Distribution of Colours in Rainbow H-free Colourings

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Outline

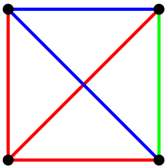
1. Preliminaries
2. $H = K_3$ / Gallai Colourings
   - Background
   - Lower Bound
   - Upper Bound
3. General H
   - Degeneracy 1 / Trees
   - Degeneracy ≥ 3
   - Degeneracy 2 & Open Questions

Definition (Colour distribution sequence)

An edge colouring of K_n using k colours has colour distribution sequence $(e_1, \cdots, e_k)$ if there are exactly e_i edges of colour i for every $1 \leq i \leq k$.

Definition (Rainbow H-free colouring)

An edge colouring of K_n is rainbow H-free if any subgraph of K_n isomorphic to H contains at least two edges of the same colour.



a rainbow K_3-free colouring of K_4

colour distribution sequence $(3, 2, 1)$

Preliminaries

Question

If an edge colouring of K_n is rainbow H-free, what could its colour distribution sequence be?

Definition ($g(H,k)$)

For any connected graph H and integer k, let $g(H,k)$ be the smallest integer N such that for all $n \geq N$ and any $(e_1, \cdots, e_k) \in \mathbb{N}^k$ satisfying $e_1 + \cdots + e_k = \binom{n}{2}$ can be realised as the colour distribution sequence of a rainbow H-free colouring of K_n.

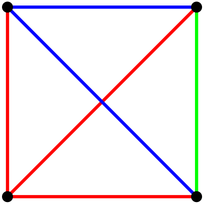
Question

- Is $g(H,k)$ finite?
- If so, what is its order of magnitude?

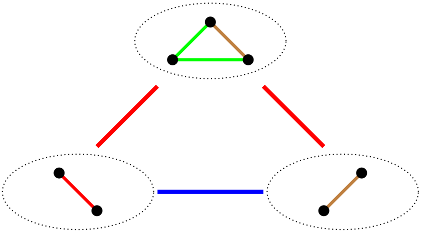
Outline

Definition

An edge-colouring of K_n using k colours is a Gallai k-colouring if it does not contain a rainbow triangle, or equivalently if it is rainbow K_3-free.



a Gallai 3-colouring of K_4



a Gallai 4-colouring of K_7

$g(K_3, k)$

Theorem (Gyárfás, Pálvölgyi, Patkós, Wales, 2020)

For every integer $k \geq 2$, there exists an integer N such that for all $n \geq N$ and any $(e_1, \cdots, e_k) \in \mathbb{N}^k$ satisfying $\sum_{i=1}^{k} e_i = \binom{n}{2}$, there exists a Gallai k-colouring of K_n with colour distribution sequence $(e_1, \cdots, e_k)$.
In other words, $g(K_3, k) < \infty$.

Bounds on $g(K_3, k)$:

- (Gyárfás, Pálvölgyi, Patkós, Wales, 2020)

$$2k - 2 \leq g(K_3, k) \leq 8k^2 + 1.$$

- (Feffer, Fu, Y., 2020)

$$\Omega(k^{1.5}/\log k) = g(K_3, k) = O(k^{1.5}).$$

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$$\Omega(k^{1.5}/\log k) = g(K_3, k) = O(k^{1.5}).$$

Theorem (Y., 2023+)

$$g(K_3, k) = \Theta(k^{1.5}/(\log k)^{0.5}).$$

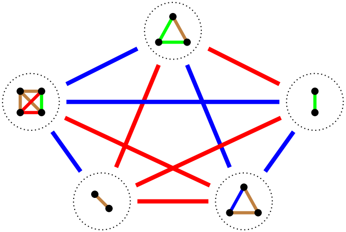
Decomposition of Gallai Colourings

Theorem (Gyárfás, Simonyi, 2004)

Given a Gallai k-colouring of K_n, we can find at most 2 colours, which we call base colours, and a decomposition of K_n into $m \geq 2$ vertex disjoint complete graphs $K_{n_1}, \cdots, K_{n_m}$, such that

- *For each $i \neq j$, there exists a base colour such that all edges between K_{n_i} and K_{n_j} have this colour.*

Conversely, any such decomposition, along with Gallai k-colourings on each K_{n_i} gives a Gallai k-colouring on K_n.



a decomposition of

a Gallai 4-colouring of K_{14}

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Strengthened Version of the Decomposition Theorem

Theorem (Feffer, Fu, Y., 2020)

Given a Gallai k-colouring of K_n, we can find at most 2 colours, which we call base colours, and a decomposition of K_n into $m \geq 2$ vertex disjoint complete graphs $K_{n_1}, \cdots, K_{n_m}$, such that

- *For each $i \neq j$, there exists a base colour such that all edges between K_{n_i} and K_{n_j} have this colour.*
- *Each base colour is used to colour at $\geq n-1$ edges between the K_{n_i}'s.*

Corollary

Suppose we have a Gallai k-colouring of K_n with colour distribution sequence $(e_1, \cdots, e_k)$, where $e_1 \geq \cdots \geq e_\ell \geq b+1 > e_{\ell+1} \geq \cdots \geq e_k$. Then colours $\ell+1, \cdots, k$ will not be used as base colours until we are decomposing a complete graph of size at most $b+1$.

Lower Bound

Example

There is no Gallai 4-colouring of K_6 with colour sequence $(4, 4, 4, 3)$.

Example

There is no Gallai 4-colouring of K_7 with colour sequence $(9, 4, 4, 4)$.

Proof.

The only possible decomposition is $K_7 \rightarrow K_6 \cup K_1$, and we have to colour all 6 edges between with colour 1. But then we need to find a Gallai 4-colouring of K_6 with colour sequence $(3, 4, 4, 4)$. □

Lower Bound

Proposition (Y., 2023+)

Let $n = k^{1.5}/10(\log k)^{0.5}$ and let $a = \Theta(k^2/\log k)$, $b = \Theta(k)$. Then there is no Gallai k-colouring of K_n with colour distribution sequence $(a, a, \cdots, a, b, b, \cdots, b)$. This shows $g(K_3, k) \geq k^{1.5}/10(\log k)^{0.5}$.

Proof (Sketch).

$$(a, a, \cdots, a, \overbrace{b, b, \cdots, b}^{\text{Not used until size } b+1}) \xrightarrow[\text{decomposition steps}]{} (\overbrace{\leq b, \leq b, \cdots, \leq b,}^{\text{Before reaching size } b+1}\ b, b, \cdots, b)$$

□

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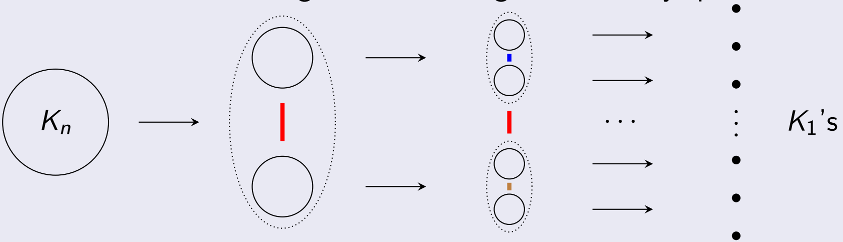
Upper Bound

Proposition (Y., 2023+)

Let $n \geq 5000000k^{1.5}/(\log k)^{0.5}$. *Then for any* $e_1 + \cdots + e_k = \binom{n}{2}$, *there is a Gallai* k*-colouring with colour sequence* $(e_1, \cdots, e_k)$. *This shows* $g(K_3, k) \leq 5000000k^{1.5}/(\log k)^{0.5}$.

Proof (Sketch).

We show that a colouring of the following form is always possible.

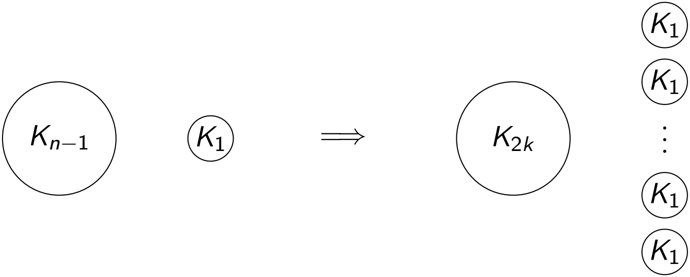


□

Upper Bound

Lemma

Suppose there exists some $e_i \geq t(n-t)$, then we can decompose K_n into K_t and K_{n-t}, and colour all $t(n-t)$ edges between them with colour i. In particular, if $n \geq 2k$, then we can decompose K_n into K_{n-1} and K_1.



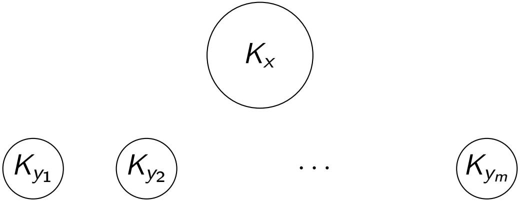
Problem: How to colour K_{2k}?

Cushions

Observation

Suppose at some stage of this process, the complete graphs remaining have sizes $x \geq y_1 \geq \cdots \geq y_m$, and we still need to colour e_i edges with colour i. Then we must have $\sum_{i=1}^{k} e_i = \binom{x}{2} + \sum_{j=1}^{m} \binom{y_j}{2}$.

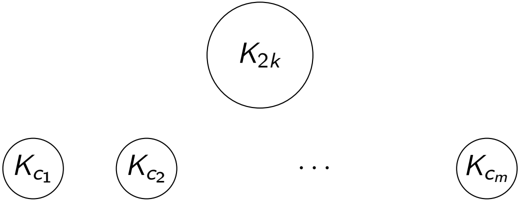
We view the quantity $\sum_{j=1}^{m} \binom{y_j}{2}$ as the cushion we have available to colour the complete graph K_x.



Creating Cushions

Lemma (Y., 2023+)

If $e_1 + \cdots + e_k \geq \binom{2k}{2} + \frac{1}{2}k^2$, then there exists a Gallai k-colouring of K_{2k} with at most e_i edges of colour i.

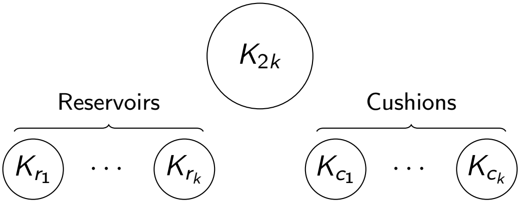


$c_1, \cdots, c_m \ll k$, but $\binom{c_1}{2} + \cdots + \binom{c_m}{2} \geq \frac{1}{2}k^2$.

Problem: How to colour these K_{c_i}?

Reservoirs

Use an absorption type argument.



$$r_1, \cdots, r_k \ll c_1, \cdots, c_m \ll k, \text{ but } \binom{c_1}{2} + \cdots + \binom{c_m}{2} \geq \frac{1}{2}k^2.$$

- Use the cushions created by $K_{c_1}, \cdots, K_{c_m}$ to colour K_{2k}.
- Use the cushions created by $K_{r_1}, \cdots, K_{r_k}$ to colour both $K_{c_1}, \cdots, K_{c_m}$ and $K_{r_1}, \cdots, K_{r_k}$.

Outline

Degeneracy

Definition (Degeneracy)

- A graph H is k-degenerate if every subgraph of H has a vertex of degree at most k.
- The degeneracy of H is the smallest integer k such that H is k-degenerate.

Example

- A connected graphs has degeneracy 1 if and only if it is a tree.
- If a graph has degeneracy at least k, then it has a subgraph with minimum degree at least k.

$H = K_{1,m}$ / Star

Proposition (Wu, Y., 2023++)

If $n \geq 10\sqrt{k}$ and $k \geq 10m^2$, then there is no rainbow $K_{1,m}$-free k-colouring of K_n with the balanced colour distribution sequence.
In particular, this shows that $g(K_{1,m}, k) = \infty$ for large k.

Proof.

Double count $N =$ the number of pairs (v, c), where v is a vertex of K_n and c is the colour of some edge adjacent to v.
If the colouring is rainbow $K_{1,m}$-free, then $N \leq n(m-1)$.
If the colouring has colour distribution sequence $(e_1, \cdots, e_k)$, then $N \geq \sum_{i=1}^{k} \sqrt{2e_i}$ as edges with colour i is incident with least $\sqrt{2e_i}$ vertices.
We have a contradiction if conditions in the proposition are satisfied. □

Theorem (Wu, Y., 2023++)

Let H be a tree on m vertices. If $n \geq 10\sqrt{k}$ and $k \geq (10m)^{10m}$, then any k-colouring of K_n with "almost balanced" colour distribution sequence contains a rainbow H.
In particular, this shows that $g(H, k) = \infty$ for large k.

Proof (Sketch).

Induction on m. Let v be a leaf of H.

- The set A of vertices in K_n adjacent to edges of at least $2m + 1$ colours has size at least $n/2$.
- Colour distribution inside A is still "almost balanced".
- Induction gives a rainbow $H - v$ in A.
- Can attach leaf v by the defining property of A. □

Outline

Degeneracy ≥ 3

Theorem (Wu, Y., 2023++)

Let H be a graph on m vertices with degeneracy at least 3, then

$$g(H, k) = \Theta_m(k).$$

Degeneracy ≥ 3 Lower Bound

Proposition (Wu, Y., 2023++)

Let H be a graph on m vertices and let $n \leq k/m^3$. Then any k-colouring of K_n with the balanced colour sequence $(e_1, \cdots, e_k)$ contains a rainbow H. In particular, this shows $g(H, k) \geq k/m^3$.

Proof (Sketch).

- Fix any balanced k-colouring of $G = K_n$.
- Let S be a size m subset of $V(G)$ chosen uniformly at random.
- Show that the expected number of edge pairs in $G[S]$ with the same colour is < 1.
- Thus, there is a realisation of S such that $G[S]$ is rainbow, and so contains a rainbow copy of H. □

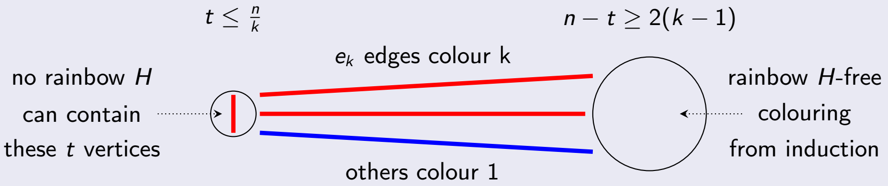
Degeneracy ≥ 3 Upper Bound

Proposition (Wu, Y., 2023++)

Let H be a graph with minimum degree at least 3. Let $n \geq 2k$, and let $e_1 \geq \cdots \geq e_k$ be such that $e_1 + \cdots + e_k = \binom{n}{2}$. Then there exists a rainbow H-free colouring of K_n with colour distribution sequence $(e_1, \cdots, e_k)$.

Proof (Sketch).

Induction on k. Let t be the smallest integer satisfying $\binom{t}{2} + t(n-t) \geq e_k$.



So this colouring is rainbow H-free. □

Degeneracy ≥ 3 Upper Bound

Proposition (Wu, Y., 2023++)

Let H be a graph with minimum degree at least 3. Let $n \geq 2k$, and let $e_1 \geq \cdots \geq e_k$ be such that $e_1 + \cdots + e_k = \binom{n}{2}$. Then there exists a rainbow H-free colouring of K_n with colour distribution sequence $(e_1, \cdots, e_k)$. Therefore, $g(H, k) \leq 2k$.

Corollary

Let H be a graph with degeneracy at least 3. Then $g(H, k) \leq 2k$.

Proof.

From definition of degeneracy, H contains a subgraph H' with minimum degree at least 3. Since rainbow H'-free implies rainbow H-free, we have $g(H, k) \leq g(H', k) \leq 2k$. □

Outline

Degeneracy 2

Let H be a graph on m vertices with degeneracy 2. From the definition of degeneracy, H contains a cycle.

- The random lower bound argument works for any graph, so $g(H,k) = \Omega_m(k)$.
- Can show that the upper bound construction for K_3 is not only rainbow K_3-free, but in fact contains no rainbow cycle. So $g(H,k) = O(k^{1.5}/(\log k)^{0.5})$.

Open Questions

- Determine the order of magnitude of $g(C_4, k)$.
- Determine the order of magnitude of $g(H, k)$ for all H with degeneracy 2.
- Better constants in the known Θ results for $g(H, k)$.
- More necessary and sufficient conditions for possible colour distribution sequence of rainbow H-free colourings of K_n when $n \leq g(H, k)$.