MA3D9 Example Sheet 1

Without otherwise mentioned, all curves are smooth and regular.

1. Complete all the exercises mentioned in the class and in the lecture notes.

2. A *tractrix* $\gamma : (0, \pi) \to \mathbb{R}^2$ is given by

$$\gamma(t) = (\sin t, \cos t + \ln(\tan \frac{t}{2})),$$

where t is the angle that the y axis makes with the vector $\gamma'(t)$.

a. Show that γ is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

3. One often gives a plane curve in polar coordinates by $r = r(\theta), a < \theta < b$.

a. Show that the arc length is

$$\int_a^b \sqrt{r^2 + (r_\theta)^2} d\theta,$$

where r_θ denote the derivative relative to θ.

b. Show that the curvature is

$$\kappa(\theta) = \frac{r^2 + 2r_\theta^2 - r r_{\theta\theta}}{(r^2 + r_\theta^2)^{\frac{3}{2}}}.$$

c. Calculate the curvature of Archimedes' spiral $r = c\theta$.

4. Let $h(s)$ be an arbitrary differentiable function on a segment (a, b). Then there is a plane curve γ for which $h(s)$ is the signed curvature function and s is the arc length parameter. The curve is determined unique up to a translation and a rotation.

5. Let $\gamma : (a, b) \to \mathbb{R}^2$ be a plane curve. Assume that there exists t_0, $a < t_0 < b$, such that the distance $||\gamma(t)||$ from the origin to the curve will be a maximum at t_0. Prove that the curvature κ of γ at t_0 satisfies $|\kappa(t_0)| \geq \frac{1}{||\gamma(t_0)||}$.

6. Prove that if a curve $\gamma(s)$ lies on a unit sphere, then the following equality holds:

$$(\kappa')^2 = \kappa^2 \tau^2 (\kappa^2 - 1),$$

where κ and τ are the curvature and the torsion of the curve. (You don't need to show: actually the inverse is also true.)