Paul Samuelson's critique and equilibrium concepts in evolutionary game theory

Reinoud Joosten
University of Twente, The Netherlands

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1 Plan of the talk

- My personal fascination.
- Samuelson's critique in economics.
- Evolutionary games.
- Main equilibrium concept.
- Samuelson's critique again.
- To the rescue?
- Other concepts.
- Why ESS?
- Concluding remarks

2 My personal fascination

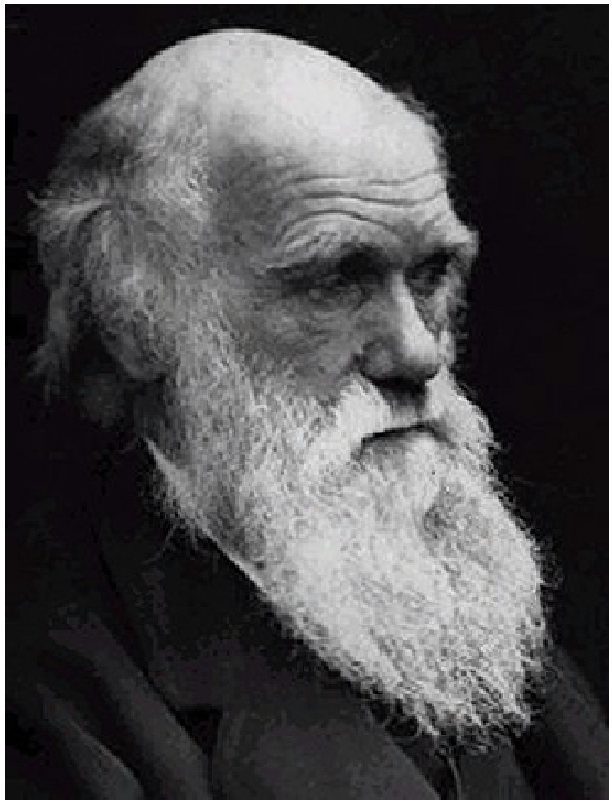

Figure 1: The ten pound note guy.

Figure 2: Leon Walras: 1834-1910. Élements d'Économie Politique Pure (1874).

3 Samuelson's critique in economics

Stability analysis should be performed directly on the dynamics instead of the underlying system.

Figure 3: Paul Samuelson, 1915-2009, winner of The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel.

- Central questions:
  - What will happen to a system in disequilibrium?
  - What will happen if an equilibrium is perturbed?

- Focus on excess demand function:

$$f : \mathbb{R}_+^{n+1} \backslash \{0^{n+1}\} \to \mathbb{R}^{n+1}.$$

- Sonnenschein-Mantel-Debreu-theorems: excess demand functions are characterized by
  - homogeneity of degree zero in prices, $f(\lambda x) = f(x)$ for all $\lambda > 0$.
  - continuity,
  - complementarity, i.e., $x \cdot f(x) = 0$ (Walras' law),
  - desirability, i.e., $f_i(x) > 0$ whenever $x_i = 0$.

- Hom $\rightarrow$ unit simplex (existence proofs a la Brouwer, Kakutani).

- Des too strong.

- Given $f : S^n \rightarrow \mathbb{R}^{n+1}$, $y \in S^n$ is a Walras equilibrium iff $f(y) \leq 0^{n+1}$.

- Implication of WARP:

$$(y - x) \cdot f(x) > 0.$$

- Price-adjustment dynamics (Samuelson [1941,1947]):

$$\frac{dx}{dt} = f(x).$$

- Sphere.

4 Evolutionary games

- Population with $n+1$ subgroups, population shares $x = (x_1, ..., x_{n+1}) \in S^n$.

- Fitness function:

$$E(x) = (E_1(x), ..., E_{n+1}(x)).$$

- **Relative fitness function:**

$$f(x) = E(x) - (x \cdot E(x))\, 1^{n+1}.$$

- $$f_i(x) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0 \iff E_i(x) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} x \cdot E(x).$$

- Continuity of E implies continuity of f.

- Complementarity, i.e., $x \cdot f(x) = 0$!

- Nash equilibrium:

$$f(y) \leq 0^{n+1}.$$

- Generalization: Saturated equilibrium if f is **not** of the type

$$f(x) = Ax - x \cdot Ax$$

5 Main equilibrium concept

Evolutionarily stable strategy (state) (Maynard Smith & Price [1973]): $y \in S^n$ is an ESS if a neighborhood $U \subseteq S^n$ containing y exists such that for all $x \in U \backslash \{y\}$:

$$y \cdot E(x) > x \cdot E(x).$$

- Rearrangeing:

$$\begin{aligned}(y-x) \cdot E(x) &> 0, \\ (y-x) \cdot f(x) &> 0.\end{aligned}$$

6 Samuelson's critique again

Stability analysis should be performed directly on the dynamics instead of the underlying system.

- Elegant way out:

Daniel Friedman [1991]

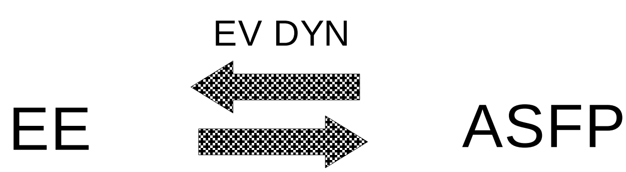

Figure 4: End of the discussion. End of this talk?

7 To the rescue?

- Replicator dynamics (Taylor & Jonker [1978]) for every $i \in I^{n+1}$ given by

$$\frac{dx_i}{dt} = x_i f_i(x) \text{ for all } x \in S^n.$$

Taylor & Jonker [1978]

Zeeman [1980,1981]

Hofbauer, Schuster, Sigmund [1979]

ESS + REPL $\Longrightarrow$ ASFP

Figure 5: Every ESS is asymptotically stable under the replicator dynamics. Not vice versa.

- CFP-dynamics (Rosenmüller [1971], Brown [1951]), BR-dynamics (Gilboa & Matsui [1991], Matsui [1992]), logit-dynamics (Fudenberg & Levine [1998]).

- BVN-dynamics (Brown & Von Neumann [1950]):

$$\frac{dx_i}{dt} = \mathsf{max}\{0, f_i(x)\} - x_i \left(\sum_h \mathsf{max}\{0, f_h(x)\} \right).$$

- Orthogonal projection dynamics of Lahkar & Sandholm:

$$h(x) = f(x) - \left(\sum_h f(x) \right) \frac{1}{n+1}^{n+1}.$$

- Ray projection dynamics of Joosten & Roorda:

$$h(x) = f(x) - \left(\sum_h f(x) \right) x.$$

- Basic idea: PROJECT Samuelson's simultaneous tâtonnement process unto the unit simplex.

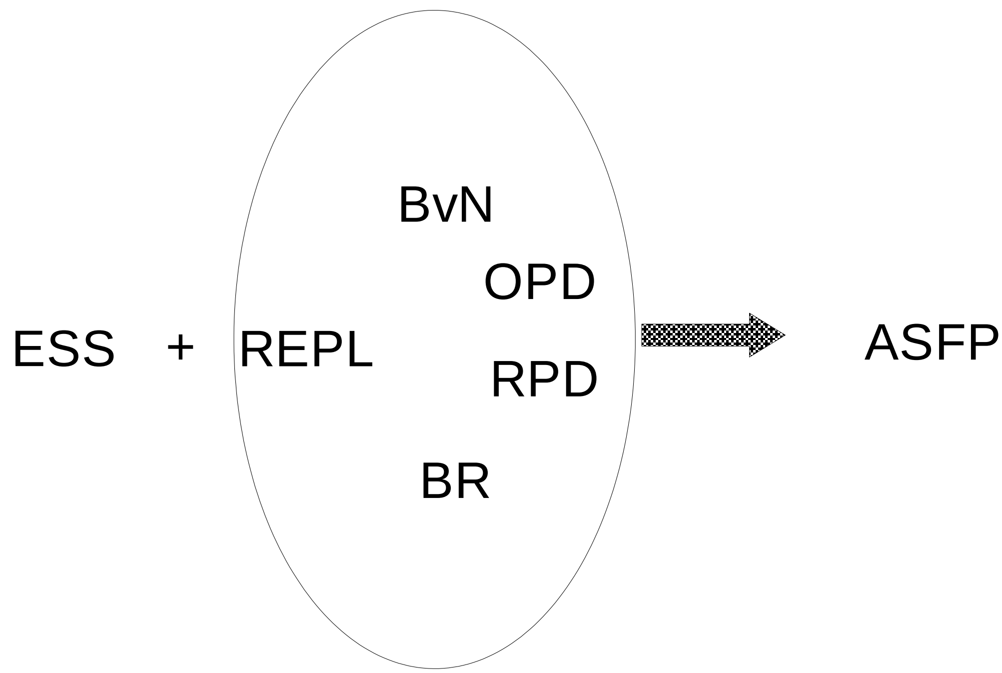

Figure 6: Extending dynamics for which ESS implies asymptotical stability. Why replicator dynamics???

8 Other point-valued concepts

- **Evolutionarily stable equilibrium**: let $\frac{dx}{dt} = h(x)$ for some $h : S^n \longrightarrow \mathbb{R}^{n+1}$, then y is an ESE iff an open neighborhood U containing y exists such that

$$(y - x) \cdot h(x) > 0 \text{ for all } x \in U \backslash \{y\}.$$

- Compare:

$$\begin{aligned} ESE &: (y - x) \cdot h(x) > 0. \\ (G)ESS &: (y - x) \cdot f(x) > 0. \end{aligned}$$

- ESE inspired by WARP (Samuelson [1941]) + Samuelson's tatonnement [1941,1947].

- Lahkar & Sandholm [2009]: Equivalence under orthogonal projection dynamics.

Hybrid ESS, EE: Joosten [1996]

$$\text{ESE} \xrightarrow{\text{EV DYN}} \text{ASFP}$$

Figure 7: ESE inspired by early work in economics.

9 Why ESS?

Definition 1 *Given relative fitness function* $f : S^n \to \mathbb{R}^{n+1}$ *and evolutionary dynamics* $h : S^n \to \mathbb{R}^{n+1}$, *let* $d : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ *be a distance function,* $\xi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ *be differentiable, and monotonically strictly either decreasing or increasing, with* $\xi(0) = \xi_0$. *Let furthermore,* $V : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ *be given by*

$$V(x, y) = \xi(d(x, y)) \text{ for all } x, y \in \mathbb{R}^{n+1}.$$

Then, $y \in S^n$ *is a* **generalized evolutionarily stable equilibrium** *if and only if an open neighborhood* $U \subseteq S^n$ *containing* y, *exists such that for all* $x \in U \backslash \{y\}$ *it holds that* $[V(x, y) - \xi_0] \cdot \dot{V}(x, y) < 0$, *where* $\dot{V}(x, y) = \sum_{i=1}^{n+1} \frac{\partial V}{\partial x_i} h_i(x)$.

- Motivation: Samuelson's critique and ESE too restrictive (Euclidean distance).

- Intuition: Monotone convergence for at least one (monotone transformation of a) distance function.

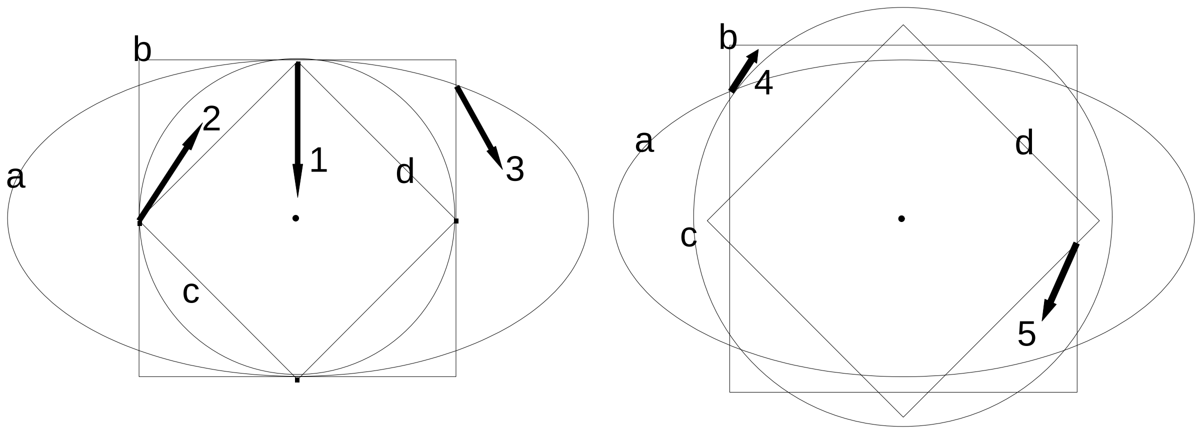

Figure 8: Nonequivalence of distance functions for monotone convergence.

Definition 2 *Let relative fitness function* $f : S^n \to \mathbb{R}^{n+1}$ *and evolutionary dynamics* $h : S^n \to \mathbb{R}^{n+1}$ *be given. Let furthermore* $C(z) = \{i \in I^{n+1} | \ z_i > 0\}$ *for all* $z \in S^n$ *and let* $S^n(S) = \{x \in S^n | \ x_i > 0$ *for all* $i \in S \subset I^{n+1}\}$*. Then, the state* $y \in S^n$ *is a* **truly evolutionarily stable state** *iff*

a. $h(y) = 0^{n+1}$;

b. a nonempty open neighborhood $U \subset S^n(C(y))$ *containing* y *exists such that*

$$\sum_{i \in C(y)} (y_i - x_i) \frac{h_i(x)}{x_i} - \sum_{i \notin C(y)} h_i(x) > 0.$$

- Motivation: Samuelson's critique and GESS too restrictive.

- Intuition: Behavior of the dynamics near TESS are similar to behavior of replicator dynamics near ESS.

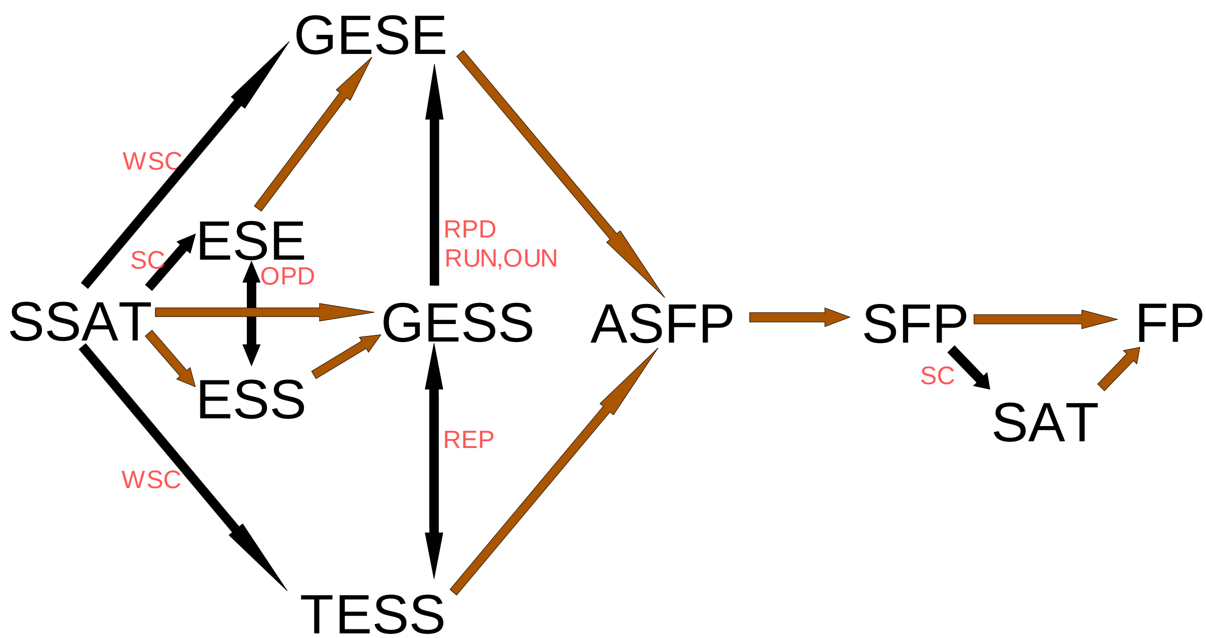

Figure 9: Overview of connections between equilibrium and fixed point concepts.

10 Concluding remarks

- We designed equilibrium concepts which can withstand Samuelson's critique.
  - The GESE generalizes the ESE of Joosten [1996].
  - The TESS generalizes the ESS of Maynard Smith & Price [1978] and GESS of Joosten [1996].
- GESE: monotone convergence in some (generalized) distance function.
- TESS: behavior of dynamics nearby similar to behavior of replicator dynamics near ESS.
- Future research:
  - Examine further connections under classes of dynamics.

- Design new equilibrium notions?
- Set-valued concepts.
- Global stability.

- Huge literature on Samuelson's process and related ones in economics.

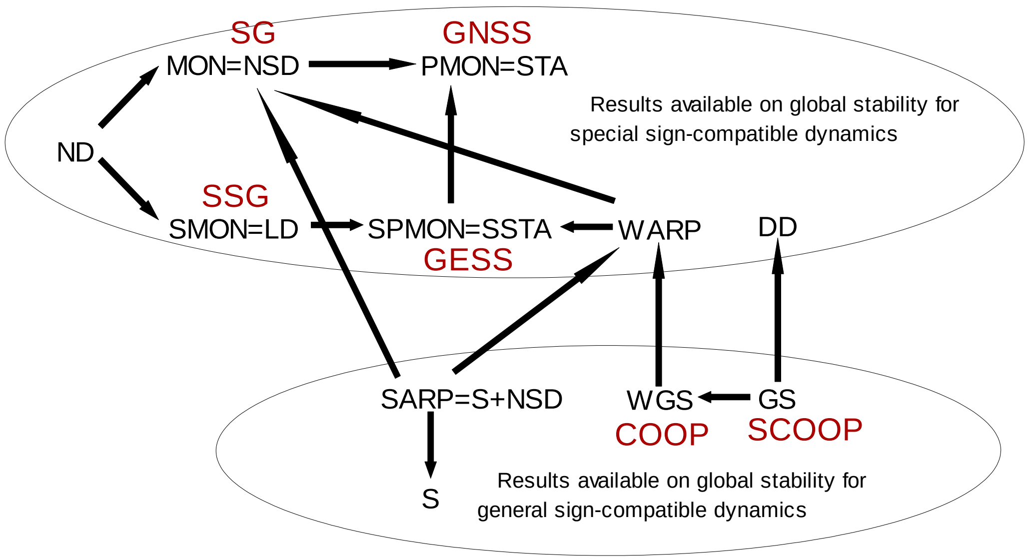

Figure 10: Overview from Joosten [2006].

11 Generalized projection dynamics

Here, we define some $g : S^n \to \mathbb{R}^{n+1}$. Dynamics induced by g in two variants:

$$
\begin{aligned}
\overset{.}{x}{}^{r}_{g} &= \left[g(x) - \left(\sum_{i=1}^{n+1} g_i(x) \right) x \right], \\
\overset{.}{x}{}^{o}_{g} &= \left[g(x) - \left(\frac{1}{n+1} \sum_{i=1}^{n+1} g_i(x) \right) i \right].
\end{aligned}
$$

- Non-negativity implies 'nice' boundary behavior.

- g weakly compatible, then ray-projection dynamics weakly compatible as well (orthogonal ???).

- g sign-compatible, then ray-projection dynamics weakly compatible (orthogonal ???).

- Replicator dynamics: ray and orthogonal projection.

- Best-response dynamics: ray.

- Brown-von Neumann dynamics: ray.

- Generalizations of the latter: ray.

- 'Logit-type' dynamics (Fudenberg & Levine [1998], Cabrales & Sobel [1992], Björnerstedt & Weibull [1996]): ray.

- For every function a 'cousin' is generated.