Part I: Mean field game models in pedestrian dynamics

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Graduate Summer School on 'Mean field games and applications'

Mean field games - general assumptions

- *Agents are indistinguishable.*
- *Agents are perfectly rational individuals.*
- *Every agent knows the distribution of all others for all times.*

Mean field games and applications

Mean field games - general assumptions

- *Agents are indistinguishable.*
- *Agents are perfectly rational individuals.*
- *Every agent knows the distribution of all others for all times.*

Mean field game theory provides a powerful mathematical framework to analyze the dynamics of large interacting agent systems, but the underlying assumptions are often only partially consistent with reality.

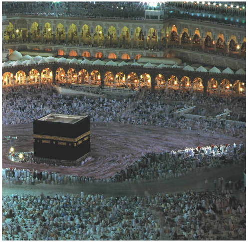
Outline

1. Pedestrian dynamics
   - Individual trajectories, the fundamental diagram, ...
   - Microscopic models
   - Kinetic models
   - Macroscopic approaches

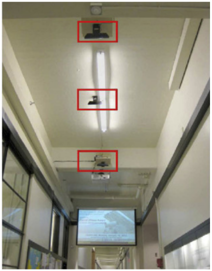
2. On a mean field model for fast exit scenarios
   - Mathematical modeling
   - Analysis of the optimal control model
   - Understanding the Hughes model for pedestrian dynamics
   - Including local vision

Pedestrian dynamics

- *Empirical studied of human crowds started about 50 years ago.*
- *Nowadays there is a large literature on different micro- and macroscopic approaches available.*
- *Challenges: microscopic interactions not clearly defined, multiscale effects, finite size effects,.....*

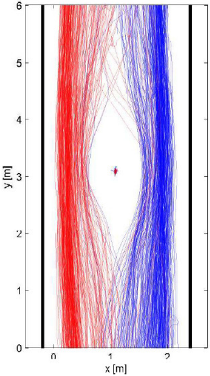


Individual trajectories - obtained from cameras[1]



(a) Kinect sensors mounted on the ceiling.

(b) Density map obtained from sensors.



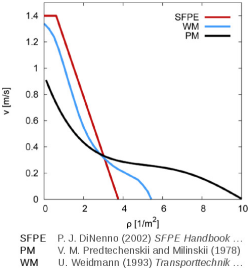
(c) Extracted trajectories.

[1] Seer et al., *Validating social force based models with comprehensive real world motion data*, Transportation Research Procedia, 2014

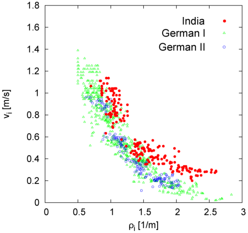
Or from sensors placed on the head ...[2]

[2]Courtesy of Armin Seyfried (Forschungszentrum Jülich), BaSiGo experiments (5 days, 31 experiments, 200 runs, 28 industrial cameras, 2200 participants in total)

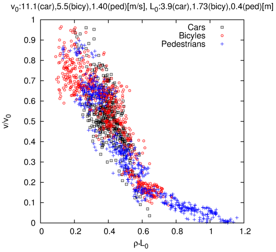
Fundamental diagram[3]



(d) Fundamental diagram.



(e) Germany vs. India



(f) Cars vs. pedestrians vs. bikes.

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[3] Courtesy of Armin Seyfried (Forschungszentrum Jülich), BaSiGo experiments (5 days, 31 experiments, 200 runs, 28 industrial cameras, 2200 participants in total)

Force based models

Newton's laws of motion: *Let $x_i = x_i(t)$ and $v_i = v_i(t)$ denote the position and velocity of the $i - th$ individual with mass m_i. Then*

$$\begin{aligned} dx_i &= v_i dt \\ m_i dv_i &= F_i(x_1, \ldots, x_N, v_1, \ldots, v_N)dt + \sigma_i dB_i^t. \end{aligned}$$

describes the dynamics driven by the forces F_i and some additive noise dB_i.

Stochastic optimal control *Let's assume that all pedestrians are perfectly rational and that the i-th individual wants to minimize a cost functional*

$$\mathbb{E}(\int_0^T L_i(x_1, \ldots, x_i, \ldots x_N, v_1, \ldots v_i, \ldots v_N) + g(x_i, (T), T)dt)$$

under the constraint that

$$dx_i = v_i dt + \sigma_i dB_i^t.$$

where L and Φ denote the running and terminal cost.

Example: Social force model [4]

Assumptions:

- *Each pedestrian wants to move at a desired velocity v_i^0 in a desired direction e_i^0..*
- *Pedestrians avoid collisions with others and obstacles (walls, ...).*
- *Individuals follow each other ....*

Equation of motion is given by

$$m_i \frac{dv_i}{dt} = m_i \frac{v_i^0 \mathbf{e}_i^0 - v_i}{\tau_i} + \underbrace{\sum_{j \neq i} f_{ij}}_{\text{interactions with others}} + \underbrace{\sum_{W} f_{i,W}}_{\text{Don't run into walls !}} ,$$

where τ_i is the relaxation time.

Interaction forces:

$$f_{ij} = \underbrace{A_i \exp\left(\frac{R_{ij} - d_{ij}}{B_i}\right) \cdot \mathbf{n}_{ij}}_{\text{repulsion}} + \underbrace{k(R_{ij} - d_{ij}) \cdot \mathbf{n}_{ij}}_{\text{body force}} - \underbrace{c_{ij}\mathbf{n}_{ij}}_{\text{attraction}} + \ldots$$

where $R_{ij} = R_i + R_j$, $d_{ij} = \|x_i - x_j\|$ and $\mathbf{n}_{ij}$ is the normalized vector pointing from pedestrian j to i.

[4] D. Helbing and P. Molnar, *Social force model for pedestrian dynamics*, Phys. Rev. E. 51, 1995

Microscopic optimal control approaches[5]

Consider an individual with position $x = x(t)$ *(state) and velocity* $v = v(t)$ *(control). Then*

$$dx(t) = vdt + \sigma dB(t), \text{ subject to } x(t) = \hat{x}$$

Constraints on the velocity: $v(t) \in \mathcal{V}(x, t) = \{v \text{ such that } \|v\| \leq v_0(x, t)\}$.

Individuals are perfectly rational and want to minimize

$$\mathbb{E}\left(\int_t^T L(s, x(s), v(s))ds + g(T, x(T))\right)$$

where L is the running cost and g is the terminal cost.

Terminal cost: Penalty if an individual does not make it to a target A at the final time, that is

$$g(T, x(T)) = \begin{cases} 0 & \text{if } x(T) \in A \\ \bar{g} & \text{otherwise.} \end{cases}$$

[5] S.P. Hoogendorn, P.H.L. Bovy, *Pedestrian route-choice and activity scheduling theory and models*, Transportation Research B 38, 2004

Microscopic optimal control approaches[6]

Running costs

1. *Expected travel time* $L_1 = c$, *where* c *is the time pressure*
2. *Don't get too close to obstacles and walls* $L_2 = ae^{-d(O,x)/b}$, *where* d *is the distance between the pedestrian and the obstacle.*
3. *Kinetic energy* $L_3 = \frac{1}{2}\|v\|^2$
4. *Expected number of pedestrian interactions - discomfort due to crowding Let* $\zeta = \zeta(x(t), t)$ *denote the expected number of interactions with others. They assume that*

$$L_4 = \zeta(\rho(x(t))$$

   where ρ *is the pedestrian density.*
5. *Benefit of walking in certain area:* $L_5 = \gamma(x(t), t)$

Optimal velocity

$$v^* = \operatorname{argmin} \mathbb{E}\left(\int_t^T L(s, x(s), v(s), \rho)ds + g(T, x(T))\right)$$

[6] S.P. Hoogendorn, P.H.L. Bovy, *Pedestrian route-choice and activity scheduling theory and models*, Transportation Research B 38, 2004

Let's go back to stochastic OC

Expected value of costs, the so-called value function

$$V(\hat{x}, t) = \mathbb{E}(\int_t^T L(s, x^*(s), v^*(s))ds + g(x^*(T), T))$$

subject to the constraint that $dx^*(t) = v^* dt + \sigma dB(t)$, $x^*(t) = \hat{x}$.

Using Bellman's principle we calculate the Hamilton-Jacobi-Bellman equation for V

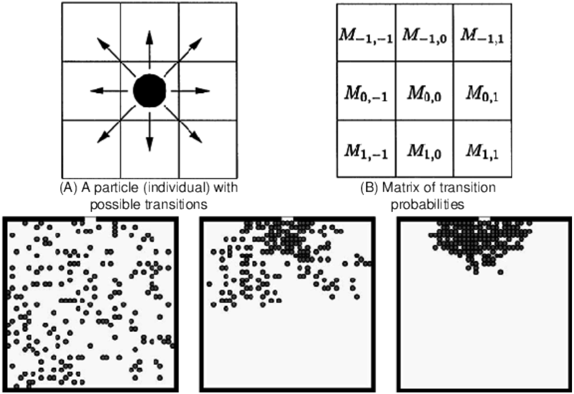
$$\frac{-\partial V}{\partial t}(x, t) = H(x, \nabla V, \Delta V)$$

where $H := \min_{v \in \mathcal{V}}(L(x, v) + \sum_i v_i \frac{\partial V}{\partial x_i} + \frac{\sigma^2}{2} \sum_{ij} \frac{\partial^2 V}{\partial_i x \partial_j x})$ and terminal condition $V(x, T) = \bar{g}$.

Optimal velocity and direction:

$$v^* = \min(\|\nabla V\|, v_0) \text{ and } e^* = \frac{\nabla V}{\|\nabla V\|}.$$

Cellular automata model



(C) Simulation of pedestrians leaving room with single door

Figure: From C. Burstedde,K. Klauck, A. Schadschneider, J. Zittarzt, *Simulation of pedestrian dynamics using a two-dimensional cellular automaton*, Physica A, 2001

Kinetic models

Aim: Describe the evolution of pedestrians with respect to their position x in space and their velocity v.
Let $f = f(x, v, t)$ denote the distribution of individuals with respect to their position and velocity. Then f solves a Boltzmann type equation of the form

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) = \mathcal{Q}(f, f)$$

where $\mathcal{Q}$ is the so-called collision operator.

The collision operator can include

- velocity changes due to possible collisions (individuals may step aside).
- adjustment of the velocity to move towards a target.
- noise, since people usually don't walk in straight lines.

PDE models for pedestrian dynamics

In the macroscopic limit $N \to \infty$ one usually obtains a nonlinear transport-diffusion equation of the form

$$\partial_t \rho = \operatorname{div}(D(\rho) \underbrace{\nabla(E'(\rho) - V + W * \rho)}_{:=v}).$$

- $V = V(x)$ *is an external potential energy (e.g. confinement,....),*
- $D = D(\rho)$ *denotes the nonlinear diffusion/mobility*
- $E = E(\rho)$ *an entropy/internal energy.*
- $W = W(x)$ *is an interaction energy.*

- *General PDE models for pedestrian flows are conservation laws.*
- *Highly nonlinear - for example nonlocal model by Colombo et al*

$$\partial_t \rho + \operatorname{div}(\rho v(\rho)(\nu(x) + \mathcal{I}(\rho)) = 0, \text{ where } \mathcal{I} = -\varepsilon \frac{\nabla(\rho * \eta)}{\sqrt{1 + \|\nabla(\rho * \eta)\|^2}}$$

The Hughes model for pedestrian flow [7]

1. *Speed of pedestrians depends on the density of the surrounding pedestrian flow*

$$v = f(\rho)u, \quad |u| = 1.$$

2. *Pedestrians have a common sense of the task (called potential ϕ)*

$$u = -\frac{\nabla\phi}{|\nabla\phi|}.$$

3. *Pedestrians try to minimize their travel time, but want to avoid high densities*

$$|\nabla\phi| = \frac{1}{f(\rho)}.$$

Hughes' model for pedestrian flow:

$$\partial_t\rho - \operatorname{div}(\rho f^2(\rho)\nabla\phi) = 0$$
$$|\nabla\phi| = \frac{1}{f(\rho)}$$

People slow down as they approach the maximum density $\rho_{\max}$: $f(\rho) = (\rho_{\max} - \rho)$.

[7] Hughes, R. *A continuum theory for the flow of pedestrians*, Transportation Research Part B, 36, 507-535, 2002

Analytic issues:

- fully coupled system; nonlinear hyperbolic conservation law.
- density dependent stationary Hamilton Jacobi equation $\Rightarrow \phi \in C^{0,1}$ only.

Let us consider the regularized system:

$$\partial_t \rho^\varepsilon - \operatorname{div}((\rho^\varepsilon f^2(\rho^\varepsilon)\nabla\phi^\varepsilon) = \varepsilon \Delta \rho^\varepsilon$$
$$-\delta_1 \Delta \phi^\varepsilon + |\nabla \phi^\varepsilon| = \frac{1}{f(\rho^\varepsilon) + \delta_2}.$$

1D : solution ρ^ε converges to an entropy solution for $\varepsilon \to 0$, but $\delta_1 > 0, \delta_2 > 0$!

Mean field games [8]

Microscopic model

N-player stochastic differential game

$$\inf_{V_i \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(t, X_i, V_i, \rho)dt + g(\rho, X_i, t = T)\right]$$
$$dX_i = V_i dt + \sigma dB_i \ , \ X_i(t=0) = x.$$

Transient macroscopic model

Calculate Nash equilibrium, limiting equations as $N \to \infty$ gives time dependent mean field game: Find (ϕ, ρ) such that

$$\partial_t \phi + \nu \Delta \phi - H(x, \nabla \phi) = 0$$
$$\partial_t \rho - \nu \Delta \rho - \mathrm{div}\left(\frac{\partial H}{\partial p}(x, \nabla \phi)\rho\right) = 0,$$

with the initial and end conditions $\phi(x,T) = g[\rho(x,T)]$, $\rho(x,0) = \rho_0(x)$, where H is the Legendre transform of the running cost f.

[8] P.-L. Lions, J.-M. Lasry, *Mean field games*, Japan. J. Math., 2, 229-260, 2007

Connection to parabolic optimal control

If the running cost f has the form

$$f(x,t,v,\rho) = L(x,t,v)\rho(x,t),$$

then the MFG can be written as an optimal control problem. For example let us consider the kinetic energy $f(x,t,v) = \frac{1}{2}\rho|v|^2$, *then*

$$\inf_v \left[\frac{1}{2}\int_0^T \int_\Omega \rho(x,t)|v(x,t)|^2 \, dxdt + g(\rho(T),T)\right]$$

under the constraint that

$$\partial_t \rho = \nu\Delta\rho - \mathrm{div}(\rho v), \quad \rho(x,0) = \rho_0(x).$$

The formal optimality condition is $v = \nabla\phi$ *and therefore the adjoint equation reads as*

$$\partial_t \phi + \nu\Delta\phi - \frac{1}{2}|\phi|^2 = 0$$

with the terminal condition $\phi(x,T) = g'(\rho(T))$.

An optimal control approach for fast exit scenarios

- *Let us consider an evacuation or fast exit scenario, i.e. a room with one or several exits from which a groups wants to leave as fast*
- *Each individual tries to find the optimal trajectory to the exit, taking into account the distance to the exit, the density of people and other costs.*

Figure: Fast-exit experiment conducted at the TU Delft

Fast exit of particles

- Let $x(t)$ denote the trajectory of a particle, the exit time is defined as:

$$T_{exit}(x) = \sup\{t > 0 \mid x(t) \in \Omega\}.$$

- Fastest path is chosen such that

$$\frac{1}{2}\int_0^{T_{exit}} |v(t)|^2 \, dt + \frac{\alpha}{2} T_{exit}(x(t)) \to \min_{(x(t),v(t))}.$$

  subject to $\dot{x}(t) = v(t)$, $x(0) = \hat{x}$.

- Let $\mu = \delta_{x(t)}$ denote a Dirac measure and the final time T be sufficiently large:

$$T_{exit} = \int_0^T \int_\Omega d\delta_{x(t)} dt.$$

- Equivalence of continuum formulation and particle formulation, i.e.

$$\int_0^T \int_\Omega |v(y,t)|^2 d\mu dt = \int_0^T \int_\Omega |v(y,t)|^2 d\delta_{x(t)} dt = \int_0^{T_{exit}} |v(x(t),t)|^2 dt.$$

  $\Rightarrow$ map Eulerian to Lagrangian coordinates.

- Hence the minimization for the particle problem can be written as a continuum problem

$$I_T(\mu, v) = \frac{1}{2}\int_0^T \int_\Omega |v(y,t)|^2 d\mu dt + \frac{\alpha}{2}\int_0^T \int_\Omega d\mu dt,$$

subject to $\partial_t \mu + \operatorname{div}(\mu v) = 0$, $\mu \,|_{t=0} = \delta_{\hat{x}}$.

If $d\mu = \rho dy$ and the final time T sufficiently large, the minimization can be written as

$$I_T(\rho, v) = \frac{1}{2}\int_0^T \int_\Omega \rho(y,t)\ |v(y,t)|^2\ dydt + \frac{\alpha}{2}\int_0^T \int_\Omega \rho(y,t)\ dydt,$$

subject to $\partial_t \rho + \operatorname{div}(\rho v) = \frac{\sigma^2}{2}\Delta\rho,\ \ \rho(x,0) = \rho_0(x)$.

Optimality conditions

- Lagrangian with dual variable ϕ:

$$L_T(\rho, v, \phi) = I_T(\rho, v) + \int_0^T \int_\Omega (\partial_t \rho + \mathrm{div}(v\rho) - \frac{\sigma^2}{2}\Delta\rho)\phi \; dy \; dt.$$

- Optimality solutions

$$0 = \partial_v L_T(\rho, v, \phi) = \rho v - \rho \nabla \phi$$

$$0 = \partial_\rho L_T(\rho, v, \phi) = \frac{1}{2}|v|^2 + \frac{\alpha}{2} - \partial_t \phi - v \cdot \nabla \phi - \frac{\sigma^2}{2}\Delta\phi,$$

plus the terminal condition $\phi(x, T) = 0$.

- Inserting $v = \nabla\phi$ we obtain the following system (with MFG structure):

$$\partial_t \rho + \mathrm{div}(\rho \nabla \phi) - \frac{\sigma^2}{2}\Delta\rho = 0$$

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + \frac{\sigma^2}{2}\Delta\phi = \frac{\alpha}{2}.$$

Mean field games and crowding

We consider the following generalization of the optimal control problem:

$$I_T(\rho, v) = \frac{1}{2}\int_0^T \int_\Omega F(\rho)|v(y,t)|^2 dy\ dt + \frac{1}{2}\int_0^T \int_\Omega E(\rho)\ dy\ dt,$$

subject to

$$\partial_t \rho + \operatorname{div}(G(\rho)v) = \frac{\sigma^2}{2}\Delta\rho, \text{ with initial condition } \rho(y, t=0) = \rho_0(y).$$

Motivation:

- $G = G(\rho)$ *is nonlinear mobility, e.g.* $G(\rho) = \rho(\rho_{\max} - \rho)$. *Hence people slow down as the density increases.*
- $F = F(\rho)$ *correspond to transport costs created by large densities. For example:*

$$F(\rho) \to \infty \text{ as } \rho \to \rho_{\max}.$$

- $E = E(\rho)$ *can model active avoidance of jams, in particular by penalizing large density regions.*

First MFG version of Hughes

Let $H(\rho) = \frac{G^2}{F} = \rho f(\rho)^2$, $E(\rho) = \rho$ *and* $\sigma = 0$.

Optimality conditions:

$$\partial_t \rho + \operatorname{div}(\rho\ f(\rho)^2 \nabla \phi) = 0$$
$$\partial_t \phi + \frac{f(\rho)}{2}(f(\rho) + 2\rho f'(\rho))|\nabla \phi|^2 = \frac{\alpha}{2}$$

Hand-waving argument: If T is large, we expect equilibration of ϕ backward in time.

'MFG Hughes system' vs. 'classical Hughes model':

$$\partial_t \rho + \operatorname{div}(\rho\ f(\rho)^2 \nabla \phi) = 0$$
$$(f(\rho) + 2\rho f'(\rho))|\nabla \phi|^2 = \frac{\alpha}{f(\rho)}$$

$$\partial_t \rho + \operatorname{div}(\rho\ f(\rho)^2 \nabla \phi) = 0$$
$$|\nabla \phi| = \frac{1}{f(\rho)}.$$

If $f(\rho) = \rho_{\max} - \rho$ and $\alpha = 1$:

$$f(\rho) + 2\rho f'(\rho) = \rho_{\max} - 3\rho \Rightarrow \text{ additional singular point if } \rho = \frac{\rho_{\max}}{3}.$$

Analysis of the optimal control model

Let $\rho_{\max} > 0$ denote the maximum density and $\Upsilon = [0, \rho_{\max}]$. Let $F = G = H^{-1}$ which satisfy the following assumptions:

(A1) $F = F(\rho) \in C^1(\mathbb{R})$, *$F$ bounded,* $E = E(\rho) \in C^1(\mathbb{R})$ *and* $F(\rho) \geq 0$, $E(\rho) \geq 0$ *for* $\rho \in \Upsilon$.

Existence of minimizers is guaranteed if

(A2) $E = E(\rho)$ *is convex.*

To ensure that the minimizers satisfy $\rho \in \Upsilon = [0, \rho_{\max}]$, we need the following assumption on F:

(A3) $F(0) > 0$ *if* $\rho \in \Upsilon$ *and* $F = 0$ *otherwise.*

Uniqueness holds for:

(A4) $F = F(\rho)$ *is concave.*

We consider the optimization problem on the set $V \times Q$, i.e. $I_T(\rho, v) : V \times Q \to \mathbb{R}$, where V and Q are defined as follows

$$V = L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \text{ and } Q = L^2(\Omega \times (0, T)).$$

We introduce another formulation based on

$$w = \sqrt{F(\rho)}v.$$

Then

$$J(\rho, w) = \frac{1}{2} \int_0^T \int_\Omega \left(|w|^2 + E(\rho)\right) \; dydt,$$

and the optimization problem formally becomes

$$\min_{(\rho,w) \in V \times Q} J(\rho, w) \text{ such that } \partial_t \rho = \frac{\sigma^2}{2} \Delta\rho - \operatorname{div}(\sqrt{F(\rho)} w).$$

To make the relation rigorous, we need to extend the domain of the velocity v to

$$\tilde{Q}_\rho := \{v \text{ measurable} \mid \sqrt{F(\rho)} v \in Q\}.$$

Moreover, for given ρ we define an extension mapping $w \in Q$ to $v \in \tilde{Q}_\rho$ via

$$R_\rho(w)(x) := \begin{cases} \frac{w(x)}{\sqrt{F(\rho(x))}} & \text{if } F(\rho(x)) \neq 0 \\ 0 & \text{else.} \end{cases}$$

Definition (Weak formulation of the alternative formulation)

Let $\rho_0 \in L^2(\Omega)$*. A pair* $(\rho, w) \in V \times Q$ *is a weak solution with initial condition* ρ_0*, if* $\rho(0) = \rho_0$ *and*

$$\langle \partial_t \rho, \psi \rangle_{H^{-1},H^1} + \int_\Omega \left(\frac{\sigma^2}{2}\nabla\rho - \sqrt{F(\rho)}w\right) \cdot \nabla\psi \; dy = -\int_{\Gamma_E} \beta\rho\psi \; ds,$$

for all $\psi \in H^1(\Omega)$*, and if*

$$J_T(\rho, w) = \min\left\{J_T(\rho, w), \; : \; (\bar\rho, \bar w) \in V \times Q, \quad (\bar\rho, \bar w) \;\; \text{satisfy the FPE}\right\}.$$

Lemma (A-priori estimates)

Let $\rho_0 \in L^2(\Omega)$*. Let (A1) and (A2) be satisfied and let* $\sigma > 0$*,* $\beta \geq 0$*. Let* $w \in Q$ *and let* $\rho \in V$ *be a weak solution of*

$$\langle \partial_t \rho, \psi \rangle_{H^{-1},H^1} + \int_\Omega \left(\frac{\sigma^2}{2}\nabla\rho - \sqrt{F(\rho)}w\right) \cdot \nabla\psi \; dy = -\int_{\Gamma_E} \beta\rho\psi \; ds,$$

for all $\psi \in H^1(\Omega)$*. Then there exist constants* $C_1, C_2 > 0$ *depending on* F*,* σ*,* Ω *and* T *only, such that*

$$\|\rho\|_V \leq C_1\|w\|_Q + C_2.$$

Existence of weak solutions

Lemma

Assume ρ and w are as before and let (A3) be satisfied. Then, $\rho(\cdot, t) \in \Upsilon = [0, \rho_{\max}]$ for all $t \in (0, T]$ if $\rho_0(x) \in \Upsilon$.

Theorem (Existence in the general case)

Let $\rho_0 \in L^2(\Omega)$. Let (A1) and (A2) be satisfied, $\sigma > 0$ and $w = \sqrt{F(\rho)}v$. Then the variational problem has at least a weak solution $(\rho, w) \in V \times Q$ with initial condition ρ_0. If in addition (A3) is satisfied, then $\rho \in \Upsilon$.

Uniqueness of solutions for the optimality system

Proposition

Let assumption (A1) and (A2) be satisfied and let ρ be such that $H(\rho) \geq \gamma$ for some $\gamma > 0$. Then the adjoint system

$$\partial_t \phi + \frac{\sigma^2}{2}\Delta\phi = \frac{1}{2}E'(\rho) - \frac{1}{2}|j|^2\frac{F'}{F^2}$$
$$\phi(x, T) = 0$$

with the appropriate adjoint boundary conditions has a unique solution $\phi \in L^q(0, T; W^{1,q}(\Omega))$ *with* $q < \frac{N+2}{N+1}$.

Theorem (Uniqueness for the optimality system)

For a fixed initial condition $\rho_0 \in L^2(\Omega)$, there exists a unique weak solution

$$(\rho, \phi) \in L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega))$$

to the optimality system.

Understanding the Hughes model

- Let us consider N particles with position $x_k = x_k(t)$ and the empirical density $\rho^N(t) = \frac{1}{N}\sum_{k=1}^{N} \delta(y - x_k(t))$.
- To define the cost functional in a proper way we introduce the smoothed approximation ρ_g^N by

$$\rho_g^N(t) = (\rho^N * g)(y,t) = \frac{1}{N}\sum_{k=1}^{N} g(y - x_k(t)),$$

where g is a sufficiently smooth positive kernel.

Let us 'freeze' the empirical density ρ^N and look for the optimal trajectory of each particle, i.e.

$$C(X;\rho_g^N(t)) = \min_{(\xi(t),v(t))} \frac{1}{2}\int_t^{T+t} \frac{|v(s)|^2}{G(\rho_g^N(\xi(s;t))}\,ds + \frac{1}{2}T_{exit}(x(t),v(t)),$$

subject to $\frac{d\xi}{ds} = v(s)$ *and* $\xi(0) = x(t)$.

Understanding the Hughes model

Let's assume that the macroscopic (rescaled) version of $\rho^N(t)$ *converges to the mean field* $\rho(t)$*, we replace it by* $\rho(t)$ *and obtain:*

$$C(X;\rho(t)) = \min_{(\xi,w)} J(\mu,w) = \frac{1}{2}\int_t^{T+t}\int_\Omega \left(\frac{w^2(x,s)}{G(\rho(\xi(s;t)))} + 1\right) d\mu ds,$$

subject to $\partial_s\mu + \operatorname{div}(\mu w) = 0$ *with* $\mu(t=0) = \delta_X$*.*

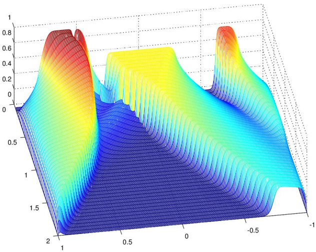
- The formal optimality conditions can be calculated via the Lagrange functional.
- For $T \to 0$ the behavior at $s = t$ represents the long-time behavior of the HJE.

Then we recover the Hughes model by choosing $G(\rho) = f(\rho)^2$ *i.e.*

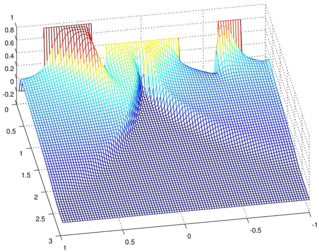
$$\partial_t\rho + \operatorname{div}(\rho f(\rho)^2 \nabla\phi) = 0,$$

$$|\nabla\phi| = \frac{1}{f(\rho)}.$$

Fast exit for three groups

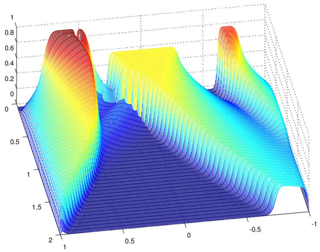


(a) Solution of the classical Hughes model

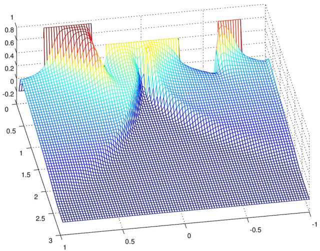


(b) Solution of the mean field optimal control approach

Fast exit for three groups



(c) Solution of the classical Hughes model



(d) Solution of the mean field optimal control approach

Thanks to collaborators: M. Burger (WWU Müenster), M. Di Francesco (L'Aquila), P.A. Markowich (Kaust and Cambridge), J.-F. Pietschmann (WWU Münster)

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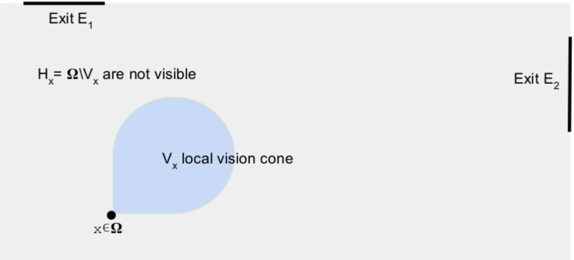
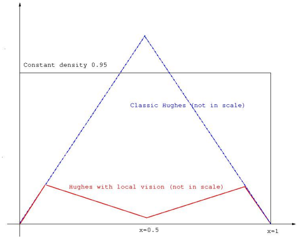
Including local vision



Modeling assumptions:

- *If a point* $y \in \Omega$ *is visible, i.e.* $y \in V_x$*, then* $\rho = \rho(y, t)$.
- *If a point is outside the visibility cone, i.e.* $y \in H_x$ *then* $\rho(y, t) = \rho_H$ *with* $\rho_H \in \mathbb{R}^+$.
  Example: assume that the area is empty, i.e. $\rho_H = 0$.
- *Angular dependent vision cone* $\Rightarrow$ *velocity dependence of the model.*
  Contradiction to the first-order character of the continuity equation.

Eikonal equation with discontinuous RHS



Potential ϕ calculated with and without vision cone

- *Consider the constant density $\rho = 0.95$ in the domain*
- *Classic model of Hughes: potential ϕ has a single turning point at $x = 0.5$.*
- *Two local vision cones ($0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$): the potential ϕ has three turning points $\Rightarrow$ shock formation.*

Low regularity of the potential $\phi \Rightarrow$ considerable problems in the numerical simulation of the nonlinear conservation law.

Exit strategy

- Exit strategy is determined by estimating the evacuation cost for each exit separately:

$$\|\nabla_y \phi_k(x,\cdot)\| = \begin{cases} \frac{1}{f(\rho(y,t))g(\rho(y,t))} & \text{for all } y \in V_x \\ \frac{1}{f(\rho_H)g(\rho_H)} & \text{for all } y \in H_x \end{cases}$$

$$\phi_k = 0 \text{ for } x \in \partial\Omega_k.$$

- It corresponds to the direction towards the exit with the minimal exit cost (weighted by the difference in the costs to the 2nd best strategy):

$$u = \frac{\nabla \phi_{k^{\text{opt}}}}{\|\nabla \phi_{k^{\text{opt}}}\|}(\phi_{k^{\text{opt}+1}} - \phi_{k^{\text{opt}}}),$$
$$k^{\text{opt}} = \operatorname{argmin}_k \phi_k,$$
$$k^{\text{opt}+1} = \operatorname{argmin}_{k \neq k^{\text{opt}}} \phi_k.$$

- The actual direction is determined by averaging the directions in the close neighborhood (weighted by the density ρ):

$$\varphi = \frac{\rho u * K}{\rho * K}$$

for a sufficiently smooth convolution kernel K.

Modified Hughes model

For every exit $\partial\Omega_k$, $k = 1, \dots M$ calculate

$$\|\nabla\phi_k\| = \begin{cases} \frac{1}{f(\rho(y,t))g(\rho,t))} \\ \frac{1}{f(\rho_H)g(\rho_H)}. \end{cases}$$

⇒ costs to each exit based on the vision cone

$$\phi_k|_{\partial\Omega_{E_k}} = 0$$

$$k^{\text{opt}}(x) = \text{argmin}_k\, \phi_k(x)$$

⇒ choose exit with the lowest costs

$$k^{\text{opt}+1}(x) = \text{argmin}_{k \neq k^{\text{opt}}}\, \phi_k(x)$$

⇒ determine exit with the 2^{nd} lowest costs

$$u = \frac{\nabla\phi_{k^{\text{opt}}}}{\|\nabla\phi_{k^{\text{opt}}}\|} \cdot (\phi_{k^{\text{opt}+1}} - \phi_{k^{\text{opt}}})$$

⇒ weigh optimal direction

$$\varphi = \frac{\rho u \star \mathcal{K}}{\rho \star \mathcal{K}}$$

⇒ smooth direction to avoid oscillations

$$\partial_t\rho - \nabla_x \cdot \left(\rho f(\rho)\frac{\varphi}{\|\varphi\|}\right) = 0$$