MA3D9 Example Sheet 5

1. Let v and w be tangent vector fields along a curve $\gamma : I \to S$. Prove that
$$\frac{d}{dt} < v(t), w(t) >=< \nabla_\gamma v, w > + < v, \nabla_\gamma w > .$$

2. Let $\gamma(s)$ be a curve parametrized by arclength s, with nonzero curvature. Consider the parametrized surface
$$\sigma(s, v) = \gamma(s) + v\mathbf{b}(s), s \in I, -\epsilon < v < \epsilon, \epsilon > 0,$$
where $\mathbf{b}$ is the binormal vector of γ. Prove that if ϵ is small, $\sigma(I \times (-\epsilon, \epsilon)) = S$ is a regular surface over which $\gamma(I) = \sigma(I \times 0)$ is a geodesic.

3. Let T be a torus of revolution which we shall assume to be parametrized by
$$\sigma(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u).$$
Prove that

   a. If a geodesic is tangent to the parallel $u = \frac{\pi}{2}$, then it is entirely contained in the region of T given by $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

   b. A geodesic that intersects the parallel $u = 0$ under an angle θ $(0 < \theta < \frac{\pi}{2})$ also intersects the parallel $u = \pi$ if $\cos\theta < \frac{a-r}{a+r}$.

   Hint: use Clairaut's relation.

4. *Surface of Liouville* are those surfaces for which it is possible to obtain a system of local coordinates $\sigma(u, v)$ such that the coefficients of the first fundamental form are given as
$$E = G = U(u) + V(v), F = 0.$$
Prove that

   a. The geodesic of a surface of Liouville may be obtained by
$$\int \frac{du}{\sqrt{U - c}} = \pm \int \frac{dv}{\sqrt{V + c}} + c_1,$$
where c and c_1 are constants that depend on the initial conditions.

   b. If θ, $0 \le \theta \le \frac{\pi}{2}$, is the angle which a geodesic makes with the curve $v = const$, then
$$U \sin^2\theta - V \cos^2\theta = const.$$
(this is the analogue of Clairaut's relation.)