MA4E0 Exercise Sheet 3

1. Verify the Baker-Campbell-Hausdorff formula to the cubic term for matrix groups. That is, for $X, Y \in \mathrm{GL}(n, \mathbb{R})$, let $\exp(\mu(X,Y)) = \exp X \cdot \exp Y$, find out the Taylor expansion of $\mu(X,Y)$ up to the cubic term.

2. $\mathrm{Sp}(1) = \{q \in \mathbb{H} : q\bar{q} = 1\}$. Identifying $\mathbb{R}^3$ with the imaginary quaternions.
   (a) Show that for $q\bar{q} = 1$, the map $v \mapsto qv\bar{q}$ maps $\mathbb{R}^3$ to itself, and is an isometry.
   (b) Verify that the resulting map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ is a $2:1$ covering map.
   (c) Show $\mathrm{Sp}(1) = \mathrm{SU}(2) = S^3$. Hence it is simply connected.

3. Look at the adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$:
$$\mathrm{Ad} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}(2, \mathbb{C})).$$
   (a) Show that this gives a $2:1$ covering $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$.
   (b) Show that $\mathrm{SL}(2, \mathbb{C})$ is simply connected.
   (c) Prove
$$\mathrm{tr}(\mathrm{Ad}(A)) = \mathrm{tr}^2(A) - 1 = \mathrm{tr}(A^2) + 1$$
   for all $A \in \mathrm{SL}(2, \mathbb{C})$.

4. Show that the universal covering space $\tilde{G}$ of a Lie group G is a Lie group. Moreover, the covering map $\pi : \tilde{G} \to G$ is a Lie group homomorphism.

5. * For Levi-Civita connection, show the Koszul formula
$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) \\ &- \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle \end{aligned}$$

6. There is a bijective correspondence between left-invariant metrics on a Lie group G and inner products on the Lie algebra $\mathfrak{g}$.
   Hint: If $\langle , \rangle$ is an inner product on $\mathfrak{g}$, set $\langle u, v\rangle_g = \langle (l_{g^{-1}})_* u, (l_{g^{-1}})_* v\rangle$, for all $u, v \in T_g G$.

7. * For a left-invariant metric $\langle , \rangle$ on G and any two left-invariant vector fields X, Y, we have $Z(\langle X, Y\rangle) = 0$ for any vector field Z.