Flexing the Default Barrier

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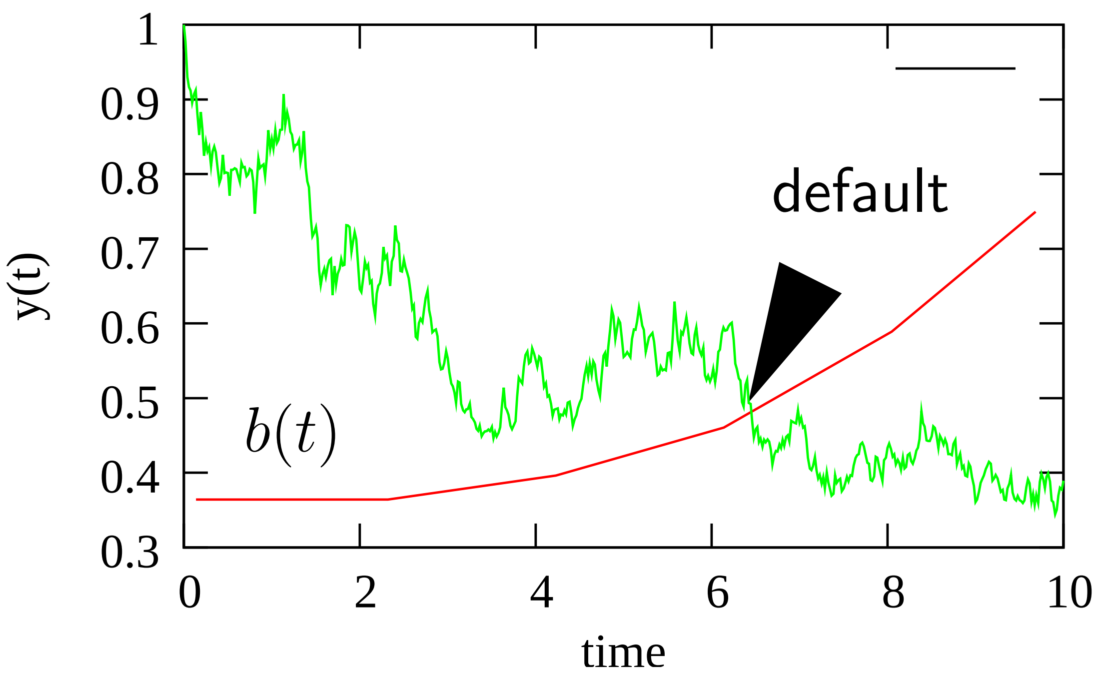
Structural Models



- ✔ Model value of assets as stochastic process y_t
- ✔ When asset value hits barrier the corporation defaults
- ✔ Quantity of interest is the *survival probability*

Structural Models cont.

- ✔ Related Models
  - ✘ Black and Cox (1976)
  - ✘ Brigo and Tarenghi (2005)
- ✔ Asset value follows Geometric Brownian Motion
- ✔ Default Barrier has specific functional form for tractability
- ✔ Closed-form expressions for survival probabilities even with time-dependent drift and diffusion coefficients

Our Contribution to Literature

Our model grants:

- ✔ Complete freedom in the shape of the default barrier
- ✔ Consistency with observable quantities (equity vol)
- ✔ Numerical stability in efficient numerical procedure
- ✔ Sequential calibration to leverage – equity – interest rate – credit
- ✔ Extendibility to inaccessible model

Our model is relevant for:

- ✔ Determination of market-implied expected default barrier
- ✔ Pricing under counterparty risk

Model Setup

- ✔ From SDE:

$$\frac{dy_t}{y_t} = (r(t) - q(t))\,dt + \sigma(t)\,dB_t\,, \qquad y_0 = 1,$$

- ✔ We look at the first passage time of $y(t)$ to some time-dependent barrier $b(t)$:

$$\tau = \inf\{t \geq 0 : y_t \leq b(t)\}\,.$$

- ✔ Survival probability is not known for general $b(t)$

Natural Barrier

✔ Survival probability is known for *natural* barrier

$$b^\diamond(s) = \exp\left\{\frac{\tilde{\alpha}(t,T)\,\tilde{\beta}(t,s) - \tilde{\alpha}(t,s)\tilde{\beta}(t,T)}{\tilde{\alpha}(t,T)}\right\}\left(\frac{d}{c}\right)^{\frac{\tilde{\alpha}(t,s)}{\tilde{\alpha}(t,T)}} c$$

for $t \le s \le T$, however, where we define

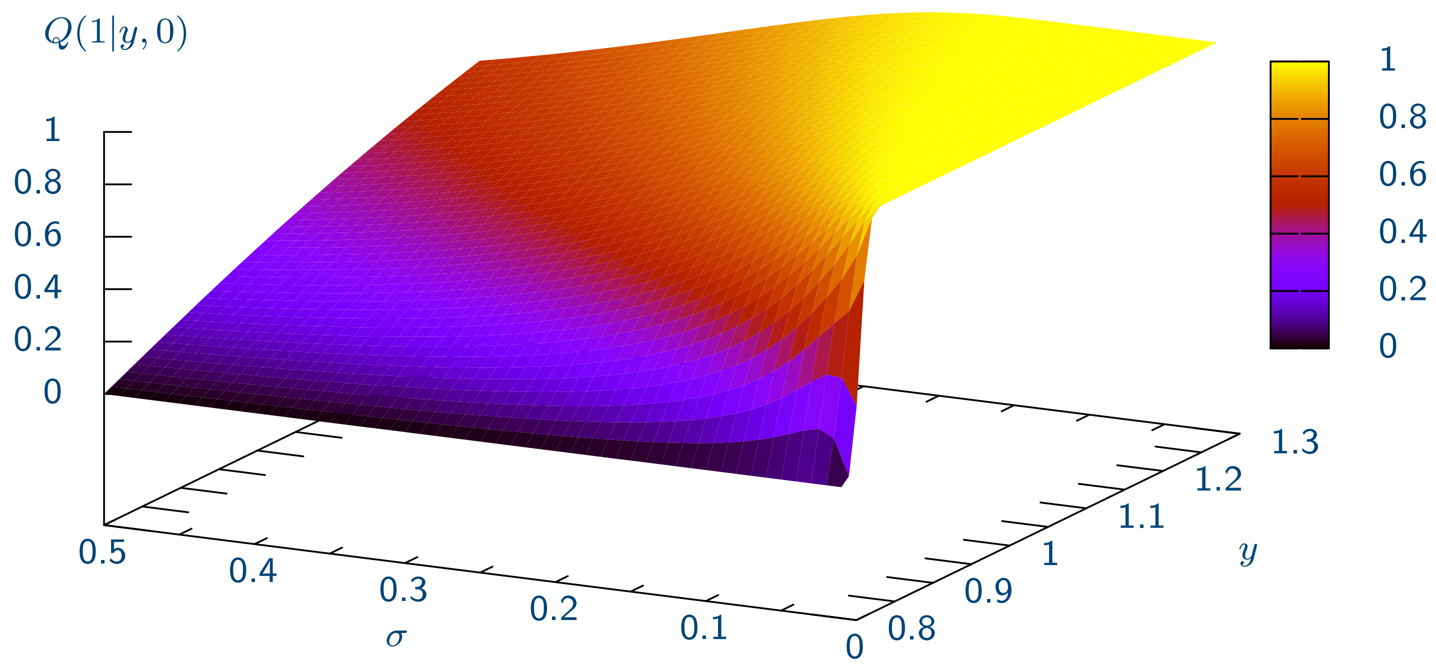
$$\tilde{\alpha}(t,T) := \int_t^T \alpha(s)\,ds,\ \tilde{\beta}(t,T) := \int_t^T \beta(s)\,ds$$

and $\tilde{r}(t,T) := \int_t^T r(s)\,ds$

for $\beta(s) := r(s) - q(s)$, $\alpha(s) = \sigma^2(s)/2$, $c = b^\diamond(t), d = b^\diamond(T)$

✔ Green function g^+ solves the survival probability

$$Q(T|y_0,0) := \mathbb{Q}(\tau > T) = \int_d^\infty g^+(x,T|y_0,0)\,dx\,.$$



Approximation to General Boundary

- ✔ g^+ available in closed-form
- ✔ *Idea:* develop approximation of general $b(t)$ using

$$b(t) \approx \sum_{i=1}^{n} b_i^\diamond(t)\, \mathbb{1}_{[t_{i-1},t_i]}(t) \quad \text{for} \quad 0 \leq t \leq T$$

- ✔ Solve a sequence of feasible $b_i^\diamond$ problems instead of the infeasible b problem

Proposition 1

$\mathbb{Q}[y_t > b(t), \forall t \in [0,T]] = \mathbb{Q}[\bar{B}_t > \bar{b}(t), \forall t \in [0,T]]$, *where*

$$\bar{b}(t) = \frac{1}{\sqrt{\frac{2}{t}\tilde{\alpha}(0,t)}} \left(\ln \frac{b(t)}{y_0} - \tilde{\beta}(0,t) + \tilde{\alpha}(0,t) \right), \; 0 \leq t \leq T$$

Proposition 2 *The approximation quality can be bounded*

1. *for the boundary itself:*

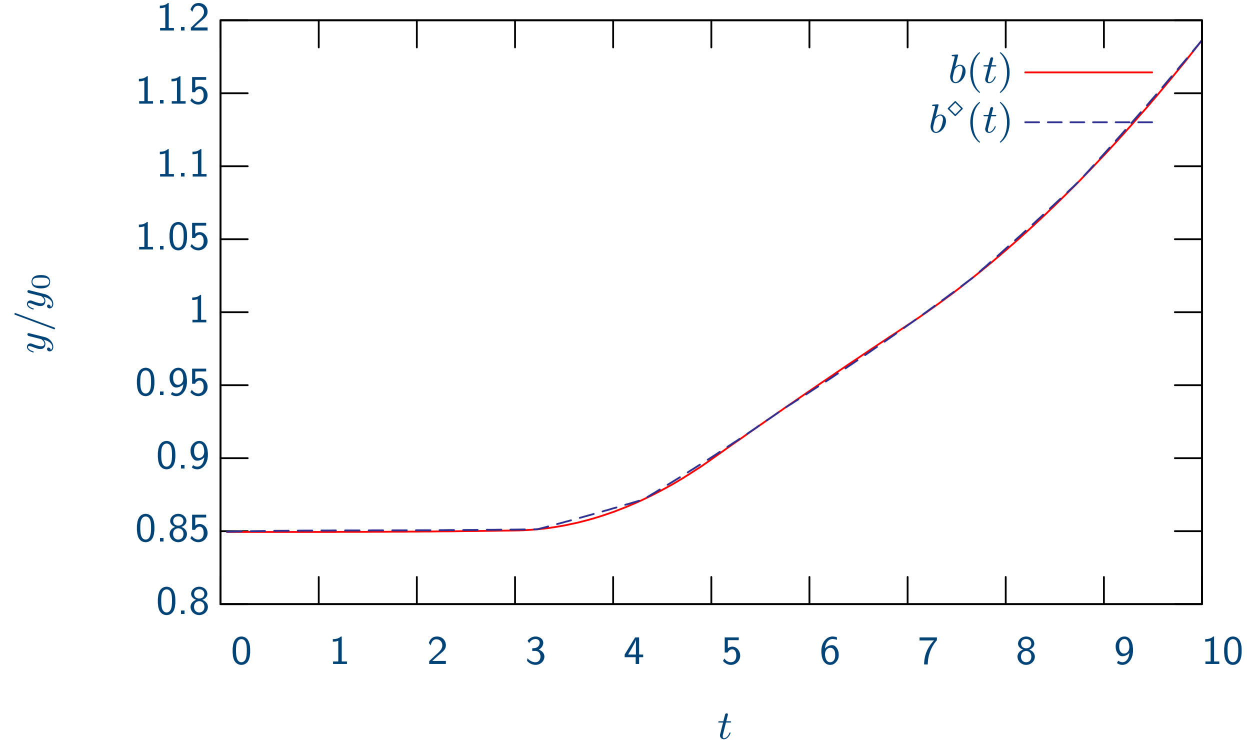
$$|\bar{b} - \bar{b}^{\diamond}|_2^2 \;\leq\; \sum_{i=1}^{n} (t_i - t_{i-1})^2 \, |\bar{b}' - (\bar{b}_i^{\diamond})'|_2^2 \, ,$$

2. *for the survival probability over* $[0, T]$*:*

$$\begin{aligned} &|\mathbb{Q}[y_t > b(t), \forall t \in [0,T]\,] - \mathbb{Q}[y_t > b^{\diamond}(t), \forall t \in [0,T]\,]\,| \\ &\qquad \leq \sqrt{\frac{2}{\pi}}\, |\bar{b}' - (\bar{b}^{\diamond})'|_2 . \end{aligned}$$

Sequence of Feasible Problems

- ✔ Fix start and end points at prescribed times t_i , i.e. $b_i^\diamond(t_{i-1}) = b(t_{i-1})$ and $b_i^\diamond(t_i) = b(t_i)$ such that $\int_{t_{i-1}}^{t_i} (b_i^\diamond(t) - b(t))^2 \, dt \leq \varepsilon^2$, $i = 1, \ldots, n$



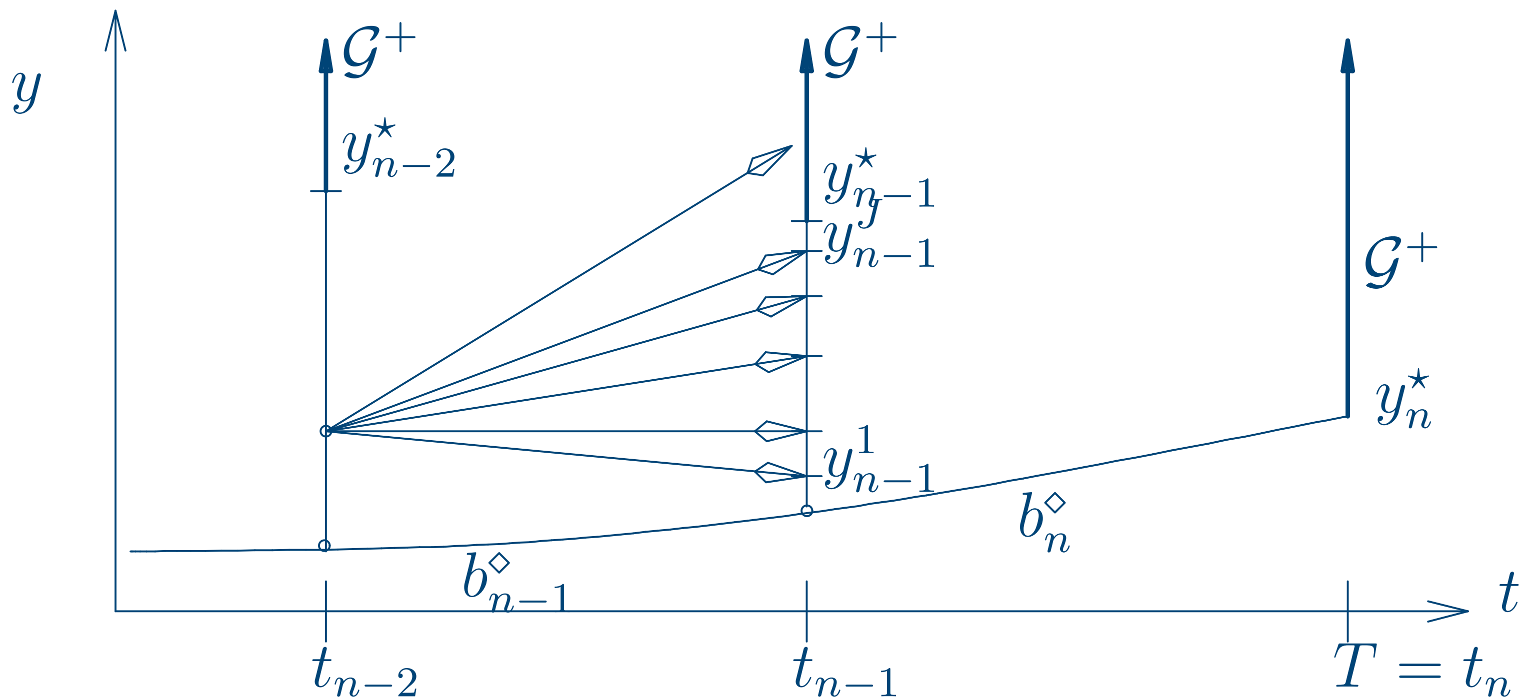
Recursion

- ✔ We can approximate the b problem to arbitrary precision backward-propagating the sequence of $b_i^{\diamond}$ problems
- ✔ At an intermediate point in time the survival probability up to time t_n is given by

$$\begin{aligned} Q(t_n|y,t_i) &= \int_{b(t_{i+1})}^{\infty} Q(t_n|x,t_{i+1})\, g^{+}(x,t_{i+1}|y,t_i)\, dx \\ &\approx \int_{b(t_{i+1})}^{y^{\star}_{i+1}} Q(t_n|x,t_{i+1})\, g^{+}(x,t_{i+1}|y,t_i)\, dx \\ &\quad + \mathcal{G}^{+}(y^{\star}_{i+1},t_{i+1}|y,t_i)\,, \text{where} \end{aligned}$$

$$\mathcal{G}^{+}(y^{\star}_{i+1},t_{i+1}|y,t_i) := \int_{y^{\star}_{i+1}}^{\infty} g^{+}(x,t_{i+1}|y,t_i)\, dx$$

Closed-Form Recursive Backward Solution



- ✔ Everything available in closed-from $\Rightarrow$ *very fast*, algorithm converges

Credit Default Swap (CDS)

Through CDS protection against default can be bought/sold

$$V^{\text{fix}}(T) = \sum_{i=1}^{N} (T_i - T_{i-1}) P(T_i|0) \mathbb{Q}\left[\tau > T_i\right] + \int_0^T (s - T_{I(s)}) P(s|0) d\mathbb{Q}[\tau \leq s]$$

$$V^{\text{def}}(T) = (1 - R^{\mathbb{Q}}) \int_0^T P(s|0) d\mathbb{Q}[\tau \leq s].$$

$$s(T) = \frac{V^{\text{def}}(T)}{V^{\text{fix}}(T)}.$$

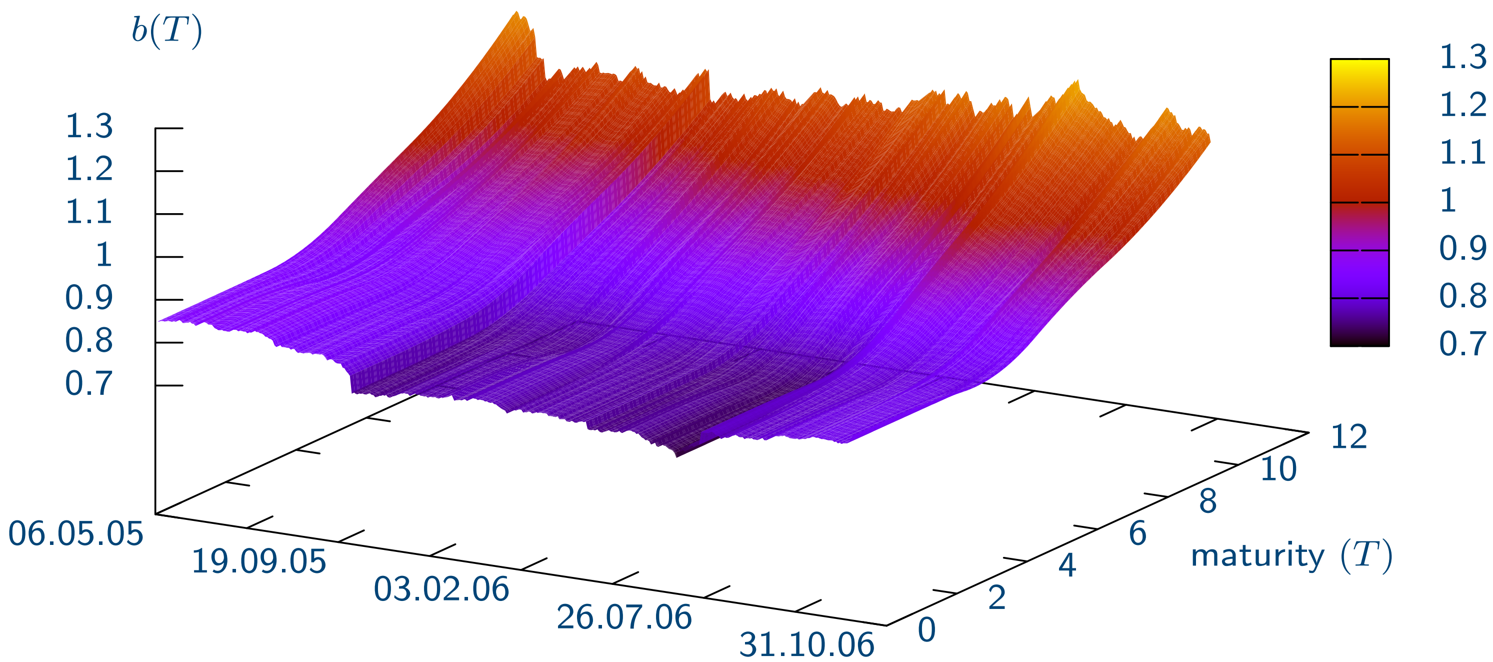
Implied Volatility, Zero Yields and Capital Structure

- ✔ We observe equity implied volatilities and estimates of the unconditional equity volatility
- ✔ We observe term-structure of zero yields
- ✔ We observe estimates of debt-to-asset ratios
- ✔ Putting everything together we get estimates for instantaneous asset volatilities and instantaneous short rates as functions of time
- ✔ We fit Nelson-Siegel polynomials to market data

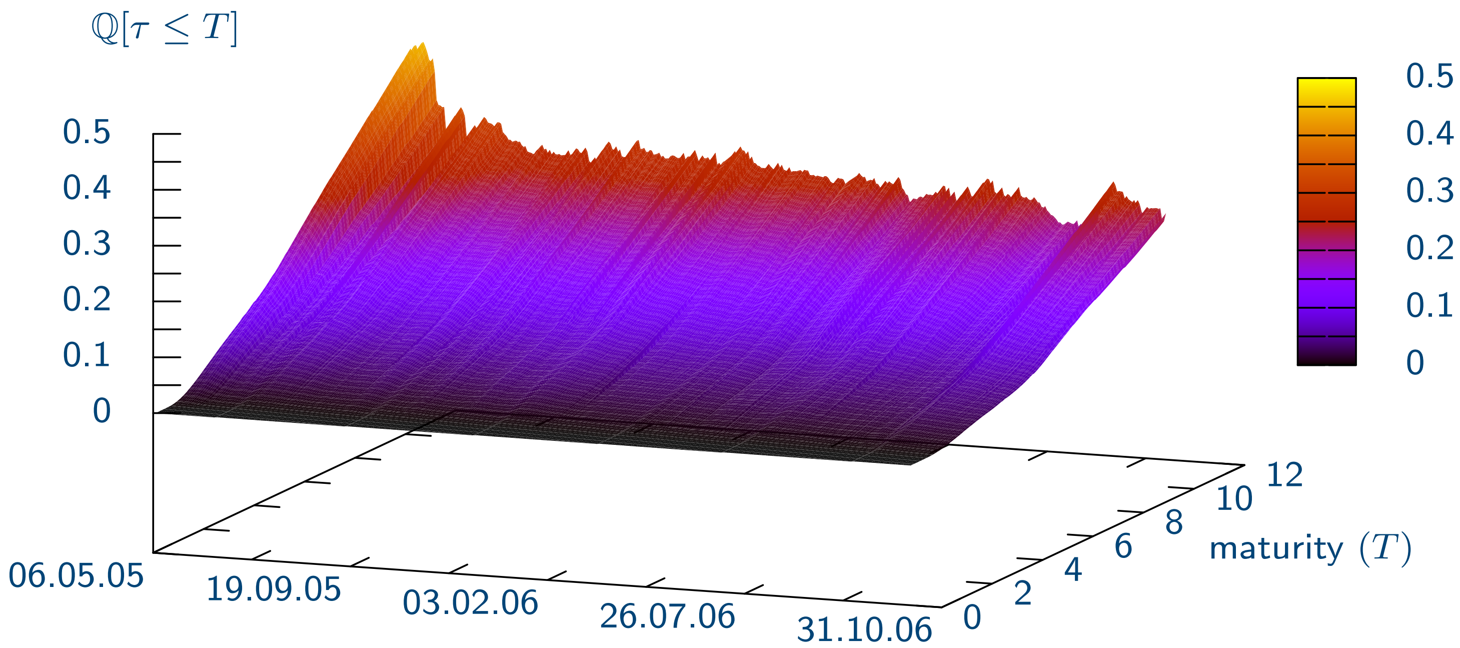
Implied Default Barrier

- ✔ There are 5 canonical maturities for CDS contracts, 1y, 3y, 5y, 7y, 10y
- ✔ Barrier can be specified as function of 5 parameters
- ✔ Barrier is smooth function of time, and specific functional form admits sequential calibration, starting from 1y CDS
- ✔ All 5 liquid CDS contracts are fitted exactly

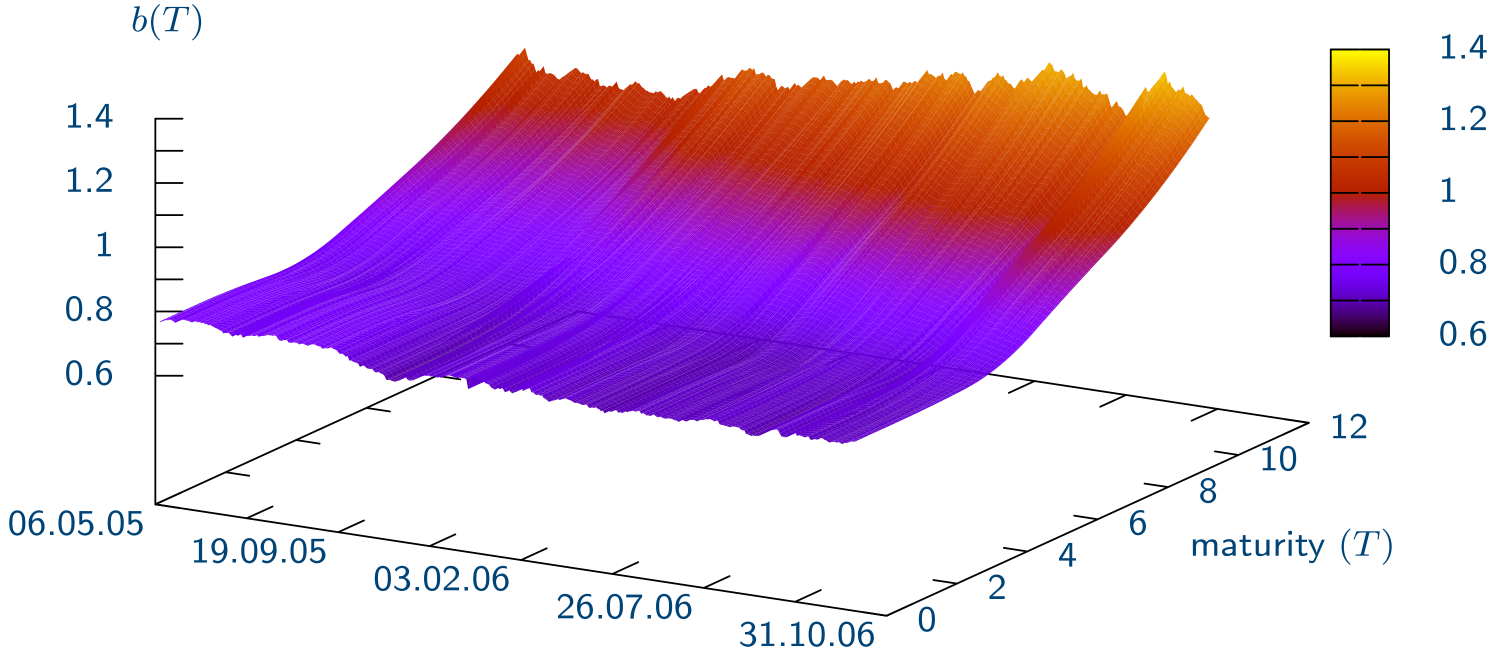
Daimler Chrysler Implied Default Barrier



Daimler Chrysler Survival Probabilities



Telecom Italia Implied Default Barrier



Conclusion and Prospects

- ✔ What we have done so far:
  - ✘ General structural model that nests Black and Cox (1976) as well as Brigo and Tarenghi (2005)
  - ✘ Green function approach to solve the model; fast and efficient numerical procedure
  - ✘ Prices are fitted via changing the shape of the default barrier
- ✔ What is possible (future research):
  - ✘ Extension to jump-to-default model
  - ✘ Explanation of CDS premia written on investment-grade obligors