MA3D9 Example Sheet 2

Without otherwise mentioned, all curves are smooth and regular.

1. Complete all the exercises mentioned in the class and in the lecture notes.

2. Let $\alpha(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented (i.e. $\kappa_s > 0$). The curve $\beta(s) = \alpha(s) - r\mathbf{n}(s)$, where r is a positive constant, is called a parallel curve of α. Show that

   a. The length $l(\beta) = l(\alpha) + 2\pi r$.

   b. The area $A(\beta) = A(\alpha) + rl + \pi r^2$.

   c. $\kappa_\beta = \frac{\kappa_\alpha}{1+r\kappa_\alpha(s)}$.

   Remarks for problem 2: If $\alpha(s)$ is a curve of constant width R. Then $\beta(s) = \alpha(s) + R\mathbf{n}(s)$ is its opposite. We could use this idea and calculations in the last problem to show properties of a curve of constant width:

   (a) opposite points of these curve have opposite normals;

   (b) $\frac{1}{\kappa_\alpha} + \frac{1}{\kappa_\beta}$ is a constant.

3. Let $\alpha(s)$ be a regular plane curve with unit-speed prametrization. Assume that κ_s is nonzero everywhere. The curve

$$\beta(s) = \alpha(s) + \frac{1}{\kappa_s}\mathbf{n_s}$$

   is called the *evolute* of α.

   a. Show that the tangent at s_0 of the evolute of α is the normal to α at s_0.

   b. Show that a plane curve α has a vertex at s_0 if and only if the evolute β of α has a singular point at s_0.

4. Let $\gamma(s)$, $s \in [0, l]$, be a plane simple closed curve. Assume that the curvature $\kappa(s) \leq c$, where c is a constant. Prove that

$$l(\gamma) \geq \frac{2\pi}{c}.$$

5. Find all the vertices for the ellipse $\gamma(t) = (p\cos t, q\sin t)$ when $p \neq q$.