MA3D9 Example Sheet 4

Without otherwise mentioned, all curves are smooth and regular.

1. Complete all the exercises mentioned in the class and in the lecture notes.

2. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^\pi \kappa_n(\theta) d\theta,$$

where $\kappa_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

3. If the surface S_1 and S_2 intersect along a regular curve C, then the curvature k of C at p is given by

$$k^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos \theta,$$

where λ_1 and λ_2 are the normal curvatures at p, along the tangent line to C, of S_1 and S_2, respectively, and θ is the angle made up by the normal vectors of S_1 and S_2 at p.

4. Show that the Gaussian curvature of the surface $z = f(x, y)$, where f is a smooth function, is

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

5. **a.** Show that if σ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then the Gaussian curvature

$$K = -\frac{1}{2\lambda} \Delta(\ln \lambda),$$

where $\Delta\phi$ denotes the Laplacian $\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$ of the function ϕ.

b. Calculate the Gaussian curvature of the surface (upper half-plane model) with first fundamental form

$$\frac{dv^2 + du^2}{u^2}.$$

6. Show that there exists no surface $\sigma(u, v)$ such that $E = G = 1$, $F = 0$ and $L = 1$, $M = 0$ and $N = -1$.

7. Find the Gaussian curvature of each surface:

   a. Paraboloid $x^2 + y^2 = 2pz$.

   b. Torus $\sigma(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u); 0 < b < a, 0 \leq u, v \leq 2\pi$.

8. When $E = G = 1$ and $F = \cos\theta$, show that

$$K = -\frac{\theta_{uv}}{\sin\theta}.$$

9. Suppose that the first and second fundamental forms of a surface patch are $Edu^2 + Gdv^2$ and $Ldu^2 + Ndv^2$. Show that the principal curvatures $\kappa_1 = \frac{L}{E}$ and $\kappa_2 = \frac{N}{G}$ satisfy the equations

$$(\kappa_1)_v = \frac{E_v}{2E}(\kappa_2 - \kappa_1), (\kappa_2)_u = \frac{G_u}{2G}(\kappa_1 - \kappa_2).$$