^k\rfloor(e_j\wedge\bar{e}_j\wedge\alpha)) = -\sum_{j\neq k} \bar{e}^k\rfloor(e_j\wedge(e^k\rfloor(\bar{e}_j\wedge\alpha))) + \sum_j \bar{e}^j\rfloor(e^j\rfloor(e_j\wedge\bar{e}_j\wedge\alpha)).$$

The first terms on the right hand sides cancelled. Without loss, we can assume $\alpha = e_I \wedge \bar{e}_J$. The second term on the right hand side in the first formula counts the number of $j \in I\cap J$; the one in the second formula counts the number of j with $j \notin I \cup J$. Since $|I \cap J| + |I \cup J| = |I| + |J|$, we have $|I \cap J| - |(I \cup J)^c| = k - n$, and hence the third identity. □

Lemma 3.2.3.
- $*\bar{e}_j \wedge * = 2e^j \rfloor$,
- $*^{-1}(\bar{e}_j \wedge)* = 2(-1)^{k+1} e^j \rfloor$,
- $*e_j \wedge * = 2\bar{e}^j \rfloor$,
- $*^{-1}(e_j \wedge)* = 2(-1)^{k+1} \bar{e}^j \rfloor$,
- $*^{-1}(e_j \wedge \bar{e}_j \wedge)* = -4\bar{e}^j \rfloor e^j \rfloor$.

Proof. For the first one, apply both sides to $\alpha = e_j \wedge \beta$, we have $*2e^j \rfloor \alpha = 2 * \beta$. It could be written as $\bar{e}_j \wedge \bar{\gamma}$ where $\beta \wedge e_j \wedge \gamma = 2h(\beta, \beta)\frac{\omega^n}{n!}$. As $\beta \wedge e_j \wedge \gamma = (-1)^k \alpha \wedge \gamma$, we have $*\alpha = (-1)^{k-1}\bar{\gamma}$ as $h(\alpha, \alpha) = 2h(\beta, \beta)$. So $\bar{e}_j \wedge *\alpha = (-1)^{k-1} \bar{e}_j \wedge \bar{\gamma}$. Then the second one also follows immediately.

The next two are similar. The last one apply the second and fourth and $(-1)^{k+1} \cdot (-1)^k = -1$. □

Hence, L, Λ, H defines a natural $\mathfrak{sl}(2, \mathbb{C})$ representation on $\Lambda^* T^* M$, with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L = X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Lambda = Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence, it is a direct sum of irreducible submodules. We recall some basic fact of (finite dimensional) $\mathfrak{sl}(2, \mathbb{C})$ representations.

Let (ρ, V) be an irreducible complex representation. Then the action H on V is diagonalizable. Thus we have $V = \oplus V_\lambda$, where λ runs over a collection of complex numbers. It is easy to compute that $X(V_\lambda) \subset V_{\lambda+2}, Y(V_\lambda) \subset V_{\lambda-2}$. Then for any $v \in V$ with $X(v) = 0$, we know V is generated as a vector space by $\{v, Yv, Y^2(v), \cdots\}$. The key formula used to show it is

$$XY^m(v) = m(n - m + 1)Y^{m-1}(v), v \in V_n \tag{3.1}$$

A similar result is true when we start with some v with $Y(v) = 0$. This would immediately imply that each V_λ is one-dimensional, $H(V_\lambda) = V_\lambda, X(V_\lambda) = V_{\lambda+2}, Y(V_\lambda) = V_{\lambda-2}$. And as $\dim V < \infty$, all the eigenvalues of H are integers, and we have

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n}.$$

In other words, the eigenvalues are symmetric with respect to the origin and are appearing with multiplicity 1.

To summarize, all the finite dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ are indexed by integers $n \geq 0$. Explicitly, $V(n) = Sym^n(\mathbb{C}^2)$ which has dimension $n+1$. Hence, we can define the Lefschetz decomposition of a (not necessarily irreducible, but finite dimensional) $\mathfrak{sl}(2, \mathbb{C})$ module V. Let $PV = \ker \rho(X)$, then $V = PV \oplus YPV \oplus Y^2 PV \oplus \cdots$. With respect to each eigenspace V_m, we see $Y^m : V_m \to V_{-m}$ are isomorphisms. Finally, in general, $\ker X \cap V_k = \ker(Y^{k+1} : V_k \to V_{-k-2})$. Similar when we swap the roles of X and Y.

There are two applications (to the triple $(-H, \Lambda, L)$. Applying to $\Lambda^* M = \Lambda^* T^* M$ for an (almost) Hermitian manifold M, we have a direct sum decomposition of vector bundles

$$\Lambda^k M = \oplus_{i\geq 0} L^i(P^{k-2i}M),$$

where $P^{k-2i}M := \ker(\Lambda : \Lambda^{k-2i}M \to \Lambda^{k-2i-2}M)$.

Second, notice all the operators L, Λ, H commute with Δ_d by Kähler identities, so they act on the space of harmonic forms $\mathcal{H}_d^*(M) = \ker \Delta_d$. As by Hodge theory, $\mathcal{H}_d^*(M) \cong H^*(M)$, we have the following

Theorem 3.2.4 (Hard Lefschetz Theorem)**.** *Let M be a compact Kähler manifold. The map $L^k : H_{dR}^{n-k}(M) \to H_{dR}^{n+k}(M)$ is an isomorphism, $L^l : H_{dR}^{n-k}(M) \to H_{dR}^{n-k+2l}(M)$ is injective when $l < k$. If we define the primitive cohomology*

$$P^{n-k}(M) = \ker(L^{k+1} : H_{dR}^{n-k}(M) \to H_{dR}^{n+k+2}(M)) = \ker \Lambda \cap H_{dR}^{n-k}(M),$$

then we have $H_{dR}^m(M) = \oplus_k L^k P^{m-2k}(M)$.

We denote the projection to (p,q) component by $\Pi^{p,q}$. Then there is a linear map

$$\mathbb{I} = \sum_{p,q} i^{p-q}\Pi^{p,q}$$

multiplicatively extending the almost complex structure J.

When the compatible form ω is closed, we call (M, J, ω) an almost Kähler manifold. We can formulate following identities (due to Cirici and Wilson with a correction by Tom Holt), which are called Kähler identities when J is integrable.

Proposition 3.2.5. *The following identities hold on any almost Kähler manifold.*

1. $[L, \mu] = [L, \partial] = [L, \bar{\partial}] = [L, \bar{\mu}] = 0$
2. $[\Lambda, \mu] = -i\bar{\mu}^*, \quad [\Lambda, \partial] = i\bar{\partial}^*, \quad [\Lambda, \bar{\partial}] = -i\partial^*, \quad [\Lambda, \bar{\mu}] = i\mu^*$
3. $[L, \mu^*] = -i\bar{\mu}, \quad [L, \partial^*] = i\bar{\partial}, \quad [L, \bar{\partial}^*] = -i\partial, \quad [L, \bar{\mu}^*] = i\mu$
4. $[\Lambda, \mu^*] = [\Lambda, \partial^*] = [\Lambda, \bar{\partial}^*] = [\Lambda, \bar{\mu}^*] = 0$

Proof. The first item follows from $[L, d](\alpha) = \omega\wedge d\alpha - d(\omega\wedge\alpha) = -d\omega\wedge\alpha = 0$ since $d\omega = 0$, and then look at each bidegree component separately.

For the second set, we first show $[\Lambda, d] = *\mathbb{I}^{-1} \circ d \circ \mathbb{I}*$. By Lefschetz decomposition of form bundle, it is enough to prove the assertion for forms of type $L^j\alpha$ with α a primitive k-form. We write $d\alpha = \alpha_0 + L\alpha_1 + \cdots$, with $\alpha_j \in P^{k+1-2j}(M)$. Since $[d, L] = 0$, we have

$$0 = L^{n-k+1}\alpha_0 + L^{n-k+2}\alpha_1 + \cdots .$$

This implies $L^{n-k+j+1}\alpha_j = 0$ for each j. On the other hand, L^l is injective on $\Omega^i(M)$ for $l \leq n-i$. Hence we have $\alpha_j = 0$ for $j \geq 2$. Thus $d\alpha = \alpha_0 + L\alpha_1$ with $\Lambda\alpha_0 = \Lambda\alpha_1 = 0$.

We compute $\Lambda dL^j\alpha = \Lambda L^j d\alpha = \Lambda L^j\alpha_0 + \Lambda L^{j+1}\alpha_1$ for $\alpha \in P^k(M)$. By (3.1), it is

$$j(n-k-1-j+1)L^{j-1}\alpha_0 + (j+1)(n-k+1-j)L^j\alpha_1.$$

And

$$d\Lambda L^j\alpha = j(n-k-j+1)L^{j-1}d\alpha = j(n-k-j+1)(L^{j-1}\alpha_0 + L^j\alpha_1).$$

Therefore,

$$[\Lambda, d]L^j\alpha = -jL^{j-1}\alpha_0 + (n-k-j+1)L^j\alpha_1.$$

On the other hand, by the following exercise, we can compute

$$*\mathbb{I}^{-1} \circ d \circ \mathbb{I} * L^j\alpha = -jL^{j-1}\alpha_0 + (n-k-j+1)L^j\alpha_1.$$

Then for any (p,q)-form α and any linear map $D : \Omega^{*,*} \to \Omega^{*,*}$ with bidegree (r,s) we have

$$\mathbb{I}^{-1} \circ D \circ \mathbb{I}(\alpha) = i^{(q+s)-(p+r)}D(i^{p-q}\alpha) = i^{s-r}D\alpha.$$

Applying this to $d = \mu + \partial + \bar{\partial} + \bar{\mu}$ gives us

$$\mathbb{I}^{-1} \circ d \circ \mathbb{I} = i(\mu - \partial + \bar{\partial} - \bar{\mu})$$

and thus

$$[\Lambda, d] = i(\mu^* - \partial^* + \bar{\partial}^* - \bar{\mu}^*).$$

The statement follows by equating bidegree components.

Remaining two identities follows from the first two. Item four follows from

$$[\Lambda, D^*] = - *^{-1} L * *\bar{D} * + * \bar{D} * *^{-1} L* = - * (L\bar{D} - \bar{D}L)* = 0.$$

Item three follows from $[L, D^*] = - * [\Lambda, \bar{D}]* = \pm i\bar{D}$. □

Exercise: *For all $\alpha \in P^k$, we have*

$$*L^j\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j}\mathbb{I}(\alpha).$$

This is a fundamental but somewhat mysterious identity. It is equivalent to

$$*\exp(L)\alpha = (-1)^{\frac{k(k+1)}{2}} \exp(L)\mathbb{I}(\alpha).$$

In fact, this can be understood as the consequence that the Hodge star is essentially the Chevalley τ-operator, which is given by $\tau = \exp(L)\exp(-\Lambda)\exp(L)$. Using product property and decomposition of vector space $V = T_x^*M$ into one dimensional. We can $\tau = (-1)^{\frac{k(k+1)}{2}} *^{-1} \mathbb{I}$ on Ω^k.

By definition, $*\exp(L) = \exp(\Lambda)*$. And for primitive α, we get $\exp(-\Lambda)\alpha = \alpha$. Hence

$$*\exp(L)\alpha = (-1)^{\frac{k(k+1)}{2}} \exp(L)\exp(-\Lambda)\mathbb{I}(\alpha) = (-1)^{\frac{k(k+1)}{2}} \exp(L)\mathbb{I}(\alpha).$$

On a complex manifold, as $d = \partial + \bar\partial$, we have

$$\Delta_d = \Delta_\partial + \Delta_{\bar\partial} + \partial\bar\partial^* + \bar\partial^*\partial + \bar\partial\partial^* + \partial^*\bar\partial.$$

For Kähler manifolds, we have more

Theorem 3.2.6. *On a Kähler manifold,* $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

Proof. Since $[\Lambda, \partial] = i\bar\partial^*$, we have

$$i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial = 0.$$

Take complex conjugate, we have $\bar\partial\partial^* + \partial^*\bar\partial = 0$.

To prove $\Delta_\partial = \Delta_{\bar\partial}$, we have

$$-i\Delta_\partial = \partial(\Lambda\bar\partial - \bar\partial\Lambda) + (\Lambda\bar\partial - \bar\partial\Lambda)\partial = \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial,$$

and

$$i\Delta_{\bar\partial} = \bar\partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\bar\partial = \bar\partial\Lambda\partial - \bar\partial\partial\Lambda + \Lambda\partial\bar\partial - \partial\Lambda\bar\partial.$$

Then $\Delta_\partial = \Delta_{\bar\partial}$ since $\bar\partial\partial = -\partial\bar\partial$. □

For almost Kähler manifolds, we also have corresponding identities. In particular, we have $\Delta_\partial + \Delta_{\bar\mu} = \Delta_{\bar\partial} + \Delta_\mu$.

We also have the following.

Corollary 3.2.7. *For any almost Kähler manifold, the following hold:*

1. $[L, \Delta_{\bar\partial}] = [L, \Delta_{\bar\mu}] = -[L, \Delta_\partial] = -[L, \Delta_\mu] = -i(\bar\partial\partial + \partial\bar\partial) = i(\bar\mu\mu + \mu\bar\mu)$.
2. $[\Lambda, \Delta_{\bar\partial}] = [\Lambda, \Delta_{\bar\mu}] = -[\Lambda, \Delta_\partial] = -[\Lambda, \Delta_\mu] = -i(\bar\partial^*\partial^* + \partial^*\bar\partial^*) = i(\bar\mu^*\mu^* + \mu^*\bar\mu^*)$.

Proof. $[L, \Delta_{\bar\partial}] = [\bar\partial, [L, \bar\partial^*]]^\pm = -i(\bar\partial\partial + \partial\bar\partial) = i(\bar\mu\mu + \mu\bar\mu) = [\bar\mu, [L, \bar\mu^*]]^\pm = [L, \Delta_{\bar\mu}]$. Here $[]^\pm$ means the signed commutator. All the others follow from taking conjugates or adjoints. □

In particular, for Kähler manifolds, $[L, \Delta_d] = [L, \Delta_{\bar\partial}] = 0$.

3.3 Hodge theorem

We known that the Hodge theorem for compact orientable Riemannian manifolds states that each de Rham cohomology class contains a unique harmonic representative, *i.e.* $\Delta_d \alpha = 0$. For Hermitian manifolds and Dolbeault cohomology class, we have a similar statement. It follows from the following Hodge theorem.

Theorem 3.3.1. *Let M be a compact, complex manifold.*

- *The space of $\bar{\partial}$-harmonic (p,q)-forms $\mathcal{H}^{p,q}(M)$ is of finite dimension.*
- *The orthogonal projection $\mathcal{H} : \Omega^{p,q}(M) \to \mathcal{H}^{p,q}(M)$ is well defined, and there is a unique Green's operator, $G : \Omega^{p,q} \to \Omega^{p,q}$ with $G(\mathcal{H}^{p,q}(M)) = 0$, $\bar{\partial} G = G \bar{\partial}$, $\bar{\partial}^* G = G \bar{\partial}^*$ and*

$$\psi = \mathcal{H}(\psi) + \bar{\partial}(\bar{\partial}^* G \psi) + \bar{\partial}^*(\bar{\partial} G \psi)$$

  for $\psi \in \Omega^{p,q}(M)$.

In the above, $\bar{\partial}$ could be any differential operator P, *e.g.* d, ∂, or ∂_E, as long as Δ_P is elliptic.

Exercise: State the theorem for Δ_d.

A related decomposition in analysis is the Helmholz decomposition for a vector field on a bounded domain $V \subset \mathbb{R}^3$. It is a sum of a curl-free vector field and a divergence-free vector field.

$$F = -\nabla \Phi + \nabla \times \mathbb{A}.$$

If $V = \mathbb{R}^3$ and F vanishes faster than $\frac{1}{r}$ as $r \to \infty$, one has

$$\Phi(r) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla' \cdot F(r')}{|r - r'|} dV', \quad \mathbb{A}(r) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla' \times F(r')}{|r - r'|} dV',$$

and ∇' is the nabla operator with respect to r'. In analysis, $\frac{1}{4\pi|r-r'|}$ is the Green function for the 3-dimensional Laplacian, which means it solves $\Delta G(r, r') = \delta(r - r')$. Green operator is an integral operator $G\phi(x) = \int_M G(x, y)\phi(y) dV(y)$.

In fact, it follows from Hodge decomposition, although it has a much simpler and direct argument. We assume $\mathbb{R}^3$ have coordinate x, y, z and standard metric. A vector field $F = (F_1, F_2, F_3)$ is equivalent to a 1-form $\Psi = F_1 dx + F_2 dy + F_3 dz$. The infinite behavior makes Ψ to be a smooth form on S^3 which has value 0 at the one adding point. By Hodge theorem for Riemannian manifolds, it could be represented as $d\alpha + d^* \beta$ since the harmonic 1-form must be zero. $d\alpha$ and $d^* \beta$ correspond to $\nabla \Phi$ and $\nabla \times \mathbb{A}$ respectively. So the first term is $dG(d^* \Psi)$ and the second is $- * d * G d \Psi$.

Note the Hodge decomposition of every $\bar{\partial}$-closed form $\phi \in Z^{p,q}_{\bar{\partial}}(M)$ is

$$\phi = \mathcal{H}(\phi) + \bar{\partial}(\bar{\partial}^* G\phi)$$

since $\bar{\partial}G\phi = G\bar{\partial}\phi = 0$. We have the following

Theorem 3.3.2. *Let (M, J, g) be a compact Hermitian manifold. Each Dolbeault cohomology class $a \in H^{p,q}(M)$ contains a unique $\bar{\partial}$-harmonic representative α, i.e. $\Delta_{\bar{\partial}}\alpha = 0$.*

Similarly, for a Riemannian manifold (M, g), each cohomology class $a \in H^k(M, \mathbb{C})$ contains a unique harmonic representative α.

In particular, the dimension $\mathcal{H}^{p,q}(M)$ is independent of the choice of the Hermitian metric. In fact, we can still define $\Delta_{\bar{\partial}}$ and the corresponding $\mathcal{H}^{p,q}(M) := \ker \Delta_{\bar{\partial}}$ for almost Hermitian manifolds. On one side, $\Delta_{\bar{\partial}}$ is still elliptic, so $h^{p,q} = \dim \mathcal{H}^{p,q}(M) < \infty$. But on the other hand, $h^{p,q}$ does depend on the choice of Hermitian structure, and can be arbitrarily large when we vary J.

When (M, J, g) is turned out to be Kähler, by Kähler identities, $\Delta_d = 2\Delta_{\bar{\partial}}$. Hence two notions of harmonic coincides.

Theorem 3.3.3. *On a complex manifold, we have the Serre duality $H^{p,q}(M) = H^{n-p,n-q}(M)$.*

On a Kähler manifold, a differential form is d-harmonic if and only if it is $\bar{\partial}$-harmonic. Hence

$$H^r(M, \mathbb{C}) = \oplus_{p+q=r} H^{p,q}(M).$$

Moreover, we have isomorphism

$$H^{p,q}(M) = H^{q,p}(M) = H^{n-p,n-q}(M) = H^{n-q,n-p}(M).$$

Serre duality follows because $\alpha \in \mathcal{H}^{p,q}$ implies $*\bar{\alpha} \in \mathcal{H}^{n-p,n-q}$ since Hodge star commutes with the Laplacian, and $\int \alpha \wedge *\bar{\alpha} = ||\alpha||^2 > 0$ when $\alpha \neq 0$. The decomposition follows since Δ_d preserves bidegree. For the last claim, the first equality follows from $\Delta_\partial = \Delta_{\bar{\partial}}$ by Kählerness and conjugation.

It follows that $\sum_{p+q=r} h^{p,q} = b_r$. In particular, we have $h^{p,q} \leq b_{p+q}$. On an almost Kähler manifold, $h^{p,q}$ could be arbitrarily large.

An immediate corollary is an important topological criterion for non-Kählerness.

Corollary 3.3.4. *If M is a compact Kähler manifold, b_{2r+1} is even for all r.*

Actually, a complex surface is Kähler if and only if b_1 is even. There are at least two proofs. First is by the Enrique-Kodaira classification of complex surfaces. Second is a classification free proof by analytic method due to Buchdahl and Lamari independently.

Remark 3.3.5. *Although each cohomology group could be written algebraically in terms of sheaf cohomology, we don't know an algebraic proof of Hodge theorem, in particular $H^r(M,\mathbb{C}) = \oplus_{p+q=r} H^{p,q}(M)$, even for projective manifold M. However, there is an algebraic proof of Deligne-Illusie to show the Hodge to de Rham spectral sequence $H^j(X,\Omega_X^i) \Rightarrow H^{i+j}(X,\mathbb{C})$ degenerates at E_1. Notice it doesn't imply the direct sum decomposition, however it does imply the above corollary for projective manifolds.*

Example 3.3.6. *The Kodaira-Thurston manifold KT^4 is defined to be the direct product $S^1 \times (H_3(\mathbb{Z})\backslash H_3(\mathbb{R}))$, where $H_3(\mathbb{R})$ denotes the Heisenberg group*

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL(3,\mathbb{R}) \right\}$$

and $H_3(\mathbb{Z})$ is the subgroup $H_3(\mathbb{R}) \cap GL(3,\mathbb{Z})$ acting on $H_3(\mathbb{R})$ by left multiplication. It will also be useful to consider the covering of this manifold by $\mathbb{R}^4$ given by identifying points with the relation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t+t_0 \\ x+x_0 \\ y+y_0 \\ z+z_0+x_0y \end{pmatrix} \tag{3.2}$$

for every choice of integers $t_0, x_0, y_0, z_0 \in \mathbb{Z}$. Vector fields $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z}$ are well defined, form a basis at each point. The Kodaira-Thurston manifold is the first example of a manifold admitting both complex and symplectic structures but has no Kähler structure.

A complex structure can be seen as following. We first make a coordinate change $(t,x,y,z) \mapsto (t+\frac{1}{4}(x^2+y^2), x, y, z-\frac{1}{2}xy)$, we have the new isomorphism

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t+t_0+\frac{1}{2}x_0x+\frac{1}{2}y_0y \\ x+x_0 \\ y+y_0 \\ z+z_0+\frac{1}{2}x_0y-\frac{1}{2}xy_0 \end{pmatrix} \tag{3.3}$$

Let $v = x+iy, w = t+iz$ in the new coordinate, and then the complex structure of KT^4 is from the quotient of the $\mathbb{C}^2$ by a non-abelian group of affine automorphisms of $\mathbb{C}^2$ with four generators

$$g_1(v,w) = (v, w+1), \quad g_2(v,w) = (v, w+i), \quad g_{3,4} = (v+\alpha_{3,4}, w+\bar{\alpha}_{3,4}v).$$

There is a non-abelian relation $g_3g_4 = g_2g_4g_3$.

We can compute its first Betti number. It is the rank of $\Gamma/[\Gamma,\Gamma]$ which is 3 *(generated by g_1, g_3, g_4, where $\Gamma \cong H_3(\mathbb{Z}) \times \mathbb{Z}$ or the above group generated by g_i. The generators in $H^1_{dR}(M)$ are dt, dx, dy. By the Corollary above,* *it is not Kähler. Although it does admit an almost Kähler structure for* $J\frac{\partial}{\partial t} = \frac{\partial}{\partial x}, J(\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}$, *with compatible symplectic form* $dt\wedge dx+dz\wedge dy$ *which is* $\frac{i}{2}(\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2)$ *where*

$$\phi_1 = dt + idx \quad \& \quad \phi_2 = dy + i(dz - xdy)$$

are generators of $T^*_{1,0}$ *at every point.*

An important corollary of Hodge theorem is Hodge index theorem. First, we have

Theorem 3.3.7 (Hodge-Riemann bilinear relation)**.** *Let* (M,ω) *be a compact Kähler manifold of dimension* n. *Then for any* $\alpha \in PH^{p,q}(M) \subset PH^k(M,\mathbb{C})$, *we have* $i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_M \alpha \wedge \bar{\alpha} \wedge \omega^{n-k} > 0$.

Proof. We choose α as the harmonic form in the class. Recall we have $*L^j\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j}\mathbb{I}(\alpha)$. In our situation,

$$L^{n-k}\alpha = (-1)^{k+\frac{k(k+1)}{2}}(n-k)!i^{p-q} * \alpha.$$

Hence,

$$\begin{aligned} i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_M \alpha \wedge L^{n-k}\bar{\alpha} &= (n-k)! \int_M \alpha \wedge *\bar{\alpha} \\ &= (n-k)!||\alpha||^2_{L^2} \\ &> 0 \end{aligned}$$

□

Theorem 3.3.8. *When* $n = \dim_{\mathbb{C}} M$ *is even, the signature of the intersection form* $Q(\alpha,\beta) = \int_M \alpha \wedge \beta$ *on* $H^n(M,\mathbb{R})$ *is equal to* $\sum_{a,b}(-1)^a h^{a,b}(M)$.

Proof. The signature is also equal to the signature of the Hermitian form $H(\alpha,\beta) = \int_M \alpha \wedge \bar{\beta}$. We use the decomposition $H^n(M) = \oplus_k L^k PH^{a,b}(M)$ for all $a+b = n-2k$. By Theorem 3.3.7, the sign of H on $L^kPH^{a,b}(M)$ is equal to $i^{a-b} \cdot (-1)^{\frac{1}{2}(n-2k)(n-2k-1)} = (-1)^a \cdot (-1)^{\frac{n-2k}{2}} \cdot (-1)^{\frac{n-2k}{2}} = (-1)^a$ as n is even.

We thus have

$$sign(Q) = \sum_{a+b=n-2k} (-1)^a h^{a,b}_{prim}.$$

But $h^{a,b}_{prim} = h^{a,b} - h^{a-1,b-1}$, so by Poincaré duality, we have $sign(Q) = 2\sum_{a+b=n-2k,k>0}(-1)^a h^{a,b} + \sum_{a+b=n}(-1)^a h^{a,b} = \sum_{a+b\equiv 0(mod2)}(-1)^a h^{a,b}$. On the other hand, by complex conjugation, it is clear that

$$\sum_{a+b\equiv 1(mod2)} (-1)^a h^{a,b} = 0.$$

So $sign(Q) = \sum_{a,b}(-1)^a h^{a,b}(M)$. □

A particular important special case of the above is the following

Corollary 3.3.9 (Hodge index theorem). *Let M be a compact Kähler surface, then the intersection pairing has index $(2h^{2,0}(X)+1, h^{1,1}-1)$. Restrict to $H^{1,1}(M)$, it is of index $(1, h^{1,1}(X)-1)$.*

Proof. All the forms in $H^{0,2}(M)$ or $H^{2,0}(M)$ are primitive, so the restriction of the Hermitian form H on them are positive definite. For the rest, the $sign(Q) = 2 + 2h^{0,2} - h^{1,1}$, so $(2h^{2,0}(X) + 1, h^{1,1} - 1)$ is the index of the intersection pairing. In fact, $H^{1,1}(M) = PH^{1,1} \oplus [\omega]\,\mathbb{R}$. □

Hodge index theorem is mostly used in the following form.

Corollary 3.3.10. *Let D and E be $\mathbb{R}$-divisors on an algebraic surface. If $D^2 > 0$ and $D \cdot E = 0$, then $E^2 \leq 0$ and $E^2 = 0$ if and only if E is homologous to 0.*

We could extend the discussion to the case of vector bundles. For example, the Hodge star is defined as

$$*_E : \Omega^{p,q}(E) \to \Omega^{n-p,n-q}(E^*)$$

by requiring for $\alpha, \beta \in \Omega^{p,q}(E)$,

$$(\alpha, \beta) = \int_M \alpha \wedge *_E \beta$$

where the wedge product maps $\Omega^{p,q}(E) \otimes \Omega^{p',q'}(E^*)$ to $\Omega^{p+p',q+q'}(M)$ by $(\eta \otimes s) \wedge (\eta' \otimes s') =< s, s' > \eta \wedge \eta'$.

The Hodge theorem still holds for E valued forms $\Omega^{p,q}(E)$. Let $\nabla = \partial_E + \bar\partial_E$, where $\bar\partial_E$ is the holomorphic $\bar\partial$ operator, be the Chern connection on E. Then we still have the Kähler identities, namely $[\Lambda, \bar\partial_E] = -i\partial_E^*$.

Choose local frame $\{e_\alpha\}$ for E. If $\theta = \theta' + \theta''$ is the connection matrix for ∇ in terms of the frame. For $\eta = \sum_\alpha \eta_\alpha \otimes e_\alpha$, we have

$$[\Lambda, \bar\partial_E]\eta = \sum_\alpha [\Lambda, \bar\partial]\eta_\alpha \otimes e_\alpha + [\Lambda, \theta'']\eta.$$

We also have

$$\partial_E^* \eta = \sum_\alpha \partial^* \eta_\alpha \otimes e_\alpha + \theta'^* \eta.$$

Hence

$$[\Lambda, \bar\partial_E] + i\partial_E^* = [\Lambda, \theta''] + i\theta'^*.$$

But we can choose a frame of E at a neighborhood of z_0 for which $\theta(z_0)$ vanishes. Hence we have the Kähler identity.

3.3.1 $\partial\bar{\partial}$-Lemma

$\partial\bar{\partial}$-Lemma is a very basic but important lemma for Kähler manifolds.

Lemma 3.3.11. *Let M be a Kähler manifold and let ϕ is a d-closed (p, q) form on M, where $p, q > 0$. If ϕ is d-, ∂-, or $\bar{\partial}$-exact, then it is $\partial\bar{\partial}$-exact, i.e. there is a $\psi \in \Omega^{p-1,q-1}$ such that $\phi = \partial\bar{\partial}\psi$. Moreover, if $p = q$ and ϕ is real, then we may take ψ such that $i\psi$ is real.*

Proof. We have Kähler identities

$$[\Lambda, \bar{\partial}] = -i\partial^*, \quad [\Lambda, \partial] = i\bar{\partial}^*, \quad \Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

Hence the Green operators also have the similar identity

$$2G_d = G_\partial = G_{\bar{\partial}}.$$

The Green operators commute with differentials $d, \partial, \bar{\partial}$ and their adjoints. We have the Hodge decomposition $I = \mathcal{H} + \Delta_D G_D$ where D is one of $d, \partial, \bar{\partial}$. Here $\mathcal{H}$ is independent of D from the three.

If ϕ is a $\bar{\partial}$-exact (p, q)-form, then

$$\phi = (\mathcal{H} + \Delta_{\bar{\partial}} G_{\bar{\partial}})\phi = \bar{\partial}\bar{\partial}^* G_{\bar{\partial}}\phi = \bar{\partial}\eta.$$

By Kähler identity, $\bar{\partial}^* = -i[\Lambda, \partial]$ anti-commutes with ∂, so $\partial\eta = -\bar{\partial}^* G_{\bar{\partial}}\partial\phi = 0$, since $d\phi = 0$ implies $\partial\phi = 0$ in particular. Thus

$$\eta = (\mathcal{H} + \Delta_\partial G_\partial)\eta = \partial\partial^* G_\partial\eta = \partial\psi.$$

Hence $\phi = \bar{\partial}\eta = -\partial\bar{\partial}\psi$.

Similarly, we can argue it when ϕ is d-exact or ∂-exact. When argue it for d-exact, we use $d^c := i(\bar{\partial} - \partial)$.

When $\phi = \bar{\phi}$, if $\phi = i\partial\bar{\partial}\psi$, then

$$2\phi = i\partial\bar{\partial}(\psi + \bar{\psi}),$$

and $\gamma = \frac{1}{2}(\psi + \bar{\psi})$ is real. □

In particular, if M is a Kähler manifold and ϕ_1 and ϕ_2 are d-cohomologous $(1, 1)$ forms. Then there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $\phi_1 - \phi_2 = \partial\bar{\partial}f$.

It is also known that the $\partial\bar{\partial}$-lemma is preserved under birational transformation. Hence in particular it holds for Moishezon manifolds.

An application of Lemma 3.3.11 is the proof of Theorem 3.1.5.

Proof. If L is a holomorphic line bundle on M and h is a Hermitian metric on L, which is a collection $\{h_\alpha\}$ of smooth positive functions $h_\alpha : U_\alpha \to \mathbb{R}$ satisfying the transformation rule $h_\alpha = |g_{\alpha\beta}|^2 h_\beta$ on $U_\alpha \cap U_\beta$ where $g_{\alpha\beta}$

is the transition function of L. The curvature $\Theta^h = -\partial\bar\partial \log h_\alpha$ on U_α (check it is well defined), and $\frac{i}{2\pi}\Theta^h$ is a real $(1,1)$-form representing $c_1(L) \in H^{1,1}(M)$. If ϕ is another real $(1,1)$-form representing $c_1(L)$, then the difference $\frac{i}{2\pi}\Theta^h - \phi = \frac{i}{2\pi}\partial\bar\partial f$ for a real-valued smooth function f on M. So $\phi = -\frac{i}{2\pi}\partial\bar\partial \log(e^f h) = \frac{i}{2\pi}\Theta^{\tilde h}$ is the $\frac{i}{2\pi}$ multiple of the curvature form of a new Hermitian metric $\tilde h$ on L, where $\tilde h = e^f h$. □

The canonical (holomorphic) line bundle of a complex manifold is defined as $\mathcal{K}_M = \det(T^*_{1,0}M)$. More explicitly, the transition function is given by $\phi_{\alpha\beta} = \det(\partial z^i_\beta / \partial z^j_\alpha)$ on $U_\alpha \cap U_\beta$, where $z^1_\alpha, \cdots, z^n_\alpha$ are coordinates on U_α. $\mathcal{K}_M^{-1}$ is called the anti-canonical bundle. $c_1(\mathcal{K}_M^{-1}) = c_1(M)$. If h is a Hermitian metric on M, then $h_\alpha = \det(h^\alpha_{i\bar j})$ defines a Hermitian metric on $\mathcal{K}_M^{-1}$. By above calculation the curvature corresponding to this Hermitian metric is just the Ricci curvature form of h

$$Ric(\omega) = -\frac{i}{2\pi}\partial\bar\partial \log\det h = -\frac{i}{2\pi}\partial\bar\partial \log \omega^n.$$

Here ω is the 2-form determined by h and the complex structure. In particular, $[Ric(\omega)] = c_1(M)$.

For any complex manifold, we can define the Bott-Chern cohomology as

$$H^{p,q}_{BC} := \{\alpha \in \Omega^{p,q} | d\alpha = 0\} / \partial\bar\partial\Omega^{p-1,q-1}.$$

Clearly, there are natural maps

$$H^{p,q}_{BC}(M) \to H^{p,q}_{\bar\partial}(M) \text{ and } \oplus_{p+q=k} H^{p,q}_{BC}(M) \to H^k(M,\mathbb{C}).$$

The $\partial\bar\partial$-lemma implies these two maps are injective. Clearly, the second map is surjective. The first map is surjective is less obvious, which follows from Lemma 5.15 and Remark 5.16 of Deligne-Griffiths-Morgan-Sullivan's 1975 *Inventiones* paper. Their argument is to introduce Aeppli cohomology

$$H^{p,q}_{A} := \{\alpha \in \Omega^{p,q} | \partial\bar\partial\alpha = 0\} / \{\partial\Omega^{p-1,q} + \bar\partial\Omega^{p,q-1}\}.$$

It is a nice exercise to show the $\partial\bar\partial$-lemma is equivalent to surjectivity or injectivity of the natural map $H^{p,q}_{\bar\partial}(M) \to H^{p,q}_A(M)$, or the surjectivity of the composition of $H^{p,q}_{BC}(M) \to H^{p,q}_{\bar\partial}(M) \to H^{p,q}_A(M)$. In particular, it implies Hodge decomposition holds when a complex manifold satisfies the $\partial\bar\partial$-lemma. In fact, the converse is also true by Proposition 5.17 of [DGMS].

As $\partial\bar\partial$-lemma preserved under birational transform, we know Hodge decomposition holds for Moishezon manifolds or Fujiki class $\mathcal{C}$ (*i.e.* some blow up of them are projective or Kähler respectively).

Example 3.3.12 (Hironaka's Moishezon non-Kähler manifold)**.** *Pick up two rational curves C and D in $\mathbb{C}P^3$ intersect transversally only at two points*

p, q (This is easy to do in any projective manifolds). Then "blow up" C and D but in a different order at p and q: first C and then D over p but first D then C over q. This manifold M is non-Kähler. Let l be a general fibre of the exceptional divisor E_1 over C and m a general fibre of the exceptional divisor E_2 over D. Over p, it is a reducible fibre, with the irreducible fibre of E_2 over it l_2 and the one from E_1 is l_1. Similarly, the irreducible fibre of E_1 over q is m_2 and the one from E_2 is m_1. Then

$$[m_2] = [l] = [l_1] + [l_2] = [l_1] + [m] = [l_1] + [m_1] + [m_2].$$

Hence $[l_1] + [m_1]$ is nullhomologous. This cannot be true if M is Kähler as the Kähler form restrict to smooth curves l_1 and m_1 need to be positive.

On the other hand, if we blow up l_1 or m_1, we have a projective manifold. Notice both of them are rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. If we blow up, say l_1, the exceptional divisor will be a ruled surface $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the normal bundle restricting on each direction of fibres are $\mathcal{O}(-1)$. If we blow down along the other direction, we will have the manifold M' which is obtained from $\mathbb{C}P^3$ by blowing up first D then C. This is certainly projective as we will show blowing up along smooth locus will preserve projectivity/Kählerness.

The process from M to M' is an example of a flop.

3.3.2 Proof of Hodge theorem

First, the Laplacian $\Delta = \Delta_d$ or $\Delta_{\bar\partial}$ is self-adjoint, *i.e.* $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$ for $\alpha, \beta \in \Omega^k(M)$. Let $\mathcal{H}^k$ be $\ker \Delta_d : \Omega^k(M) \to \Omega^k(M)$. Similarly for $\mathcal{H}^{p,q}(M)$. Then basically, the Hodge decomposition is the infinite dimensional version $\ker A \oplus \operatorname{Im} A = \mathbb{C}^n$ for an $n \times n$ Hermitian matrix A.

Roughly speaking, we decompose Ω^k (or $\Omega^{p,q}$) as $\mathcal{H}^k$ (or $\mathcal{H}^{p,q}$) and $(\mathcal{H}^k)^\perp \cap \Omega^k$ (or $(\mathcal{H}^{p,q})^\perp \cap \Omega^{p,q}$). Here the orthogonal complement is taken with respect to the L^2 norm on M. Then we would like to show that any $\alpha \in (\mathcal{H}^k)^\perp \cap \Omega^k$, there exists $\omega \in \Omega^k$ such that $\Delta\omega = \alpha$.

To find such a solution, we first look for a weak solution of the equation $\Delta\omega = \alpha$. This means a bounded linear functional $L : \Omega^k \to \mathbb{R}$ which satisfies $L(\Delta\phi) = (\alpha, \phi)$ for all $\phi \in \Omega^k$. The following regularity theorem guarantees such a weak solution is always smooth in our situation.

Theorem 3.3.13. *Let $\alpha \in \Omega^k(M)$, and let L be a weak solution of $\Delta\omega = \alpha$. Then there exists an $\omega \in \Omega^k(M)$ such that $L(\beta) = (\omega, \beta)$ for every $\beta \in \Omega^k(M)$.*

To show such a weak solution exists, we need the following lemmas.

Lemma 3.3.14. *Let $\{\alpha_n\}$ be a sequence of uniformly bounded smooth k-forms on M such that $\{||\Delta\alpha_n||\}$ is uniformly bounded. Then there exists a Cauchy subsequence of $\{\alpha_n\}$.*

We will postpone to explain its proof later.

Lemma 3.3.15. *There exists a constant $C > 0$ such that $||\beta|| \leq C||\Delta\beta||$ for all $\beta \in (\mathcal{H}^k)^\perp \cap \Omega^k$.*

Proof. If not, there exists a sequence $\beta_n \in (\mathcal{H}^k)^\perp \cap \Omega^k$ such that $||\beta_n|| = 1$ for all n and $\lim_{n\to\infty} ||\Delta\beta_n|| = 0$. By Lemma 3.3.14, we can choose a Cauchy subsequence, still denoted by $\{\beta_n\}$. For each $\psi \in \Omega^k$, define $F(\psi) := \lim_{n\to\infty}(\beta_n, \psi)$. F is a bounded linear functional and $F(\Delta\phi) = \lim_{n\to\infty}(\beta_n, \Delta\phi) = \lim_{n\to\infty}(\Delta\beta_n, \phi) = 0$ for every $\phi \in \Omega^k$. Hence F is a weak solution of $\Delta\beta = 0$. By Theorem 3.3.13, there exists a smooth solution $\beta \in \mathcal{H}^k$. It follows that $\beta_n \to \beta$ and hence $||\beta|| = 1$. But $(\mathcal{H}^k)^\perp \cap \Omega^k$ is closed subspace of Ω^k and hence $\beta \in (\mathcal{H}^k)^\perp$. Hence $\beta = 0$, a contradiction. □

Now we can complete the proof of Hodge theorem. First $\mathcal{H}^k$ is finite dimensional by Lemma 3.3.14 applying to otherwise an infinite sequence of orthonormal elements.

Now we will show $(\mathcal{H}^k)^\perp = \Delta(\Omega^k(M))$. Choose $\alpha \in (\mathcal{H}^k)^\perp$, the above $L(\Delta\phi) = (\alpha, \phi)$ is well defined. Moreover, L is a bounded linear functional on $\Delta(\Omega^k)$ by the above lemma, and hence extend to a weak solution by Hahn-Banach theorem. Hence there exists a unique smooth solution of $\Delta\omega = \alpha$ for $\alpha \in (\mathcal{H}^k)^\perp \cap \Omega^k(M)$ by Theorem 3.3.13. Hence the Green operator is defined as $G(\alpha) = G(\alpha - \alpha_h) = \omega$ where $\omega \in (\mathcal{H}^k)^\perp$ solves $\Delta\omega = \alpha - \alpha_h$. Here α_h is the projection of α to $\mathcal{H}^k$. Or by first choosing a basis $w_1, \cdots, w_n$ of $\mathcal{H}^k$, we have $\alpha_h = \sum_i (\alpha, w_i) w_i$.

For the remaining two unproved results, we can use Sobolev spaces. For Lemma 3.3.14, by partition of unity, it suffices to show the statement in open subsets of $\mathbb{R}^n$ or $T^n = \mathbb{R}^n / \mathbb{Z}^n$. In addition, for Theorem 3.3.13 and any $p \in M$, we can find a small neighborhood V of p and a differential operator $\tilde{L}$ on T^n which agrees with L on V. Hence, we reduce our discussions to analysis on T^n. The advantage of this space is we can use Fourier series.

Let $\mathcal{S}$ be the space of all sequences in $\mathbb{C}$ indexed by n-tuples of integers, or in other words, the space of formal Fourier series $u = \sum_{\xi\in\mathbb{Z}^n} u_\xi e^{i<\xi,x>}$. The Sobolev space

$$H_s := \{u \in \mathcal{S} | \sum_\xi (1 + |\xi|^2)^s |u_\xi|^2 < \infty\}.$$

Since $||D^\alpha\phi||^2 = \sum_\xi \xi^{2\alpha}|\phi_\xi|^2$ by Parseval identity, we know the norm $\sum_{[\alpha]\leq s} \int_T |D^\alpha\phi|^2 dx$ is equivalent to the above one. Here $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index and $[\alpha] = \sum \alpha_j$, $D^\alpha u = \frac{\partial^{[\alpha]} u}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$. Apprently $C^k \subset H_k$. More crucially, we have the following Sobolev lemmas.

Lemma 3.3.16. *1. (Sobolev) For $k > l + \frac{n}{2}, H_k \subset C^l$.*

2. *(Rellich) For any integers $l > k$, the inclusion $H_l \to H_k$ is a compact operator, i.e. given a sequence $\{u_i\}$ in H_l, we can find a convergent subsequence in H_k.*

3. *D^α is a bounded operator from $H_{s+[\alpha]}$ to H_s for each s.*

In particular, $\cap H_s = C^\infty$. This lemma holds on any manifold M, but can be seen in a very explicitly way for T^n.

A differential operator P of order 2 on smooth functions is $P = \sum_{[\alpha]=0}^2 a^\alpha(x) D^\alpha$ where a^α are smooth with at least one $a^\alpha \neq 0$ for some α with $[\alpha] = 2$. It is called elliptic at x if $\sum a^\alpha(x)\xi^\alpha \neq 0$ for all $\xi \in \mathbb{R}^n$. For standard Laplacian on $\mathbb{R}^n$, this sum is the square sum. The ellipticity could be formulated for differential operators $P : \Gamma(M, E) \to \Gamma(M, F)$ for smooth sections of Hermitian bundles E, F of any order. Now $Pu(x) = \sum_{[\alpha]\leq k} a^\alpha(x) D^\alpha u$ where a^α are rank $E \times$ rank F matrices. And the signature is a polynomial map from $T^*M \to Hom(E, F)$ by $\sigma_P(x, \xi) = \sum_{[\alpha]=k} a^\alpha(x)\xi^\alpha$ where $\xi \in T_x^*M$. Ellipticity just means $\sigma_P(x, \xi) \in Hom(E_x, F_x)$ is injective for every $x \in M$ and $\xi \neq 0$.

Example 3.3.17. *We would like to calculate the signature of Δ_d acting on $\Omega^k(M)$ of a Riemannian manifold M. That is, $E = F = \Omega^k$. Choose an orthonormal basis $\{\xi_1, \cdots, \xi_n\}$ of T_xM and dual basis ξ_i^*.*

Let $u = \sum u_I \xi_I^*$ with $|I| = k$. Then

$$du = \sum_{I,j} (\xi_j \cdot u_I)\xi_j^* \wedge \xi_I^* + \sum_I u_I d\xi_I^*, \quad d^*u = -\sum_{I,j} (\xi_j \cdot u_I)\xi_j \rfloor \xi_I^* + \sum_{I,K} \alpha_{I,K} u_I \xi_K^*$$

for some smooth coefficients $\alpha_{I,K}$ with $|K| = k-1$. Hence the principal part of Δ is the same as that of

$$u \mapsto -\sum_I (\sum_j \xi_j^2 \cdot u_I)\xi_I^*.$$

So the signature $\sigma_P(x, \xi)$ is the diagonal matrix $-\sum_j \xi_j(\xi)^2 \cdot Id$ where $\xi_j(\xi)$ denotes the jth component of ξ in terms of the basis ξ_i at x. Hence it is elliptic.

The key property we need from ellipticity is Gårding's inequality.

Theorem 3.3.18. *Let P be an elliptic operator of order d. Then given any $s \in \mathbb{Z}$, there exists a constant $C > 0$ such that*

$$||u||_{s+d} \leq C(||Pu||_s + ||u||_s)$$

for all $u \in H_{s+d}$.

Now Lemma 3.3.14 follows from Gårding's inequality and item 2 of Lemma 3.3.16. Theorem 3.3.13 actually holds for any elliptic operators P in place of Δ. It follows from applying Gårding's inequality in a boot-strapping way. In fact, if $Pu = v$ for $u \in H_s$, $v \in H_{s-1}$, then $u \in H_{s+1}$. We can use difference quotient to bound the $s+1$ norm of u.

Since the weak solution L is a bounded, it extends to a bounded linear functional on $H_0 = L^2$. It follows from Riesz representation theorem that there is an element $\omega \in H_0$ such that $L(\beta) = (\omega, \beta)$. Hence $P\omega = \alpha$ for $\omega \in H_0$ and smooth α. Hence apply the above for $s = 0$, we know $\omega \in H_1$. Applying this process for $s = 1, 2, \cdots$, we get $\omega \in H_n$ for any n and thus by Sobolev embedding $\omega \in C^\infty$.

Let us prove Gårding's inequality for order 2 elliptic operator with constant coefficient on T^n to see how the ellipticity is used. General case on T^n follows from a slight perturbation.

Proof. Ellipticity implies $|P(\xi)u|^2 > c|\xi|^4|u|^2$. Write $\phi(x) = \sum_{\xi \in \mathbb{Z}^n} \phi_\xi e^{i<\xi,x>}$, then

$$\begin{aligned}
(||P\phi||_s + ||\phi||_s)^2 &\geq ||P\phi||_s^2 + ||\phi||_s^2 \\
&= \sum_\xi (|P(\xi)\phi_\xi|^2 + |\phi_\xi|^2)(1 + |\xi|^2)^s \\
&\geq \sum_\xi (1 + c|\xi|^4)|\phi_\xi|^2(1 + |\xi|^2)^s \\
&\geq c\sum_\xi (1 + |\xi|^2)^{s+2}|\phi_\xi|^2
\end{aligned}$$

□

A general form of Hodge decomposition is the following.

Theorem 3.3.19. *Let $P : \Gamma(M, F) \to \Gamma(M, F)$ be an elliptic operator between smooth sections of an Hermitian bundle F. Then*

1. $\ker P$ *and* $\ker P^*$ *are finite dimensional.*

2. $P(\Gamma(M, F))$ *is closed and of finite codimension. Furthermore, there is a decomposition*

$$\Gamma(M, F) = P(\Gamma(M, F)) \oplus \ker P^*$$

as orthogonal direct sum in $L^2(M, F)$.

The proof is almost identical to the above special case. We could also use a slightly different strategy by first showing $H_k(M, F) = P(H_{k+d}(M, F)) \oplus \ker P^*$, and then pass to the smooth decomposition. Again the proof is very similar. When $P = P^*$, a Green operator as in Hodge decomposition exists.

3.4 Divisors and line bundles

A divisor D on M is a locally finite formal linear combination $D = \sum a_i V_i$ where V_i are irreducible analytic hypersurfaces of M and $a_i \in \mathbb{Z}$. A divisor is called effective if $a_i \geq 0$ for all i. The set $Div(M)$ of divisors is a group under addition in the obvious way. There is a basic correspondence between divisors and holomorphic line bundles. First, for a meromorphic section s of a holomorphic line bundle (*i.e.* a locally defined holomorphic function with values in $\mathbb{C}P^1$), we can associate a divisor $(s) := \sum_V ord_V(s)V$ by a weighted sum of its zeros and poles, where V are irreducible hypersurfaces. Let f_α be local defining functions of D over some open cover $\{U_\alpha\}$ of M. Then the functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ are holomorphic and nonzero in $U_\alpha \cap U_\beta$ with $g_{\alpha\beta}g_{\beta\alpha} = 1$, and in $U_\alpha \cap U_\beta \cap U_\gamma$ we have $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$. With these identities for $\{g_{\alpha\beta}\}$, we can construct a line bundle L by taking the union $\cup_\alpha U_\alpha \times \mathbb{C}$ with points $(x, \lambda) \in U_\beta \times \mathbb{C}$ and $(x, g_{\alpha\beta}(x)\lambda) \in U_\alpha \times \mathbb{C}$ identified. The line bundle given by the transition functions $\{g_{\alpha\beta}\}$ is called the associated line bundle of D and denoted by L_D (or $\mathcal{O}(D)$). It is easy to check that it is well defined. Denote by $Pic(M)$ the set of isomorphism classes of holomorphic line bundles. In sheaf theory language, $Pic(M) \equiv H^1(M, \mathcal{O}^*)$.

Recall that collection of transition functions define the same line bundle if and only if there exists non-vanishing holomorphic functions f_α on U_α such that $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$. Tensor product makes $Pic(M)$ into an abelian group, called the Picard group of M ($L \otimes L^* = End(L)$ is a trivial line bundle because identity $L \to L$ gives a nowhere zero section). The kernel of the homomorphism $Div(M) \to Pic(M)$ is those divisors (f) where f is a meromorphic section of the trivial bundle, *i.e.* a meromorphic function on M. In fact, if D is given by f_α and L_D is trivial, then there exists $h_\alpha \in \mathcal{O}^*(U_\alpha)$ such that $\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$. Then $f = f_\alpha h_\alpha^{-1}$ is a global meromorphic function on M with divisor D. Two divisors are called linearly equivalent if $D \sim D'$, *i.e.* when $D - D' = div(f)$. Thus the group homomorphism factors through an injection $Div(M)/\sim \to Pic(M)$. This homomorphism need not to be surjective, although it is true when M is a projective manifold. The following Poincaré-Lelong theorem is fundamental.

Theorem 3.4.1. *For any divisor D on a compact complex manifold, $c_1(L_D) = PD[D]$.*

Proof. As the general argument is the same, we could assume $D = V$ an irreducible subvariety. We are amount to show that

$$\frac{i}{2\pi} \int_M \Theta \wedge \psi = \int_V \psi$$

for every real closed form $\psi \in \Omega^{2n-2}(M)$.

Now let s be a global section $\{f_\alpha\}$ of L_D vanishing exactly on V. Set $D(\epsilon) = (|s(z)| < \epsilon) \subset M$ to be a tubular neighborhood around V in M.

Recall that on $M\backslash D(\epsilon)$, $\Theta = -\partial\bar{\partial}\log|s|^2 = \frac{1}{2}idd^c\log|s|^2$. Here $d^c = -i(\partial - \bar{\partial})$. We have

$$\int_M \Theta\wedge\psi = \lim_{\epsilon\to 0}\frac{i}{2}\int_{M\backslash D(\epsilon)} dd^c\log|s|^2\wedge\psi = \lim_{\epsilon\to 0}\frac{1}{2i}\int_{\partial D(\epsilon)} d^c\log|s|^2\wedge\psi.$$

In $U_\alpha\cap D(\epsilon)$, write $|s|^2 = f_\alpha\bar{f}_\alpha h_\alpha$ with $h_\alpha > 0$. As $d^c\log|s|^2 = \frac{i}{4\pi}(\bar{\partial}\log\bar{f}_\alpha - \partial\log f_\alpha + (\bar{\partial}-\partial)\log h_\alpha)$, we have

$$\lim_{\epsilon\to 0}\frac{1}{2i}\int_{\partial D(\epsilon)} d^c\log|s|^2\wedge\psi = \lim_{\epsilon\to 0} -iIm\int_{\partial D(\epsilon)}\partial\log f_\alpha\wedge\psi,$$

since $d^c\log h_\alpha$ is bounded and $\lim_{\epsilon\to 0} vol(\partial D(\epsilon)) = 0$.

If we have local coordinate $w = (w_1,\cdots,w_n)$ with $w_1 = f_\alpha$. Write $\psi = \psi(w)dw'\wedge d\bar{w}' + \phi$, where $w' = (w_2,\cdots,w_n)$ and ϕ contains dw_1 or $d\bar{w}_1$. Then in any polydisc Δ around $z_0\in V\cap U_\alpha$,

$$\lim_{\epsilon\to 0}\int_{\partial D(\epsilon)\cap\Delta}\partial\log f_\alpha\wedge\psi = \int_{|w_1|=\epsilon}\frac{1}{w_1}dw_1\wedge\psi(w)dw'\wedge d\bar{w}' = 2\pi i\int_{V\cap\Delta}\psi$$

as it is $2\pi i\int_{w'}\psi(0,w')dw'\wedge d\bar{w}'$. Hence

$$\int_M\Theta\wedge\psi = \frac{2\pi}{i}\int_V\psi.$$

□

If D is any divisor such that $L_D = L$, there exists a meromorphic section s of L with $(s) = D$. It is uniquely determined up to a scaling since a nowhere zero holomorphic section of the trivial bundle is a constant when the base is compact. On the other hand, for any meromorphic section s of L, $L = L_{(s)}$. In particular, it is the line bundle of an effective divisor if and only if it has a nontrivial global holomorphic section.

When M is projective, then every line bundle is of the form L_D, *i.e.* $Div(M)\to Pic(M)$ is surjective. This essentially follows from the Kodaira vanishing and embedding. By Poincaré-Lelong theorem, it suffices to show that every holomorphic line bundle over a projective manifold admits a meromorphic section. Choose an ample bundle H over M, then for sufficiently large N, both $L\otimes H^{\otimes N}$ and $H^{\otimes N}$ are very ample, thus have non-zero holomorphic sections s_1, s_2. Then $s = \frac{s_1}{s_2}$ is a meromorphic section of L.

Similar arguments, but using Grassmannian, lead to the following.

Theorem 3.4.2. *Let M be a projective manifolds. A cohomology class in $H^{2k}(M,\mathbb{Z})$ is the cohomology of a subvariety if and only if it is the Chern class of a holomorphic vector bundle.*

Furthermore, by the following generalization of Proposition 3.1.5, we can prescribe any closed $(1,1)$ form in $H^2(M,\mathbb{Z})$ as the curvature form of a Hermitian holomorphic line bundle.

Proposition 3.4.3. *Let M be a compact Kähler manifold and ω be a smooth closed real $(1,1)$-form such that $[\omega] \in H^2(M, \mathbb{Z})$. Then there exists a Hermitian line bundle L such that the curvature form (of the Chern connection) is $\Theta = \frac{2\pi}{i}\omega$.*

Proof. We give a conceptual proof. For a concrete proof of this proposition, see [**?**] V.13.9.b.

We first prove that any class in $H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{Z})$ is the Chern class of a holomorphic line bundle. It follows from the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

and its associated cohomology sequence

$$H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}) = H^{0,2}(M).$$

Then we apply Proposition 3.1.5 to show there is an Hermitian connection on L to realize $\frac{2\pi}{i}\omega$ as the curvature. □

Theorem 3.4.1 and Proposition 3.4.3 imply (integral) Hodge conjecture holds for $(1,1)$-classes. That is, on projective variety M, every cohomology class $\gamma \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$ is $c_1(L_D) = PD[D]$ for some divisor D on M.

In general, Hodge conjecture need to be stated for rational coefficients.

Conjecture 3.4.4. *Let M be a projective manifold, and $\alpha \in H^{2k}(M, \mathbb{Q}) \cap H^{k,k}(M)$. Then α is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of M.*

By Hard Lefschetz Theorem, $L^{n-1} : H^{1,1}(M, \mathbb{Q}) \to H^{n-1,n-1}(M, \mathbb{Q})$ is an isomorphism. Hence, the Hodge conjecture also holds for $H^{2n-2}(M, \mathbb{Q}) \cap H^{n-1,n-1}(M)$. In particular, Hodge conjecture holds up to complex dimension three. However, unlike $(1,1)$ classes, integral Hodge conjecture does not hold for $(n-1, n-1)$-classes, even for 3-folds (*e.g.* see Kollár's example). In particular, these are also examples where the Hard Lefschetz Theorem fails for integral coefficients.

However, when M is merely Kähler, the group homomorphism $Div(M) \to Pic(M)$ is not necessarily surjective. The main point is Kähler manifolds may not have enough subvarieties. Let M be a general complex 4-torus, then M does not have any curve, while $H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{Z})$ is of dimension two. Hence Proposition 3.4.3 implies $Div(M) \to Pic(M)$ is not surjective. This in turn implies the naive generalization of the Hodge conjecture to Kähler manifolds is not true. We can further ask whether the Chern classes of vector bundles (or coherent sheaves) would generate all the Hodge classes, but the answer is still no by Voisin. We should also remark that $Div(M) \to Pic(M)$ might not be surjective when M is Moishezon.

The most basic and important example of a positive line bundle is the hyperplane bundle $\mathcal{O}(1)$. First, every point on the total space of $\mathcal{O}(-1)$ could be seen as a point on $\mathbb{C}^{n+1}$. We can thus put a Hermitian metric on $\mathcal{O}(-1)$ by setting $||(z_0, z_1, \cdots, z_n)||^2 = \sum |z_i|^2$. Hence the curvature form is

$$\Theta^* = -\partial\bar{\partial} \log ||z||^2.$$

The curvature form of the dual metric on $\mathcal{O}(1)$ is consequently

$$\Theta = -\Theta^* = \partial\bar{\partial} \log ||z||^2.$$

Thus $\frac{i}{2\pi}\Theta$ is just the Fubini-Study form, which is positive. It implies that the Fubini-Study form is in the class $PD[\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n, \mathbb{Z})$ by Theorem 3.4.1 as a section of $\mathcal{O}(1)$ vanishes on a hyperplane in $\mathbb{C}P^n$.

Exercise: Let E be the exceptional divisor of a blow-up $\tilde{M}^n$ along a point. Then $L_E|_E$ is $\mathcal{O}(-1)$ on $\mathbb{C}P^{n-1}$.

Now, we discuss the adjunction formula. Recall that the normal bundle N_V of the smooth irreducible hypersurface V in a compact complex manifold M is the holomorphic line bundle given by taking the quotient of $T^{1,0}M|_V$ by $T^{1,0}V$. The conormal bundle N_V^* is the dual of N_V, it is the subbundle of $T^*_{1,0}M$ consisting of cotangent vectors to M that are zero on $T^{1,0}V$. We claim

Proposition 3.4.5. $N_V = L_V|_V$.

Proof. Suppose on U_α, V is locally defined by $f_\alpha \in \mathcal{O}(U_\alpha)$. Then the transition functions of L_V are $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$. Since $f_\alpha = 0$ on $V \cap U_\alpha$, we have $df_\alpha|_{TV} = 0$, thus the differential $df_\alpha = \partial f_\alpha$ is a section of N_V^* over $V \cap U_\alpha$. Since V is smooth, df_α is nowhere zero on V.

At points of $V \cap U_\alpha \cap U_\beta$, the locally defined 1-forms df_α satisfy $df_\alpha = g_{\alpha\beta} df_\beta$ as $f_\beta = 0$ on it. This means that the df_α fit together to get a global section of $N_V^* \otimes L_V|_V$ over V. Moreover, as explained above, it is a nowhere vanishing holomorphic section of $N_V^* \otimes L_V|_V$. Hence $N_V^* \otimes L_V|_V = \mathcal{O}$ or $N_V = L_V|_V$. □

In particular, it implies the normal bundle of the exceptional divisor is $\mathcal{O}(-1)$.

Now we have the adjunction formula.

Proposition 3.4.6. *Let $V \subset M$ be a smooth irreducible hypersurface. Then*

$$\mathcal{K}_V = \mathcal{K}_M|_V \otimes L_V|_V.$$

Proof. We have $N_V = T^{1,0}M|_V / T^{1,0}V$. Take the determinant bundle of each and use the above relation $N_V = L_V|_V$, we have the relation for line bundles $\mathcal{K}_V = \mathcal{K}_M|_V \otimes L_V|_V$. □

The corresponding map on sections $H^0(M, \mathcal{K}_M \otimes \mathcal{O}(V)) \to H^0(V, \mathcal{K}_V)$ can be seen in an explicit way. An element in $H^0(M, \mathcal{K}_M \otimes \mathcal{O}(V))$ is a meromorphic n-form with a single pole along V and holomorphic elsewhere with $n = \dim_{\mathbb{C}} M$. We write $\omega = \frac{g(z)}{f(z)} dz_1 \wedge \cdots \wedge dz_n$. Then using the isomorphism in Proposition 3.4.6, we have a form ω' such that $\omega = \frac{df}{f} \wedge \omega'$. Since $df = \sum \frac{\partial f}{\partial z_i} dz_i$, we can take

$$\omega' = \sum (-1)^{i-1} \frac{g(z) dz_1 \wedge \cdots \wedge \hat{dz_i} \wedge \cdots \wedge dz_n}{\frac{\partial f}{\partial z_i}}.$$

The map $\omega \to \omega'|_{f=0}$ is the residue map.

Corollary 3.4.7. *If $X \subset \mathbb{C}P^n$ is a smooth hypersurface of degree d, i.e. defined by a section $s \in H^0(\mathbb{C}P^n, \mathcal{O}(d))$, then $\mathcal{K}_X \cong \mathcal{O}(d-n-1)|_X$.*

Proof. By Euler sequence, we know $\mathcal{K}_{\mathbb{C}P^n} \cong \mathcal{O}(-n-1)$. So it follows from Proposition 3.4.6. □

In particular, if C is a smooth plane curve of degree d, then $\mathcal{K}_C = \mathcal{O}(d-3)|_C$. Hence the Euler characteristic is $\chi(C) = -c_1(\mathcal{K}_C) = (3-d)d$. So the genus $g = \frac{(d-1)(d-2)}{2}$.

More generally, when M is a complex surface and C is a smooth complex curve in it, pairing the above adjunction formula with the class $[C]$, we have

$$K_M \cdot [C] + [C] \cdot [C] = K_C = -c_1(TC) = 2g(C) - 2.$$

As we have shown in Example 3.1.3, for the curvature of the tangent bundle of a Riemann surface $i\Theta = K\omega$ where K is the Gauss curvature. It then follows from Gauss-Bonnet theorem that $c_1(TC) = \int_C \frac{i}{2\pi}\Theta = \frac{1}{2\pi}\int_C K dA = \chi(C) = 2 - 2g$.

3.5 Lefschetz hyperplane theorem

It also follows from Proposition 3.4.6 that a smooth irreducible representative of the anti-canonical divisor in a smooth Fano variety (*i.e.* a complex manifold whose anticanonical bundle $\mathcal{K}^*$ is ample) of complex dimension 3 is a simply connected Calabi-Yau manifold (*i.e.* whose canonical bundle is trivial). The canonical bundle is trivial follows from Proposition 3.4.6 and it is simply connected follows from Lefschetz hyperplane theorem and Theorem 3.8.1.

Theorem 3.5.1 (Lefschetz Hyperplane Theorem)**.** *Let M be an n-dimensional compact, complex manifold and $V \subset M$ a smooth hypersurface with $L = L_V$ positive. Then the natural map*

$$\pi_k(V) \to \pi_k(M)$$

is an isomorphism for $k < n-1$ and is surjective for $k = n-1$.

In fact, by Kodaira embedding theorem 3.6.1, such an M is projective.

We apply Morse theory, or more precisely the Morse-Bott theory. A Morse-Bott function is a smooth function ϕ on a manifold whose critical set is a closed submanifold and whose Hessian $(\frac{\partial^2\phi}{\partial x_i \partial x_j})$ is non-degenerate in the normal direction. Equivalently, the kernel of the Hessian at a critical point equals the tangent space to the critical submanifold. When the critical manifolds are zero-dimensional (so the Hessian at critical points is non-degenerate), it is the classical Morse function. The dimension of a a maximal subspace of T_xM at a critical point x on which the Hessian is negative definite is called the index of the critical point.

Let ϕ be a Morse-Bott function on a compact manifold M. Then the homotopy type of $M_t = \{x \in M | \phi(x) \leq t\}$ remains the same as long as t does not cross a critical value, and changes by attaching a cell of dimension k when we cross a critical value whose index is k.

So if M_* is the set on which ϕ takes on its absolute minimum. then $M = M_* \cup e_1 \cup e_2 \cup \cdots \cup e_r$, $\dim e_i \geq \min \operatorname{ind} \phi$. Hence, to deduce Lefschetz hyperplane theorem, it will be sufficient to construct a Morse-Bott function on M with $M_* = V$ and $\min \operatorname{ind} \phi \geq \dim_{\mathbb{C}} M$.

Proof. Let s be a holomorphic section with $(s) = V$. We can cover V by finitely many open sets U_i of M such that in these open sets, V is $\{z_1 = 0\}$ and $s = z_1 \cdot s_i$ where $s_i \neq 0$ on U_i. We choose $\phi(x) = |s|^2$ as our first candidate of Morse-Bott function. First, ϕ is Morse-Bott along V. For any $p \in V \cap U_i$, we have $|s_i|^2 \neq 0$. Clearly, V is in its critical locus. For the normal direction, the coordinates are x_1 and x_2 where $z_1 = x_1 + ix_2$. And the Hessian in the normal direction is $H_p(\partial_{x_\alpha}, \partial_{x_\beta}) = 2|s_i|^2\delta_{\alpha\beta}$ for $\{\alpha, \beta\} \subset \{1, 2\}$, which is non-degenerate.

To show for any critical point of ϕ on $M \setminus V$, the index is no less than n, we look at $f(x) = \log |s|^2$. We are amount to show that the index of the Hessian of ϕ, which is equal to that of f, is at least n. The corresponding Hermitian extension $\tilde{H}$ to complexification of T_xM (Levi form) has matrix

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log |s|^2.$$

This is negative definite, since the curvature is positive means $\frac{i}{2\pi}\partial\bar{\partial} \log |s|^{-2}$ is a positive form (thus $g(u, v) = \frac{i}{2\pi}\Theta(u, Jv)$ defines an Hermitian metric; or more explicitly when we write $\frac{i}{2\pi}\Theta = \frac{i}{2\pi}\sum h_{ij} dz_i \wedge d\bar{z}_j$, the matrix h_{ij} is positive definite). Then the Hessian cannot be positive definite on a complex line, since on a complex line with $z = x + iy$, $\tilde{H} = \frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \operatorname{tr} H$. Hence the index of H is no less than $n = \dim_{\mathbb{C}} M$.

Finally, we need a perturbation of ϕ such that we will have the nondegeneracy on $M \setminus V$. It is easy to see that after small perturbations, the Hessian will have the same index if the manifold is compact. □

This theorem could be stated also for (co)homology.

Theorem 3.5.2. *Under the above condition,*

- *The natural map* $H_k(V,\mathbb{Z}) \to H_k(X,\mathbb{Z})$ *is an isomorphism for* $k < n-1$ *and is surjective for* $k = n-1$.
- *The natural map* $H^k(X,\mathbb{Z}) \to H^k(V,\mathbb{Z})$ *is an isomorphism for* $k < n-1$ *and is injective for* $k = n-1$.

Exercise: State and prove the corresponding version of Lefschetz hyperplane theorems for complete intersections.

It also has a dual form which applies to any complex affine variety. The argument, which is similar to the above one, also applies to Stein manifolds, where is equivalent to admitting smooth real function ψ with $i\partial\bar{\partial}\psi > 0$, such that $\{\psi(z) < c\}$ is compact for any real number c. We can similarly apply Morse theory to this function ψ, and since $(\frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j})$ is positive definite, we know the Hessian of it would have at least n positive eigenvalues with multiplicities. Hence the index at any critical point of ψ (possibly with small perturbations) would be at most n.

Theorem 3.5.3. *If* S *is a Stein manifold of (complex) dimension* n*, then* S *has the homotopy type of a CW complex of real dimension* $\leq n$*. Therefore,*

- $\pi_i(S) = 0$ *for* $i > n$.
- $H^i(S,\mathbb{Z}) = 0$ *for* $i > n$.
- $H_i(S,\mathbb{Z}) = 0$ *for* $i > n$.

By Kodaira embedding theorem, we can view $V = X \cap H$ (possibly with some positive multiplicity) in the statement of Theorem 3.5.1 where X is embedded in $\mathbb{C}P^N$ and H is a hyperplane section in $\mathbb{C}P^N$. That means, from L_V, we get a very ample line bundle $L_V^{\otimes k}$ with (V,k) as the zero locus of a holomorphic section. Hence $X - V$ is an affine variety and thus Stein. By applying the corresponding long exact sequences for pair (X,V), we know the statements of Lefschetz hyperplane theorem are equivalent to $H_k(X,V;\mathbb{Z}) = 0$ for $k < n$ *etc.* Theorem 3.5.2 then follows from the above theorem for Stein manifolds (in fact, we only need its special case for smooth affine varieties) by Lefschetz duality for (co)homology $H_k(X,V;\mathbb{Z}) = H^{2n-k}(X \setminus V;\mathbb{Z})$.

Grothendieck stated a version of Lefschetz hyperplane theorem for constructible sheaf. In particular, if X is a projective manifold and Y a smooth complete intersection subvariety of X, then $Pic(X) = Pic(Y)$ if $\dim_{\mathbb{C}} Y \geq 3$. The cohomology statement could also be replaced by Dolbeault cohomology groups.

If we merely assume the ambient manifold X is symplectic. *i.e.* a manifold with a closed, non-degenerate 2-form (but this form are not necessarily compatible with any complex structure), then it is an open question whether symplectic divisors (real codimension two symplectic submanifolds) whose homology class is Poincaré dual to the cohomology class of a symplectic form would satisfy Lefschetz hyperplane theorem. In particular, it is not known whether there are non-simply connected symplectic hypersurfaces in $\mathbb{C}P^n$ when $n > 2$. Whether the complement of it admits Stein structures?

3.6 Kodaira embedding theorem

Unlike the real manifolds, where Whitney theorem asserts that any n-manifolds embed to $\mathbb{R}^{2n+1}$ as submanifolds, not every complex manifold embeds in $\mathbb{C}P^N$ as an analytic subvariety. Chow's theorem asserts that any analytic subvariety in projective space is actually algebraic. First, such complex manifolds have to be Kähler since the pullback of the Fubini-Study form on projective space would give rise a Kähler form. Furthermore, this Kähler form has rational cohomology class. Reversely, this would guarantee a complex manifold is algebraic projective, which is asserted by Kodaira embedding theorem.

Theorem 3.6.1. *Let M be a compact complex manifold with complex dimension n. The following conditions are equivalent.*

1. *M is projective algebraic, i.e. M can be embedded as an algebraic submanifold of the complex projective space $\mathbb{C}P^N$ for N large.*
2. *M carries a positive line bundle L.*
3. *M caries a Hodge metric, i.e. a Kähler metric determined from a Kähler form ω with rational cohomology class.*

Actually, the Whitney type theorem holds for projective manifolds. In other words, N can be taken as $2n+1$. For this, we use the so-called generic projection. Assume $N > 2n+1$. Let $a \in \mathbb{C}P^N \backslash M$. The projection map

$$p_a : M \to Y \subset \mathbb{C}P^{N-1}$$

is an isomorphism unless either a lies on a secant of M or in the tangential variety of M. Meanwhile, the secant variety $Sec(M)$ has dimension no greater than $2n+1 < N$ and $\dim Tan(M) \leq 2n < N$. Hence we can always find a point $a \notin M$ for which the map p_a is an isomorphism. Continuing this process we get N could be chosen $2n+1$. This bound is clearly sharp when $n = 1$ as plane curves are very rare and has strong restrictions on the genus. But is $2n+1$ always a sharp bound? This is related to Castelnuovo bounds.

Exercise: A similar argument can be used to show that Stein manifolds of complex dimension n can be properly holomorphically embedded in $\mathbb{C}^{2n+1}$ (in fact, the optimal space is $\mathbb{C}^q$ for minimal integer $q > \frac{3n+1}{2}$).

In fact, the study of maps to projective spaces are very important. For example, (rational) maps to $\mathbb{C}P^1$ leads to the theory of Lefschetz fibrations (pencils). The original Lefschetz's proof of Theorem 3.5.1 uses Lefschetz pencil with complex Morse theory. Holomorphic maps onto $\mathbb{C}P^{\dim_{\mathbb{C}} M}$ is the theory of branched covering, namely every projective manifold M is a branched covering of $\mathbb{C}P^{\dim_{\mathbb{C}} M}$.

For the proof of Theorem 3.6.1, firstly, (2) $\Rightarrow$ (3) is easy since the curvature of this positive bundle gives a Hodge metric. Conversely (3) $\Rightarrow$ (2) follows from Proposition 3.4.3 on $N\omega$ where $N \in \mathbb{Z}$ and $N[\omega] \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{Z})$.

Now, let us prove (1) is equivalent to (2). It is easy to see that (1) $\Rightarrow$ (2): if $f : M \to \mathbb{C}P^N$ is an embedding, the pull back of the Fubini-Study metric on $f^*\mathcal{O}(1)$ is positively curved. The basic idea of (2) $\Rightarrow$ (1) is to choose a basis $s_0, \cdots, s_N$ of $H^0(M, L^k)$ for some integer $k > 0$ and define the map $f : M \dashrightarrow \mathbb{C}P^N = \mathbb{P}((H^0(M, L^k))^*)$ by $f(x) = [s_0(x), \cdots, s_N(x)]$. The dashed arrow means f may not be defined at some points, *i.e.* there might be x such that $s_0(x) = \cdots = s_N(x) = 0$. These are called base locus B. On $M \setminus B$, since $s_j(x)$ takes value in L_x, we have to choose an identification of L_x with $\mathbb{C}$. However, a different identification would differ only by a multiplication by $\lambda \in \mathbb{C}^* = GL(1,\mathbb{C})$. Then we want to show $B = \emptyset$ (base-point-free) and f is indeed an embedding, while it is apparently holomorphic on $M \setminus B$. When f is indeed embedding, such a line bundle L^k is called very ample and L is ample. One of the implication of Kodaira embedding is a holomorphic line bundle is positive if and only if it is ample.

Before jumping into the proof, I would like to point out that if we choose the positive bundle $\mathcal{O}(k)$ over $\mathbb{C}P^n$, then each of them are very ample and the corresponding embedding is the Veronese embedding. On the other hand, let S be the tautological (rank k) bundle over Grassmannian $Gr(k,n)$, and the line bundle $\Lambda^k S^*$ is very ample and the corresponding embedding is the Plücker embedding. Another example is the Segre map $\mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{(n+1)(m+1)-1}$ which is given by the very ample bundle $\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$.

We would show (2) $\Rightarrow$ (1) in the statement of Kodaira embedding in the following by Hörmander's L^2 technique. More precisely, we need to prove the following fact.

Theorem 3.6.2. *Let $x \in M$, L a positive bundle over M, and $V_x \subset H^0(M, L^k)$ the subspace of all sections vanishing at x. Then for all large k, V_x has codimension* 1. *Write $s_{k,x}$ for a generator of the L^2-orthogonal complement of V_x, with unit length in L^2. Then*

- $|s_{k,x}(x)|^2 = k^n + O(k^{n-1})$

- *for $y \neq x$, $|s_{k,x}(y)|$ decays faster than any polynomials, denoted $O(k^{-\infty})$.*

Once we have Theorem 3.6.2, we can show (2) $\Rightarrow$ (1). V_x has codimension one implies the base locus of L^k is empty, *i.e.* the map $M \to \mathbb{P}(H^0(M, L^k)^*)$ given by $x \mapsto [s_0(x), \cdots, s_N(x)]$ is well defined. To show the map is injective, namely that if x, y are distinct then there is a section which vanishes at x but not y, we argue it as follows. Since $s_{k,x}(x) \neq 0$, we can find $a \in \mathbb{C}$, $|a| << 1$, such that $as_{k,x} + s_{k,y}$ vanishes at x. But it can't vanish at y since its growth rate will be $k^n + O(k^{n-1})$.

We also need to show that the map has injective differential everywhere. This information could be translated to its blow-up $\pi : \tilde{M} \to M$. We need two facts.

The first is the positivity of $\pi^* L^k \otimes L_{-E}$ when k is large. Let $U \subset M$ be an open neighborhood of x and $\tilde{U} = \pi^{-1}(U)$. So if we use $z = (z_1, \cdots, z_n)$ for the local coordinates on U with center x, then $\tilde{U} = \{(z, w) \in U \times \mathbb{C}P^{n-1} | z_i w_j = z_j w_i\}$. Then over (z, w), the fiber of L_E consists of points on the line w.

Let h_1 be the metric on $L_E|_{\tilde{U}}$ given by $|\lambda(w_1, \cdots, w_n)|^2 = |\lambda|^2 ||w||^2$. Let σ be the canonical section $(z_1, \cdots, z_n)$ of L_E, *i.e.* $(z, w) \mapsto z$, with $(\sigma) = E$. Let $U_\epsilon \subset U_{2\epsilon}$ be neighborhoods of E which are pulled back from neighborhoods in M with distance ϵ and 2ϵ from x. Let h_2 be the metric on $L_E|_{\tilde{M}-E}$ given by $|\sigma(z)| = 1$. Let ρ_1, ρ_2 be the partition of unity for the cover $\{\tilde{U}_{2\epsilon}, \tilde{M} - \tilde{U}_\epsilon\}$ of $\tilde{M}$, and h the global metric $\rho_1 h_1 + \rho_2 h_2$. Let $\Omega_{L_E} = \frac{i}{2\pi} \Theta_{L_E}$. On $\tilde{M} - \tilde{U}_{2\epsilon}$, $\rho_2 = 1$ so $|\sigma|^2 = 1$ and $\Omega_{L_{-E}} = 0$. On $\tilde{U}_\epsilon - E = U_\epsilon - \{x\}$, $\Omega_{L_E} = -\frac{i}{2\pi} \partial\bar{\partial} \log ||z||^2$ which is just the pullback $-pr^* \omega_{FS}$ of the Fubini-Study metric on $\mathbb{C}P^{n-1}$ under the second projection $pr(z, w) = w$. Moreover, by continuity, $-\Omega_{L_E} = pr^* \omega_{FS} \geq 0$ throughout $\tilde{U}_\epsilon$. In particular, $-\Omega_{L_E}|_E = \omega_{FS} > 0$ on E. Hence the curvature form $\Omega_{L_{-E}}$ of the dual metric on L_E^* satisfies

$$\Omega_{L_{-E}} = \begin{cases} 0 & \text{on } \tilde{M} - \tilde{U}_{2\epsilon}, \\ \geq 0 & \text{on } \tilde{U}_\epsilon, \\ > 0 & \text{on } T_x E \subset T_x \tilde{M} \text{ for all } x \in E. \end{cases}$$

On the other hand, we can choose a metric h_L on $L \to M$ such that Ω_L is a positive form. Then for the induced metric on $\pi^* L$, we have

$$\Omega_{\pi^* L} = \pi^* \Omega_L = \begin{cases} \geq 0 & \text{everywhere}, \\ > 0 & \text{on } \tilde{M} - E, \\ > 0 & \text{on } T_x \tilde{M} / T_x E \text{ for all } x \in E. \end{cases}$$

Hence $\Omega_{\pi^* L^k \otimes L_{-E}} = k \Omega_{\pi^* L} + \Omega_{L_{-E}}$ is positive in $\tilde{U}_\epsilon$ and $\tilde{M} - \tilde{U}_{2\epsilon}$. Since $\Omega_{L_{-E}}$ is bounded below in $\tilde{U}_{2\epsilon} - \tilde{U}_\epsilon$ and $\Omega_{\pi^* L}$ is strictly positive there, we see that $\Omega_{\pi^* L^k \otimes L_{-E}}$ is everywhere positive for k large.

In particular, the above argument implies Kählerness is preserved under blowup along points. It is easy to see that a similar construction would lead to general blowups along smooth centers. Hence, we have the following result which is promised in previous lectures.

Theorem 3.6.3. *Kählerness is preserved under blowing up along complex submanifolds.*

Second fact is $\mathcal{K}_{\tilde{M}} = \pi^*\mathcal{K}_M \otimes L_E^{n-1}$. As $M - x \cong \tilde{M} - E$, and $\mathcal{K}_{\tilde{M}}|_{\tilde{M}-E} = \pi^*\mathcal{K}_M|_{M-x}$, the statement is reduced to a local statement on a neighborhood U of $x \in M$ and its preimage $\tilde{U}$ in $\tilde{M}$ containing E. We can cover $\tilde{U} = \{(z,l) \in U \times \mathbb{C}P^{n-1} | z_i l_j = z_j l_i\}$ by $\tilde{U}_i = \{l_i \neq 0\}$. And E is given by $z_i = 0$ in U_i. The local coordinates are $z(i)_j$ which is $\frac{l_j}{l_i} = \frac{z_j}{z_i}$ when $i \neq j$ and $z(i)_i = z_i$. So locally, we choose a meromorphic n-form $\omega = \frac{f(z)}{g(z)} dz_1 \wedge \cdots \wedge dz_n$ on M. Under

$$\pi|_{\tilde{U}_i} : (z(i)_1, \cdots, z(i)_n) \to (z(i)_1 z_i, \cdots, z_i, \cdots, z(i)_n z_i),$$

we see that

$$\pi^*\omega = \pi^*(\frac{f}{g}) z_i^{n-1} dz(i)_1 \wedge \cdots \wedge dz(i)_n.$$

This implies the relation. We could also see the coefficient $(n-1)$ from the adjunction formula for $E \subset \tilde{M}$: $\mathcal{K}_{\tilde{M}}|_E = \mathcal{K}_E \otimes L_{-E}|_E = L_E^{n-1}|_E$.

Then we decompose $\pi^*L^k \otimes L_{-E} \otimes \mathcal{K}_{\tilde{M}}^{-1}$ as $\pi^*(L^{k_1} \otimes \mathcal{K}_M^{-1}) \otimes (\pi^*L^{k_2} \otimes L_{-nE})$. By the argument for Theorem 3.6.2, the first term is semi-positive, and the second term is positive by the above first fact. Hence the curvature of the whole is positive for large k. That's what we need to prove Theorem 3.6.2 for $\tilde{M}$ with the bundle $\pi^*L^k \otimes L_{-E}$. Especially, when $t \in E$, we get a peak section $\tilde{s}_{k,t}$. View it as a section of π^*L^k, or multiply $\tilde{s}_{k,t}$ with the standard section σ of L_E and restrict it to $\tilde{M} - E = M - \{x\}$, and then extend it by Hartogs lemma, we have section $s_{k,x,t}$ of L^k over M with $s_{k,x,t}(x) = 0$ and "peak direction" t, *i.e.* the differential $ds_{k,x,t}$ viewed as a function on projective spaces $\mathbb{P}(T_xM)$ parametrizing the tangent directions is peaked at point t in the sense of Theorem 3.6.2.

We will focus on the proof of Theorem 3.6.2 now. We begin by considering the Euclidean case. Take L be the trivial bundle over $\mathbb{C}^n$ with the metric $h(z) = e^{-\pi|z|^2}$ and consider x is the origin. This has the curvature $\Theta_h = \pi\partial\bar{\partial}|z|^2 = \pi \sum dz_j \wedge d\bar{z}_j$. The corresponding real $(1,1)$-form is $\omega = \frac{i}{2\pi}\Theta_h$ corresponds to the flat metric on $\mathbb{C}^n$. The line bundle L^k is still trivial but with metric $h^k = e^{-k\pi|z|^2}$. Among the constant sections c of L_k, we would like to find one with unit L^2-norm. Notice $\int_{\mathbb{C}^n} k^n e^{-k\pi|z|^2} dx_1 \cdots dx_{2n} = 1$, we find that the constant section $s_k(z) = k^{\frac{n}{2}}$ satisfies the both bullets of Theorem 3.6.2 since the Gaussian distributions converge to a Dirac delta centre at the origin. Our construction later will be a modification of this local model.

For general L over M and $x \in M$, we choose a small ball $x \in B$ over which L is trivial. And we fixed a Hermitian metric h such that $\omega = \frac{i}{2\pi}\Theta_h$ is a positive form. We fix ω as the Kähler form on M. The geometry of $(M, L^k, h^k, k\omega)$ approaches the flat model above as k grows. The precise meaning of it is: in geodesic balls of radius $k^{-\frac{1}{4}}$ in terms of metric ω (or equivalently ball of radius $\frac{1}{\sqrt{k}}$ in terms metric $k\omega$), the metric ω approaches to flat metric as $k \to \infty$. We use a cut-off function $\psi(x)$ in $\mathbb{C}^n$, for example one with derivatives supported in the annulus of radius between $\frac{1}{4}k^{-\frac{1}{4}}$ and $\frac{1}{2}k^{-\frac{1}{4}}$, to multiply the local section $s_{k,x}$ from the above paragraph. The resulting section $\tilde{s}_{k,x}$ of L^k is no longer holomorphic. $\bar\partial \tilde{s}_{k,x}$ is supported in the annulus and it is small in L^2, as $O(k^{-\infty})$ (a polynomial of k times $e^{-\pi k^{\frac{1}{2}}}$). Now we want to solve

$$\bar\partial f_k = -\bar\partial \tilde{s}_{k,x}$$

and set $s'_{k,x} = \tilde{s}_{k,x} + f_k$. We want f_k as small as possible. Here we apply the Hörmander's technique.

We need to introduce the Bochner-Kodaira-Nakano identity. The setting is a holomorphic Hermitian vector bundle (E, h_E) over a Kähler manifold (Here the Kähler form is specified as $\omega = \frac{i}{2\pi}\Theta_h$ as above for metric h on L). We recall some general facts for (E, h_E). The Chern connection ∇ splits as $\partial_E = \pi^{1,0} \circ \nabla$ and $\bar\partial_E = \pi^{0,1} \circ \nabla$. If Θ is the curvature of the Chern connection in E, then

$$\Delta_{\bar\partial_E} = \Delta_{\partial_E} + [i\Theta, \Lambda]. \tag{3.4}$$

The proof of it still relies on the Kähler identities. But now $(\partial_E + \bar\partial_E) \circ (\partial_E + \bar\partial_E)$ is no longer zero, it is the curvature Θ of the Chern connection. Since Θ is of type $(1,1)$, *e.g.* see Proposition 3.1.2, we have $\partial_E\bar\partial_E + \bar\partial_E\partial_E = \Theta$. Since the Kähler identities $[\Lambda, \bar\partial_E] = -i\partial_E^*$ and $[\Lambda, \partial_E] = i\bar\partial_E^*$ still hold, we have

$$i\Delta_{\partial_E} = -\partial_E(\Lambda\bar\partial_E - \bar\partial_E\Lambda) - (\Lambda\bar\partial_E - \bar\partial_E\Lambda)\partial_E, \quad i\Delta_{\bar\partial_E} = \bar\partial_E(\Lambda\partial_E - \partial_E\Lambda) + (\Lambda\partial_E - \partial_E\Lambda)\bar\partial_E.$$

Hence the identity (3.4) holds. There are more general version of this type of identities, where the base manifold is merely Hermitian. Then the identity will also involve the torsion term, see [**?**].

Later, I will simply write $\bar\partial$ for the corresponding vector bundle version. Now $\mathcal{K}^{-1} \otimes L^k$ has curvature $\Theta = -2\pi i(k\omega + Ric(M, \omega))$. On (p,q)-forms $[\omega, \Lambda] = p + q - n$ by Proposition 3.2.2. It follows that on (n,q)-forms with values in $\mathcal{K}^{-1} \otimes L^k$, or equivalently, on $(0,q)$-forms with values in L^k,

$$\Delta_{\bar\partial} = \Delta_{\partial} + 2\pi qk + 2\pi[Ric, \Lambda].$$

Hence there is a constant C such that for $f \in \Omega^{0,q}(M, L^k)$,

$$< \Delta_{\bar\partial} f, f >_{L^2} \geq (2\pi qk - C)||f||^2_{L^2}. \tag{3.5}$$

It implies the Kodaira vanishing.

Theorem 3.6.4. *Let L be a positive line bundle over M. There is a constant C such that for all $q > 0$ and all sufficiently large k, the lowest eigenvalue ν of $\Delta_{\bar\partial}$ acting on $\Omega^{0,q}(M, L^k)$ satisfies $\nu \geq 2\pi qk - C$. In particular $\Delta_{\bar\partial}$ is invertible for large k and hence $H^q(M, L^k) = 0$ for all $q > 0$.*

Moreover, the first non-zero eigenvalue μ of the operator $\Delta_{\bar\partial}$ acting on sections of L^k satisfies $\mu \geq 2\pi k - C$.

The last statement is because if $f \in \Omega^0(M, L^k)$ is a eigenvector with eigenvalue λ, then so is $\bar\partial f \in \Omega^{0,1}(M, L^k)$. We remark that the only facts we have used are the $L^k \otimes \mathcal{K}^{-1}$ is positive when k is large as well as $p+q > n$. So a general form of Kodaira-Nakano vanishing theorem says that if $L \to M$ is a positive line bundle, then $H^{p,q}_{\bar\partial}(M, L) = H^q(M, \Omega^p(L)) = 0$ for $p+q > n$. Same argument also applies to $L^k \otimes E$ where L is a positive bundle and E is an arbitrary holomorphic line bundle.

Theorem 3.6.5. *For all large k, given $g \in \Omega^{0,1}(M, L^k)$ with $\bar\partial g = 0$ then there is a section $f \in \Omega^0(M, L^k)$ such that $\bar\partial f = g$. Moreover, there is a constant C, independent of g, k such that the above solution satisfies $||f||_{L^2} \leq Ck^{-1}||g||_{L^2}$.*

Proof. Since $H^q(M, L^k) = 0$ for $q = 1$, we have $\bar\partial f = g$ for $f = G_{\bar\partial}(\bar\partial^* g) \in \Omega^0(M, L^k)$ by (vector bundle version of) Hodge decomposition. This f is orthogonal to harmonic (or equivalently holomorphic) sections of L^k with respect to L^2 norm, as $< f, h >_{L^2} = < \bar\partial^* g, \Delta_{\bar\partial} h >_{L^2} = 0$.

The estimate on $||f||_{L^2}$ is because of the eigenvalue estimate in the last part of Theorem 3.6.4:

$$(2\pi k - C)||f||_{L^2} \leq < \Delta_{\bar\partial} f, f >_{L^2} = < \bar\partial^* g, f >_{L^2} = ||g||_{L^2}.$$

Since $\Delta_{\bar\partial} f = \bar\partial^* g$, and g is C^∞, the ellipticity of $\Delta_{\bar\partial}$ shows f is a smooth section. □

Now return to the construction of section $s'_{k,x}$. First we have $||\bar\partial \tilde s_{k,x}||_{L^2} = O(k^{-\infty})$. Now, Theorem 3.6.5 provides us with a solution to $\bar\partial f_k = -\bar\partial \tilde s_{k,x}$ with $||f_k||_{L^2} \leq Ck^{-1} \cdot O(k^{-\infty})$. Since $||\tilde s_{k,x}||_{L^2} = 1$, the holomorphic section $s'_{k,x} = \tilde s_{k,x} + f_k$ is very close to the Euclidean model section in L^2. By applying the Gårding's inequality, we have $||f_k||_{l+2} \leq C(||f_k||_l + ||\bar\partial \tilde s_{k,x}||_0)$. By (weak) Sobolev inequality $||u||_{C^{l-[\frac{n}{2}]-1}} \leq C||u||_l$, this would show f_k is also small as $O(k^{-\infty})$ in C^l norms.

V_x is clearly a complex vector space. Since $s'_{k,x}$ is non-zero at x and any two such sections will have a linear combination in V_x. So V_x is a codimension 1 subspace in $H^0(M, L^k)$. Moreover, $s'_{k,x}$ is asymptotically L^2-orthogonal to V_x as $k \to \infty$, because it concentrates at x. We get $s_{k,x}$ by projecting $s'_{k,x}$ to $V_x^\perp$.

3.6.1 Proof of Newlander-Nirenberg theorem

The Hörmander technique could also be used to prove the Newlander-Nirenberg theorem. We follow the original presentation of Hörmander. We assume X is an integrable almost complex manifold. Let U be a coordinate patch in X where there is a C^∞ orthonormal basis $\omega^1, \cdots, \omega^n$ for the forms of type $(1,0)$. We write $\partial\bar{\partial}w = \sum w_{jk}\omega^j \wedge \bar{\omega}^k$. Then, firstly, we need a similar estimate as Equation (3.5). We use $||\cdot||_\phi$ to denote the weighted L^2 norm with weight $e^{-\phi}$.

Theorem 3.6.6. *There exists a continuous function C on X such that*

$$\int (\lambda - C)|f|^2 e^{-\phi} dV \leq 4 < \Delta_{\bar{\partial}} f, f >_\phi .$$

Here ϕ is an arbitrary function in $C^2(X)$, f is a $(p, q+1)$-form and λ is the lowest eigenvalue of the Hermitian symmetric form $\sum \phi_{jk} t_j \bar{t}_k$.

The function ϕ is plurisubharmonic if the Hermitian form is positive definite, *i.e.* > 0 when $t \neq 0$. For almost complex structures, it still makes sense. And the above theorem still holds if we have $\partial\bar{\partial} = -\bar{\partial}\partial$ which is true for integrable almost complex structures. We could think $e^{-\phi}$ gives a Hermitian metric on the trivial bundle $X \times \mathbb{C}$. When ϕ is plurisubharmonic, the curvature is positive. With this understood, we will need the key statement corresponding to Theorem 3.6.5.

Theorem 3.6.7. *Let X be an integrable almost complex manifold where there exists a strictly pluriharmonic function ϕ such that $\{z; z \in X, \phi(z) < c\} \subset\subset X$ for every $c \in \mathbb{R}$. Then the equation $\bar{\partial}u = f$ has a solution $u \in L^2_{p,q}(X, loc)$ for every $f \in L^2_{p,q+1}(X, loc)$ such that $\bar{\partial}f = 0$.*

Especially, this holds for Stein manifold. And it gives another proof of $H^r(X, \mathbb{C}) = 0$ for $r > \dim X$ for Stein manifolds. For the proof, we need to replace the function ϕ by $\chi(\phi)$ where χ is a convex increasing function such that $\chi'(\phi)\lambda - C \geq 4$. Then we will have $||u||_{\chi(\phi)} \leq ||f||_{\chi(\phi)}$.

Then return to our discussion. Let $\psi(x) = |x|^2$, we have the Hermitian form is positive definite at 0. Hence we could choose $\delta > 0$ such that the Hermitian form is uniformly positive definite in the ball $B_{0,\delta} = \{x; |x| < \delta\}$. So we could let $\phi = \frac{1}{\delta^2 - \psi}$. Then for every f of type $(0,1)$ in $B_{0,\delta}$ with $\bar{\partial}f = 0$, we will obtain a solution of $\bar{\partial}u = f$ and $||u||_\phi \leq ||f||_\phi$.

Let $u^1, \cdots, u^n$ be linear functions with $du^j = \omega^j$ at 0. Let $\pi_\epsilon(x) = \epsilon x$ and consider the almost complex structure defined by $\pi_\epsilon^*\omega^1, \cdots, \pi_\epsilon^*\omega^n$ in $B_{0,\delta}$. Since $du^j - \frac{1}{\epsilon}\pi_\epsilon^*\omega^j$ together with all its derivatives are $O(\epsilon)$, we conclude that $D^\alpha \bar{\partial}_\epsilon u^j = O(\epsilon)$ for all multivector α. It follows that we can find v_ϵ^j so that $\bar{\partial}_\epsilon v_\epsilon^j = \bar{\partial}_\epsilon u^j$ and $||v_\epsilon^j||_\phi = O(\epsilon)$. By Sobolev lemma, all derivatives of v_ϵ^j at 0 are $O(\epsilon)$. Hence $U^j = u^j - v_\epsilon^j$ are linearly independent at 0 if ϵ is sufficiently small. We know $\bar{\partial}U^j(\frac{x}{\epsilon}) = 0$ since $\bar{\partial}_\epsilon U^j = 0$. Hence U^j are our analytic coordinates.

3.7 Kodaira dimension and classification

Kodaira dimension provides a very successful classification scheme for complex manifolds depending on the positivity of the canonical bundle. Its first definition is to measure the growth rate of the sections of the pluricanonical bundle.

Definition 3.7.1. *Suppose (M, J) is a complex m-fold. The holomorphic Kodaira dimension $\kappa^h(M, J)$ is defined as follows:*

$$\kappa^h(M,J) = \begin{cases} -\infty & \text{if } P_l(M,J) = 0 \text{ for all } l \geq 1, \\ 0 & \text{if } P_l(M,J) \in \{0,1\}, \text{ but } \not\equiv 0 \text{ for all } l \geq 1, \\ k & \text{if } P_l(M,J) \sim cl^k;\ c > 0. \end{cases}$$

Here $P_l(M, J)$ is the l-th plurigenus of the complex manifold (M, J) defined by $P_l(M, J) = h^0(\mathcal{K}_J^{\otimes l})$, with $\mathcal{K}_J$ the canonical bundle of (M, J).

This is a birational invariant, *i.e.* it is invariant under blow-ups and blow-downs, which could be checked by Hartogs lemma.

The second, equivalent, definition is by the mth pluricanonical map $\Phi_m(x) = [s_0(x) : \cdots : s_N(x)]$, where s_i are a basis of $H^0(X, \mathcal{K}^{\otimes m})$. Even if $\mathcal{K}$ is not ample, $\Phi_m(x)$ is a holomorphic map and its image is a projective subvariety of $\mathbb{C}P^N$. Hence, the Kodaira dimension $\kappa_J(X)$ of (X, J) is defined as:

$$\kappa^h(X) = \begin{cases} -\infty, & \text{if } P_m = 0 \text{ for any } m \geq 0 \\ \max\dim\Phi_m, & \text{otherwise.} \end{cases}$$

The both definitions could be stated in almost complex setting, although it is not known whether they are equivalent as in the integrable case.

The abundance conjecture suggests the Kodaira dimension could be calculated in terms of the degeneracy of the powers of canonical class $K_M = c_1(\mathcal{K}_J)$ at least when K_M is nef. This suggests the following symplectic Kodaira dimension.

Definition 3.7.2. *For a minimal symplectic 4-manifold (M^4, ω) with symplectic canonical class K_ω, the Kodaira dimension of (M^4, ω) is defined in the following way:*

$$\kappa^s(M^4,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } & K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega > 0. \end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

Here K_ω is defined as the first Chern class of the cotangent bundle for any almost complex structure compatible with ω.

3.7.1 Complex dimension one

A one-dimensional complex manifold is called a *Riemann surface.*

Actually, there is no difference between almost complex structures and complex structures in complex dimension one. In other words, every almost complex structure is integrable. It automatically follows from the Newlander-Nirenberg theorem since the vanishing of this tensor is automatic in complex dimension one. However, this essentially follows from the existence of Isothermal coordinates.

Proof. Let S be a 2-dimensional almost complex manifold, we want to show that for any point $x \in S$, we can choose an open neighbourhood U and an open embedding $U \to \mathbb{C}$ of almost complex structures.

In complex dimension one, the almost complex structure tells us how to rotate 90 degrees counterclockwise. Hence it is given by an orientation along with a conformal structure, which is given a Riemannian metric say g. From the viewpoint of structure group, the structure group of S is $GL(1,\mathbb{C}) = \mathbb{C}^*$. While a conformal structure gives the reduction of the structure group $GL(2,\mathbb{R})$ to $\mathbb{R}_{>0} \times O(2)$. An orientation further reduces it to $\mathbb{R}_{>0} \times SO(2)$ which is C^*.

Now, our assertion is equivalent to choosing an open neighbourhood of x such that g is conformally flat, *i.e.* it has the form λg_0 where g_0 is the standard metric on $\mathbb{R}^2 = \mathbb{C}$. The latter is the statement of the existence of Isothermal coordinate for surfaces. $\square$

Let us calculate the notions mentioned before in this particular case.

If $\mathcal{L}$ is a holomorphic line bundle over the Riemann surface S, the Riemann-Roch theorem states that

$$h^0(S,\mathcal{L}) - h^0(S,\mathcal{K}\otimes\mathcal{L}^{-1}) = \deg(\mathcal{L}) + 1 - g(S),$$

where $h^0(S,\mathcal{L}) = \dim H^0(S,\mathcal{L})$ and $g(S)$ is the genus of S.

Recall that the Kodaira vanishing theorem applies to Riemann surfaces, stating that $H^0(S,\mathcal{L}) = 0$ if $\mathcal{L}^*$ is positive. It can be seen in two ways. First, we have Serre duality, $H^i(X,\mathcal{L}) \cong H^{n-i}(X,\mathcal{K}_X \otimes L^*)^*$. Moreover, a particular case of Kodaira-Nakano vanishing implies $H^i(X,\mathcal{K}\otimes\mathcal{L}^*) = 0$ for $i > 0$ and $\mathcal{L}^*$ positive. These two facts together apply to Riemann surface S where $n = 1$, we have the result.

For the second way, by virtue of Proposition 3.1.5, a line bundle $\mathcal{L}^*$ on S is positive if and only if $c_1(\mathcal{L}) = \deg\mathcal{L} < 0$. But if there is an $s \neq 0 \in H^0(M,\mathcal{L})$, then $\mathcal{L}$ is the line bundle associated to the effective divisor $D = (s)$, and we have $c_1(\mathcal{L}) \geq 0$, a contradiction.

When $\mathcal{L} = \mathcal{K}_J$ is the canonical bundle, we have $\deg(\mathcal{K}_J) = c_1(\mathcal{K}_J) = -\chi(M) = 2g-2$. In particular, when $S = S^2$, we have $g = 0$ and $P_l(M, J) = 0$ for all l, and hence $\kappa^h(S^2, J) = -\infty$. When $S = T^2$, the canonical bundle is trivial since there is a nowhere vanishing $(0, 1)$ form dz. Hence $P_l(T^2, J) \equiv 1$ and $\kappa^h(T^2, J) = 0$. Finally, when $S = \Sigma_g$ with $g \geq 2$, we know $\mathcal{K}_J$ is positive and thus $H^0(\Sigma_g, \mathcal{K}_J^{\otimes m}) = 0$ when $m < 0$. Applying Riemann-Roch theorem, we have

$$h^0(\mathcal{K}_J^{\otimes l}) = h^0(\mathcal{K}_J^{\otimes l}) - h^0(\mathcal{K}_J^{\otimes(1-l)}) = l \deg(\mathcal{K}_J) - g + 1 = (2l-1)(g-1)$$

when $l > 1$. Hence its Kodaira dimension is 1. In particular, the Kodaira dimension only depends on the topology, *i.e.* the genus, of the surface S.

3.7.2 Complex Surfaces

The classification of complex surfaces with respect to the Kodaira dimension is also very effective. Let us start with a rough list of classification.

When $\kappa^h = -\infty$, a complex surface is rational (birational to $\mathbb{C}P^2$ with the unique complex structure) or ruled (birational to $S^2 \times \Sigma_g$ with any product complex structure) if it is Kähler (and hence projective since $p_g = 0$); or is of Class VII which is the most mysterious class of complex surfaces.

When $\kappa^h = 0$, a complex surface is K3, Enriques surface, hyperelliptic or abelian when Kähler. The canonical bundle is torsion. However, the first Chern class is zero. There are non-Kähler ones which are called second Kodaira surfaces or Kodaira-Thurston manifolds.

All complex surfaces with $\kappa^h = 1$ are elliptic surfaces. That is, they admit elliptic fibrations.

The surfaces with $\kappa^h = 2$ are called surfaces of general type. These are actually very wild class of surfaces without a reasonable classification. There are several important properties. First, all surfaces of general type are Kähler. Moreover, the complex structures could be deformed to projective surfaces. Second, there are a couple of important general inequalities which are especially important in the geography problem. The first is called the Bogomolov-Miyaoka-Yau inequality, $c_1^2 \leq 3c_2$. The equality holds if and only if the surface is a complex ball quotient. The second is called the Noether's inequality for minimal surface: $p_g = h^{0,2} \leq \frac{1}{2}c_1^2 + 2$. Surfaces where the equality holds are called Horikawa surfaces. We can rewrite the Noether's inequality as $b^+ \leq 2c_2 + 3\sigma + 5$ as $c_1^2 = 2c_2 + 3\sigma$ and $b^+ = 1 + 2p_g$. This is equivalent to $b^- + 4b_1 \leq 4b^+ + 9$. Combining with Noether formula $12\chi_h = 6(1 - b_1 + b^+) = c_1^2 + c_2$, we have $5c_1^2 - c_2 + 36 \geq 6b_1 \geq 0$. These two Chern numbers inequalities give the usual picture of geography of complex surfaces with x-axis c_2 and y-axis c_1^2.

In dimension 4, it is a gauge theory result that if X_1 and X_2 are two diffeomorphic complex surfaces, then $\kappa^h(X_1) = \kappa^h(X_2)$. For higher dimensions, this is no longer true. Let us take $M = \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$ and N is the

Barlow surface which is a complex surface of general type homeomorphic to M. Then $M \times \Sigma_g$ is diffeomorphic to $N \times \Sigma_g$. This is because that, at first, they are $h-$cobordant because M and N are so. Second, they are $s-$cobordant because the Whitehead group $Wh(M \times \Sigma_g) = Wh(N \times \Sigma_g) = Wh(\Sigma_g) = 0$. Then by the $s-$cobordism theorem proved independently by Mazur, Stallings, and Barden, they are diffeomorphic to each other. On the other hand, they are not the same as complex manifolds since they have distinct Kodaira dimensions. The Kodaira dimension $\kappa^h(M \times \Sigma_g) = -\infty$ as $\kappa^h(M) = -\infty$. However, $\kappa^h(N \times \Sigma_g) = 2$ when $g = 1$ and $\kappa^h(N \times \Sigma_g) = 3$ when $g > 1$. In [**?**], there are various examples of diffeomorphic manifolds with different Kodaira dimensions constructed. It is worth noting that none of them is simply connected.

However, at least when X is a projective variety, it is known that the Kodaira dimension is invariant under the deformation of complex structures which follows from Siu's invariance of plurigenera.

3.7.3 BMY line

We say a bit more on the equality $c_1^2 = 3c_2$. Hirzebruch first observes that for any ball quotient of complex dimension two, the equality holds by his proportionality. Then, Yau shows that $c_1^2 = 3c_2$ if and only if it is a ball quotient (and the canonical bundle is ample) as a corollary of his solution of Calabi's conjecture. We briefly explain Hirzebruch's proportionality.

In differential geometry, a symmetric space is a Riemannian manifold in which around every point there is an isometry reversing the direction of every geodesic. It turns out that symmetric spaces are constructed mainly as homogeneous spaces of Lie groups. For example, let G be a non compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let θ be an involution of G such that the fixed subgroup K of θ is compact. We can decompose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as ± 1 eigenspaces of θ on $\mathfrak{g}$. It is clear that $\mathfrak{g}_c = \mathfrak{k} + i\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ whose complexification is $\mathfrak{g}_{\mathbb{C}}$ as well. We assume $\mathfrak{g}_c$ is the Lie algebra of a compact Lie group G_c. The homogeneous space $X_c = G_c/K$ is also a symmetric space, which is called the dual of $D = G/K$. The key example in our mind is $G = SU(n,1)$, $K = S(U(n) \times U(1))$, the involution is conjugation by $\mathrm{diag}(I_n, 1)$, and $G_c = SU(n+1)$. So G/K is a complex ball of dimension n, and $X_c = \mathbb{C}P^n$. We further choose a discrete torsion-free cocompact group Γ of automorphisms of D, and let $X = D/\Gamma$.

Hirzebruch's main result is the Chern numbers of X are proportional to the Chern numbers of $X_c = G_c/K$ (the constant of proportionality being the volume of X. Apply to our example, we have $\frac{c_2(X)}{c_1^2(X)} = \frac{c_2(\mathbb{C}P^2)}{c_1^2(\mathbb{C}P^2)} = \frac{1}{3}$.

For the proof of Hirzebruch proportionality, we use Chern curvature form to compute Chern numbers. The key observation is the Chern form is invariant under the both group actions. Therefore it is sufficient to compute it at one point. We choose the distinguished points $(1 : 0 : \cdots : 0)$ on $\mathbb{C}P^n$

and 0 on ball. By our previous computation, the standard ($SU(n+1)$ and $SU(1,n)$ invariant) Hermitian metrics on them ($||\sigma(z)||^2 = 1+|z|^2$ on U_0 of $\mathbb{C}P^n$ and $||\sigma(z)||^2 = 1-|z|^2$ on B) have the opposite curvature form at these two points, which are $\pm$ the standard Kähler form. Hence, the ratio is $(-1)^{\dim_{\mathbb{C}} X}$ times the ratio of the volumes.

3.8 Hirzebruch-Riemann-Roch Theorem

We keep our notation that $H^i(X,E) := H^{0,i}(X,E)$ is the Dolbeault cohomology of X with values in E, *i.e.* the cohomology of the complex $(\Omega^{0,\cdot}(X,E), \bar\partial_E)$. By Hodge theory, the cohomology $H^i(X,E)$ can be represented by E-valued harmonic $(0,i)$-forms for any Hermitian metric on X. We define $\chi(X,E) = \sum_{i=0}^{\dim_{\mathbb{C}} X}(-1)^i \dim_{\mathbb{C}} H^i(X,E)$. The Hirzebruch-Riemann-Roch Theorem applies to any holomorphic vector bundle E on a compact complex manifold X. It states that $\chi(X,E)$ is computable in terms of a polynomial on Chern classes of X and E. In a rougher form, there is a polynomial $P(x_1,\cdots,x_n,y_1,\cdots,y_n)$ depending only on n such that P is homogeneous of degree $2n$ if we set $\deg x_i = \deg y_i = 2i$, and

$$\chi(X,E) = \int_X P(c_1(X),\cdots,c_n(X),c_1(E),\cdots,c_n(E)).$$

Here, when $i > rankE$, we set $c_i(E) = 0$.

For a precise form, we have the polynomial P is $\sum ch_{n-j}(E)td_j(X)$, or equivalently, $\chi(X,E) = \int_X ch(E)td(X)$, where T_j is the Todd polynomial and ch is the Chern character. The Chern characters could also be defined using (Chern) curvature form $\Omega = \frac{i}{2\pi}\Theta_h$. Then $ch(E,h) := \sum_j \frac{1}{j!}tr(\Omega\wedge\cdots\wedge\Omega)$ and $ch(E)$ denotes its cohomology class. It could also be understood as the the homogeneous polynomials generated by expansion of $tr(e^A)$ for matrix A. Similarly, The Todd classes are generated by the homogeneous polynomials determined by $\frac{\det(tA)}{\det(I-e^{-tA})} = \sum_k P_k(A)t^k$. And $td_k(E) = [P_k(\Omega)]$. $td(X) = td(TX) = td_0(TX)+\cdots$. They have the following formal power series in terms of the Chern classes

$$ch(E) = rk(E) + c_1(E) + \frac{c_1^2 - 2c_2}{2} + \frac{c_1^3 - 3c_1c_2 + 3c_3}{6} + \cdots,$$

$$td(E) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1c_2}{24} + \frac{-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4}{720} + \cdots.$$

When $\pi : \tilde X \to X$ is a finite covering, we have $\pi^*(c_i(X)) = c_i(\tilde X)$, hence $\chi(X,E)$ is multiplicative, thus $\chi(\tilde X, \pi^*E) = \deg(\tilde X/X)\chi(X,E)$.

Exercise: Check the Hirzebruch-Riemann-Roch Theorem for vector bundles over curves is the original Riemann-Roch.

When M is a complex surface, applying HRR to trivial bundle, we have the holomorphic Euler number $\chi_h = \chi(\mathcal{O}_M) = \frac{c_1^2(M)+c_2(M)}{12}$. This is called Noether's formula. By definition of χ_h, we know $\chi_h = \frac{1-b_1+b^+}{2}$ for a Kähler surface.

Exercise: Derive Hirzebruch signature formula $c_1^2 = 3\sigma + 2c_2$ for complex surfaces from Noether's formula.

Apply HRR to a line bundle, we have Riemann-Roch for surfaces

$$\chi(L) = \chi_h + \frac{c_1(L)\cdot c_1(L) - c_1(L)\cdot K_M}{2}.$$

In particular, this gives a clear look of Kodaira dimension by setting $L = \mathcal{K}^{\otimes m}$. For higher dimensions, we also have the asymptotic Riemann-Roch for a positive bundle L,

$$h^0(X, L^{\otimes m}) = \frac{c_1^n(L)}{n!}\cdot m^n + O(m^{n-1}).$$

There is another application in higher dimensions.

Theorem 3.8.1. *Any Fano manifold is simply-connected.*

Proof. Fans means the anti-canonical bundle $\mathcal{K}_M^{-1}$ is ample. By Kodaira vanishing, for any $q > 0$,

$$h^{0,q} = h^q(\mathcal{O}_M) = h^{n-q}(\mathcal{K}_M^{-1}) = 0.$$

So the holomorphic Euler number $\chi_h = \sum_{q=0}^n (-1)^q h^{0,q} = h^{0,0} = 1$.

On the other hand, Calabi-Yau theorem says that M admits a Kähler metric of positive Ricci curvature. So $\pi_1(M)$ is finite by Myers' theorem. The universal covering $\tilde{M}$, the $d = |\pi_1(M)|$-fold cover of M, is still a Fano manifold, and by Hirzebruch-Riemann-Roch with $E = \mathcal{O}$, χ_h behaves multiplicatively under holomorphic covering. Hence $1 = \chi_h(\tilde{M}) = d\chi_h(M) = d$. That is M is simply connected. $\square$

More generally, we have an algebraic geometry proof given by Campana shows that a rationally connected manifold is simply connected. There also exist differential geometry proofs of this.

3.9 Kähler-Einstein metrics

A Riemannian metric g on a smooth manifold M is said to be Einstein if it has constant Ricci curvature, or in other words if

$$r = cg.$$

The existence of Kähler-Einstein metric on a complex manifold is a central problem in Kähler geometry. Since in this case, the Ricci curvature is proportional to the Kähler metric, or $Ric(\omega)$ is proportional to ω, the first Chern class is either negative, zero, or positive. We can reformulate it in equivalent ways. First, this is equivalent to solving the following Monge-Ampère equation of ϕ

$$\frac{(\omega - \frac{i}{2\pi}\partial\bar{\partial}\phi)^n}{\omega^n} = e^{c\phi+F}. \tag{3.6}$$

Here ϕ, F are determined by dd^c-lemma, and c is the constant such that $[Ric(\omega)] = c[\omega]$. We have

$$Ric(\omega) - c\omega = \frac{i}{2\pi}\partial\bar{\partial}F, \quad \omega = \omega' + \frac{i}{2\pi}\partial\bar{\partial}\phi.$$

Then ω' is Kähler-Einstein if and only if $Ric(\omega') - Ric(\omega) = c\omega' - Ric(\omega)$. The right hand side is $-\frac{i}{2\pi}(\partial\bar{\partial}(F + c\phi))$. Then it is equivalent to Equation (3.6) because $\partial\bar{\partial}h = 0$ if and only if h is a constant. And it is fixed because the integration of $e^{c\phi+F}$ is 1.

There is another geometrically useful formulation. We do it only for the Fano case, *i.e.* when the anti-canonical bundle $K^{-1} > 0$. Given a volume form Ω on (M, J) is equivalent to giving a Hermitian metric on K^{-1}. More precisely, let h be a Hermitian metric on K^{-1}, then

$$\Omega_h = |\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}|_h^2 (\frac{i}{2})^n dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n.$$

On the other hand, the metric on K^{-1} coupled with its Chern connection gives the Ricci curvature form ω_h. Kähler-Einstein metrics correspond to metrics h such that $\omega_h > 0$ and $\Omega_h = \omega_h^n$. This is because Yau's solution to the Calabi conjecture says there is a unique Kähler form ω, cohomologous to ω_h such that $\omega^n = \Omega$ is the prescribed volume form. Hence $\omega = \omega_h$ which is Fano Kähler-Einstein.

When the Chern class is negative, Aubin and Yau proved that there is always a KE metric. When the first Chern class is zero, Yau proved there are always Ricci flat Kähler metric (the $c_1 = 0$ case of Calabi conjecture), thus a KE metric. The positive Chern class case, *i.e.* the Fano case, is the most difficult case. The case of complex surfaces has been settled by Gang Tian. The complex surfaces with positive Chern class are either a product of two copies of a projective line (which obviously has a Kähler-Einstein metric) or a blowup of the projective plane in at most 8 points in "general position", in the sense that no 3 lie on a line and no 6 lie on a quadric. The projective plane has a Kähler-Einstein metric, and the projective plane blown up in 1 or 2 points does not, as the Lie algebra of holomorphic vector fields is not reductive. This actually follows from a more general theorem of Matsushima.

Theorem 3.9.1. *If X is a compact Kähler manifold of constant scalar curvature and $H^1(X) = 0$ then $Aut_0(X)$ (the component of identity in $Aut(X)$) is reductive, i.e. it is the complexification of a compact group.*

Translate to Lie algebra, it is reductive if the Lie algebra is isomorphic to the direct sum of its center and a semi-simple Lie algebra. Recall that a Lie algebra $\mathfrak{g}$ is semi-simple (*i.e.* it is a direct sum of simple Lie algebra) if and only if $\mathfrak{g}$ has no non-zero abelian ideals.

The Lie algebra of $Aut(\mathbb{C}P^2)$ is $sl(3, \mathbb{C})$, which is semi-simple. For $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, then $Aut(X)$ is the group of projective transformations which fix a point on the projective plane, say $[1 : 0 : 0]$. The Lie algebra $\mathfrak{a}$ of $Aut(X)$ consists of 3×3 complex matrices of trace 0 whose last two entries of the first column are 0. The center is trivial. Moreover, the space of matrices in $\mathfrak{a}$ with zero entries in second and third rows (*i.e.* only the last two entries of first row are possibly non-zero) is an abelian ideal. Hence $\mathfrak{a}$ is not reductive and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ does not carry Kähler-Eintein metric. Similarly, $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ does not carry Kähler-Eintein either. The matrix is

$$\begin{pmatrix} -a-b & 0 & * \\ 0 & a & * \\ 0 & 0 & b \end{pmatrix}.$$

Tian showed that the projective plane blown up in 3, 4, 5, 6, 7, or 8 points in general position has a Kähler-Einstein metric.

Back to Kodaira dimension. When the first Chern class is negative, it has to be minimal otherwise $c_1 \cdot E > 0$ where E is the class of an exceptional divisor. Hence $\kappa^h = \kappa^s = 2$, it is of general type. However, surfaces with negative first Chern class, or with ample canonical bundle, consist a small portion of surfaces with general type. In general, one could contract all the -2 rational curves on a surface to get the canonical model of the surface. By a result of Miyaoka, these orbifolds admit Kähler-Einstein metric.

When the first Chern class is zero, again the surface is minimal. It is of Kodaira dimension 0. Moreover, these are called the Calabi-Yau surfaces.

Finally, the first Chern class negative case corresponds to the del Pezzo surfaces as we have discussed above.

We notice that there are no complex surfaces of Kodaira dimension 1 admits any Kähler-Einstein metric. Actually, no such surfaces could admit Einstein metric which follows from the following Hitchin-Thorpe theorem.

Theorem 3.9.2. *Any compact oriented Einstein 4-manifold (M, g) satisfies $2\chi + 3\sigma \geq 0$. The equality holds if and only if (M, g) is finitely covered by a Calabi-Yau K3 surface or by a 4-torus.*

Proof.

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W^+|^2 + |W^-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

$$\sigma(M) = \frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2) d\mu$$

where $\mathring{r} = r - \frac{s}{4}g$ denotes the trace-free part of the Ricci curvature. Hence for an Einstein metric, $\mathring{r} = 0$ and the inequalities $2\chi \pm 3\sigma \geq 0$ are apparent. □

We remark that, for higher dimensions, it is still unknown whether there are obstructions for a manifold to admit Einstein metric.

Recall that if there is an almost complex structure, by Wu's theorem, $2\chi + 3\sigma = K^2$. Here K is the canonical class. If M admits a complex or symplectic structure, then $K^2 \leq 0$ with equality if and only if it is minimal. However, by Theorem 3.9.2, $K^2 = 0$ if and only if it is of Kodaira dimension 0. Hence, we have shown there is no symplectic or complex 4-dimensional Einstein manifold with Kodaira dimension 1.

A similar discussion could be extended to see what is the maximal k such that $M\#k\overline{\mathbb{C}P^2}$ is Einstein when M is a surface of general type. It is still unknown whether we could realize the optimal bound.