Convergence Rates
of
Sparse Tensor FEM for elliptic sPDEs

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1. Monte Carlo for Elliptic BVP with stochastic data
2. Karhunen-Loève (KL) Expansion
3. Infinite - Dimensional Parametric Deterministic BVP
4. Stochastic Regularity
5. Convergence Rates of sGFEM - analytic Covariance
6. Convergence Rates of sGFEM - $H^{t,t}_{pw}(D \times D)$ Covariance
7. Number Theory
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Problem Formulation

$$D \subset \mathbb{R}^d \quad \text{bounded, Lipschitz,} \quad d = 2, 3.$$

Consider (model) deterministic Boundary Value Problem (dBVP):

given

$$a \in L^\infty(D), \quad \operatorname*{ess\,inf}_{x \in D} a(x) \geq a_0 > 0, \quad f \in H^{-1}(D) = (H_0^1(D))',$$

find $u \in H_0^1(D)$ such that

$$b(u,v) := \int_D a(x) \nabla_x u \cdot \nabla_x v dx = \int_D f(x) v(x) dx \text{ in } D \quad \forall v \in H_0^1(D). \tag{1}$$

Existence, Uniqueness, Regularity, AFEM, ...:

What to do if $a(x)$ is "uncertain" ?

- Accurate numerical solutions for *one* $a(x)$ are of limited use.
- **Assume** complete statistics (joint pdf's) on data $a(x)$ available.
- Reformulate (1) as sPDE.
- Reconsider numerical solution methods for (1):
  - **Given** statistics (law) of random input data $a(\omega, x)$ (KL-expansion)
  - **compute** statistics (law) of random solution $u(\omega, x)$ ('gPC'-expansion)
  - **sampling** vs. **parsimonious param. representation of joint pdf's of** $u(\omega, x)$
  - trade **randomness** for **high-dimensionality**

Elliptic BVP with stochastic data

Given:

- probability space (Ω, Σ, P) on data space $X(D) \subseteq L^\infty(D)$, $V \subseteq H^1(D)$,
- random diffusion coefficient $a(x,\omega) \in L^\infty(\Omega, dP; X(D))$,
- deterministic source term $f \in H^{-1}(D) = (H_0^1(D))'$,

(sBVP) Find $u(x,\omega) \in L^2(\Omega, dP; H_0^1(D))$ such that

$$\mathbb{E}\left[\int_D a(x,\cdot)\nabla_x u(x,\cdot) \cdot \nabla_x v(x,\cdot)dx\right] = \mathbb{E}\left[\int_D f(x)v(x,\cdot)dx\right]$$

for all $v \in L^2(\Omega, dP; H_0^1(D))$

$a \in L^\infty(\Omega, dP; X(D))$ and essinf$a(\cdot,\cdot) \geq a_0 > 0 \Rightarrow \exists u \in L^2(\Omega, dP; H_0^1(D))$.

Monte Carlo

Sampling (sBVP): Each 'sample' $=$ 1 deterministic BVP

1. Generate (in parallel) N_Ω data "samples" $\{a_j(x)\}_{j=1}^{N_\Omega}$,

2. Solve (exactly and in parallel) the N_Ω dBVPs

$$-\nabla_x \cdot (a_j(x)\nabla_x u_j) = f \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D$$

3. Estimate k-**point correlations** $\mathcal{M}^k u$ from N_Ω solution samples $\{u_j(x)\}_{j=1}^{N_\Omega}$ ($k = 1$: estimate mean field $\mathbb{E}[u]$ from sample average $\mathbb{E}^{N_\Omega}[u]$).

Theorem 0: Assume $u \in L^\alpha(\Omega, V)$ for some $\alpha \in (1,2]$ with $V = H_0^1(D)$.

Then ex. $C(\alpha)$ such that for every $N_\Omega \geq 1$ and every $0 < \varepsilon < 1$

$$P\left\{\|\mathbb{E}[u] - \mathbb{E}^{N_\Omega}[u]\|_V \leq C\varepsilon^{-1/\alpha} N_\Omega^{-(\alpha-1)/2} \|u\|_{L^\alpha(\Omega,V)}\right\} \geq 1 - \varepsilon$$

Karhunen-Loève expansion

- separation of deterministic and stochastic variables -

Proposition 1 (Karhunen-Loève)
If $a \in L^2(\Omega, dP; L^\infty(D))$ then in $L^2(\Omega, dP; L^2(D))$,

$$a(x,\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1} \psi_m(x)Y_m(\omega)$$

- $(\lambda_m, \varphi_m)_{m\geq 1}$ eigensequence of **covariance operator**

$$\mathcal{C}[a] : L^2(D) \to L^2(D) \quad (\mathcal{C}[a]v)(x) := \int_D C_a(x,x')v(x')dx' \quad \forall v \in L^2(D),$$

- $$C_a(x,x') = \mathbb{E}\left[\big(a(x,\cdot) - \mathbb{E}[a](x)\big)\big(a(x',\cdot) - \mathbb{E}[a](x')\big)\right]$$

- $$Y_m(\omega) := \frac{1}{\sqrt{\lambda_m}} \int_D (a(x,\omega) - \mathbb{E}[a](x))\varphi_m(x)dx : \Omega \to \Gamma_m \subseteq \mathbb{R} \quad m = 1, 2, \ldots$$

Karhunen-Loève expansion

- convergence -

$$a(x,\omega) = E_a(x) + \sum_{m\geq 1} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega)$$

KL expansion converges in $L^2(D\times\Omega)$, not necessarily in $L^\infty(D\times\Omega)$

To ensure $L^\infty(D\times\Omega)$ convergence, must

- estimate decay rate of KL eigenvalues λ_m: Schwab & Todor JCP (2006),

- bound $||\varphi_m||_{L^\infty(D)}$: Todor Diss ETH (2005), SINUM (2006)

- **assume**: bounds for $||Y_m||_{L^\infty(\Omega)}$

Karhunen-Loève expansion

- eigenvalue estimates -

Regularity of C_a ensures decay of KL-eigenvalue sequence $(\lambda_m)_{m\geq 1}$

$C_a(x,x') : D \times D \to \mathbb{R}$ is

- **piecewise analytic on** $D \times D$ if ex. **smoothness partition** $\mathcal{D} = \{D_j\}_{j=1}^J$ of D into a finite sequence of simplices D_j such that

$$\overline{D} = \bigcup_{j=1}^{J} \overline{D_j} \tag{2}$$

  and $C_a(x,x')$ analytic in open neighbourhood of $\overline{D_j} \times \overline{D_{j'}}$ for any (j,j').

- **piecewise** $H^{t,t}$ **on** $D \times D$ if

$$V_a \in H^{t,t}_{pw}(D \times D) := \bigcap_{i,j \leq J} L^2(D_i, H^t(D_j))$$

Karhunen-Loève expansion

- eigenvalue estimates -

- $(H, \langle \cdot, \cdot \rangle)$ Hilbert space,
- $\mathcal{C} \in \mathcal{K}(H)$ compact, s.a.,
- eigenpair sequence $(\lambda_m, \phi_m)_{m \geq 1}$.

If $\mathcal{C}_m \in \mathcal{B}(H)$ is any operator of rank at most m,

$$\lambda_{m+1} \leq \|\mathcal{C} - \mathcal{C}_m\|_{\mathcal{B}(H)}. \tag{3}$$

(e.g. Pinkus 1985: n-widths in Approx. Theory).

Karhunen-Loève expansion

- eigenvalue estimates -

Proposition 2 (KL-eigenvalue decay)

- ($>$ exponential KL decay: *Gaussian* $C_a(x,x')$)

$$C_a(x,x') := \sigma^2 \exp(-\gamma|x-x'|^2) \Longrightarrow \quad 0 \le \lambda_m \le c(\gamma,\sigma)/m! \quad \forall m \ge 1$$

- (exponential KL decay: *Piecewise analytic* $C_a(x,x')$)

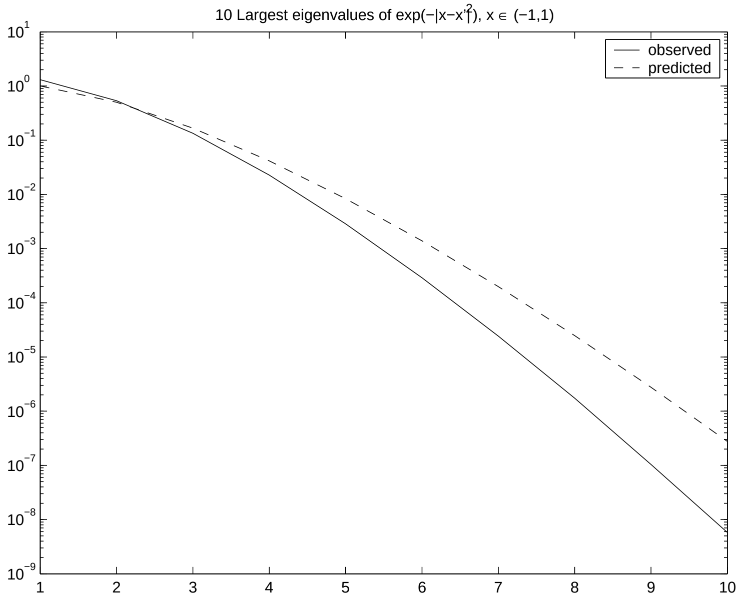
$$C_a \text{ pw analytic on } D \times D \Longrightarrow \exists c > 0 \quad 0 \le \lambda_m \le c \exp(-bm^{1/d}) \quad \forall m \ge 1$$

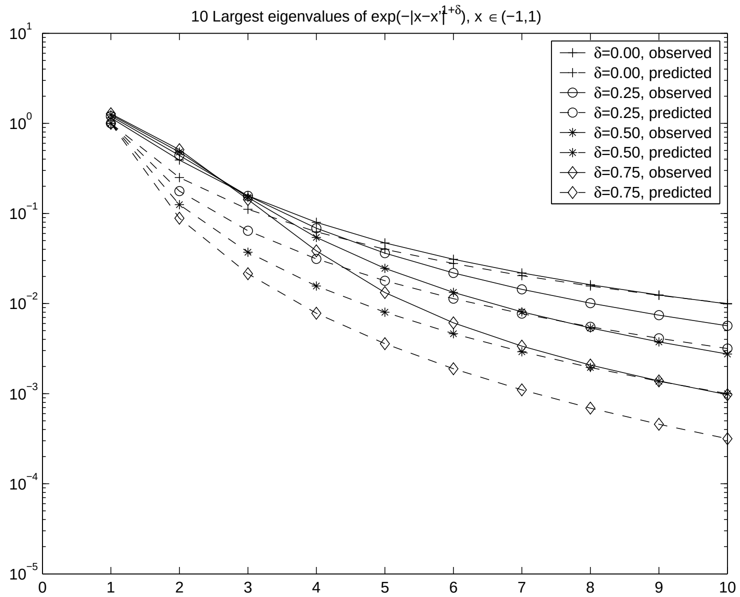
- (Algebraic KL-eigenvalue decay for p.w. $H^t(D)$-kernels)

$$C_a \in H^{t,t}_{pw}(D \times D) \ (t \ge d/2) \Longrightarrow \quad 0 \le \lambda_m \le cm^{-t/d} \quad \forall m \ge 1$$

Karhunen-Loève expansion

- eigenvalue estimates -





Karhunen-Loève expansion

- eigenfunction estimates -

Regularity of C_a ensures L^∞ bounds for L^2-scaled eigenfunctions $(\varphi_m)_{m\geq 1}$

Theorem 3 (Schwab & Todor JCP 2006)
Assume

$$C_a \in H^{t,t}_{pw}(D \times D) \quad \text{for} \quad t > d.$$

Then

$$\forall \delta > 0 \quad \text{ex.}\, C(\delta) > 0 \quad \text{s.t.} \quad \forall m \geq 1 : \qquad \|\varphi_m\|_{L^\infty(D)} \leq C(\delta)\lambda_m^{-\delta}.$$

Hence:

$$b_m := \frac{\lambda_m^{1/2}\|\varphi_m\|_{L^\infty(D)}}{\inf_{x\in D}\mathbb{E}[a]} \leq C(\delta)\lambda_m^{1/2-\delta} \leq C(\delta)m^{-t/2d-\delta}$$

Karhunen-Loève expansion

- convergence rate -

Conclusion:
KL expansion of

$$a(x,\omega) \in L^2(\Omega, dP; L^\infty(D))$$

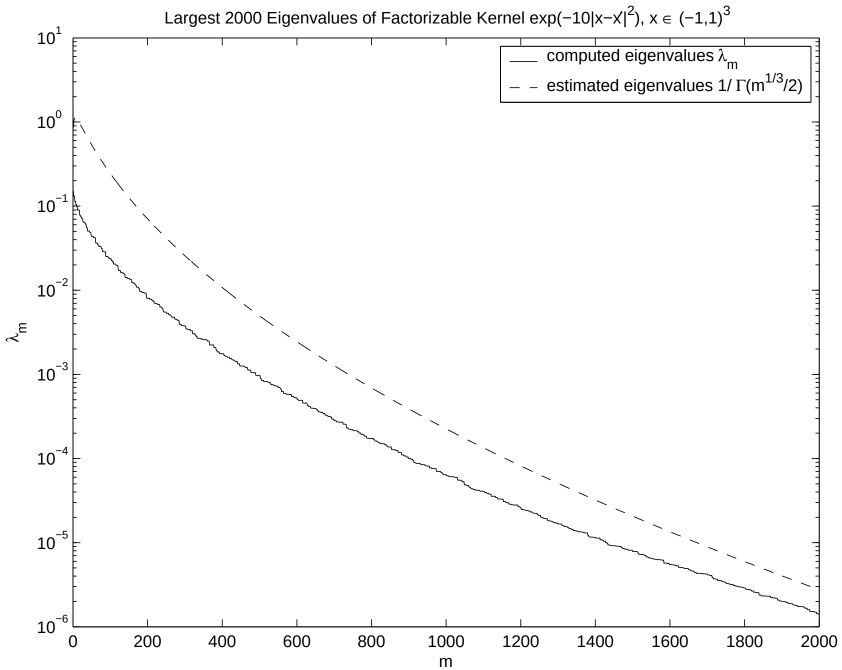
converges **uniformly and exponentially** on $D \times \Omega$ if

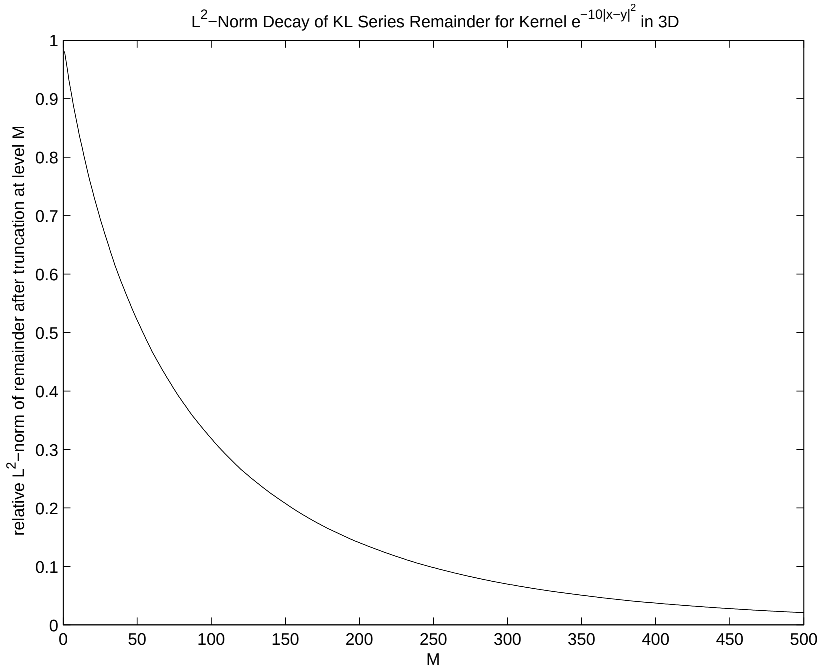
- $C_a(x,x')$ piecewise analytic

- $(Y_m(\omega))_{m\geq 1}$ uniformly bounded on Ω (e.g. $Y_m(\omega) \sim \mathcal{U}(-1,1)$)

Karhunen-Loève expansion

- convergence rate -





Karhunen-Loève expansion

- truncation from infinite to finite dimension M -

$\infty > M \in \mathbb{N}$ KL-truncation order

$$a_M(x,\omega) := \mathbb{E}[a](x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x) Y_m(\omega)$$

$(SBVP)$ with stochastic coefficient $a(x,\omega)$

$$-\text{div}(a(x,\omega)\nabla_x u(x,\omega)) = f(x) \quad \text{in } L^2(\Omega, dP; H^{-1}(D))$$

$(SBVP)_M$ with truncated stochastic coefficient $a_M(x,\omega)$

$$-\text{div}(a_M(x,\omega)\nabla_x u_M(x,\omega)) = f(x) \quad \text{in } L^2(\Omega, dP; H^{-1}(D))$$

Theorem 4 If C_a pw analytic and $(Y_m)_{m\geq 1}$ uniformly bounded, then $\forall \delta > 0$ ex. $b, C(\delta), M_0 > 0$ such that $(SBVP)_M$ well-posed for $M \geq M_0$ and

$$\|u - u_M\|_{L^2(\Omega; H_0^1(D))} \leq \begin{cases} C\exp(-bM^{1/d}) & \forall M \geq M_0 \quad \text{if } C_a \text{ pw analytic} \\ C(\delta) M^{-t/2d+1-\delta} & \forall M \geq M_0 \quad \text{if } C_a \in H_{pw}^{t,t}(D \times D) \end{cases}$$

High dimensional deterministic bvp

$$a_M : D \times \Omega \to \mathbb{R}, \quad a_M(x,\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)Y_m(\omega)$$

Assumption
$(Y_m)_{m\geq 1}$ independent, uniformly bounded family of rv's
(e.g. Y_m uniformly distributed in $\Gamma_m = I = (-1/2, 1/2)$, $m = 1, 2, 3, ...$)

$$\begin{aligned} \text{Random variable } Y_m \quad &\longrightarrow \quad \text{Parameter } y_m \in I \\ (Y_1, Y_2, \ldots, Y_M) \quad &\longrightarrow \quad y = (y_1, y_2, \ldots, y_M) \in I^M \\ dP \quad = \rho(y)dy \quad &= \bigotimes_{m\geq 1} \rho_m(y_m)dy_m \end{aligned}$$

$$\tilde{a}_M : D \times I^M \to \mathbb{R}, \quad \tilde{a}_M(x,y) = \mathbb{E}[a](x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)y_m$$

High dimensional deterministic bvp

stochastic bvp

$$-\mathrm{div}(a_M(x,\omega)\nabla_x u_M(x,\omega)) = f(x) \quad \text{in } H^{-1}(D), \quad P-\text{a.e. } \omega \in \Omega$$

M-dimensional, parametric deterministic bvp

$$-\mathrm{div}(\tilde{a}_M(x; y_1, y_2, ..., y_M)\nabla_x \tilde{u}_M(x,y)) = f(x) \quad \text{in } H^{-1}(D), \quad \forall y \in I^M$$

Proposition 5(Doob & Dynkin)
Under **Assumption**, the parametric deterministic bvp is well-posed and

$$u_M(x,\omega) = \tilde{u}_M(x, Y_1(\omega), Y_2(\omega), \ldots, Y_M(\omega))$$

High dimensional deterministic bvp

- stochastic semi-discretization -

$$\tilde{a}_M(x,y) := E_a(x) + \sum_{m\geq 1}^{M} \sqrt{\lambda_m}\varphi_m(x)y_m$$

M-dimensional, parametric deterministic bvp: find
$\tilde{u}_M \in L^2_\rho(I^M; H^1_0(D)) \simeq L^2_\rho(I^M) \otimes H^1_0(D) = S \otimes V$: $\forall v \in L^2_\rho(I^M; H^1_0(D))$ holds

$$\int_{I^M}\left(\int_D \tilde{a}_M(x,y)\nabla_x \tilde{u}_M(x,y)\cdot\nabla_x v(x,y)dx\right)\rho(y)dy = \int_{I^M}\int_D f(x)v(x)dx\rho(y)dy$$

Galerkin semi-discretization in y (sGFEM):

$$S^M \subset L^2_\rho(I^M), \quad \hat{N}_\Omega := \mathrm{dim} S^M < \infty \quad dBVPs$$

find $\tilde{U}_M \in S^M \otimes H^1_0(D)$ such that $\forall v \in S^M \otimes H^1_0(D)$:

$$\int_{I^M}\left(\int_D \tilde{a}_M(x,y)\nabla_x \tilde{U}_M(x,y)\cdot\nabla_x v(x,y)dx\right)\rho(y)dy = \int_{I^M}\int_D f(x)v(x)dx\rho(y)dy$$

sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization -

Quasi-Optimality:

$$\|u - \tilde{U}_M\|_{L^2_\rho(H^1_0)} \leq C \inf_{v \in S^M \otimes H^1_0(D)} \|u - v\|_{L^2_\rho(H^1_0)}$$

$\tilde{a}_M(x,y)$ affine in $y \Rightarrow \tilde{u}_M(x,y)$ analytic in $y \Rightarrow S^M$polyn. space w.r.to y

task: solve dbvp with KL-accuracy* $O(\exp(-cM^{1/d}))$ in "low complexity"**

*how to choose the polynomial space in $y = (y_1, y_2, \ldots, y_M)$?

**how to choose a basis $\mathcal{B}$ of $\mathcal{P}$?

sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization: p.w. analytic C_a -

'ANOVA' type Tensor Product Spaces in I^M:

For $M, \mu \geq 0$, $1 \leq \nu << M \in \mathbb{N}_0$ define index sets

$$\Lambda^M_{\mu,\nu} := \{\alpha \in \mathbb{N}_0^M \mid |\alpha|_1 \leq \mu, \quad |\alpha|_0 \leq \nu\} \subset \mathbb{N}_0^M, \tag{4}$$

polynomial subspaces (N. Wiener (1938))

$$\mathbf{y}^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_m^{\alpha_m} \ldots, \quad \mathcal{L}_\alpha(y) = L_{\alpha_1}(y_1) L_{\alpha_2}(y_2) .. \quad \alpha \in \Lambda$$

$$S^M = \mathcal{P}(\Lambda^M_{\mu,\nu}) := \text{span}\{\mathbf{y}^\alpha \mid \alpha \in \Lambda^M_{\mu,\nu}\} \subset L^2(I^M), \tag{5}$$

$$S^M = \mathcal{L}(\Lambda^M_{\mu,\nu}) := \text{span}\{\mathcal{L}_\alpha(y) \mid \alpha \in \Lambda^M_{\mu,\nu}\} \subset L^2(I^M), \tag{6}$$

sGFEM for high dimensional deterministic bvp

- stochastic semi-discretization: p.w. analytic C_a -

Theorem 6 (Todor + Sc IMA Journ Numer. Anal. (2007))
If ex. $b, C, \kappa > 0$ s.t.

$$\lambda_m \leq C \exp(-bm^\kappa) \qquad m \to \infty,$$

ex. $c_3, c_4, c_r > 0$ such that for

$$\mu = \lceil c_3 M^\kappa \rceil, \quad \nu = \lceil c_4 M^{\kappa/(\kappa+1)} \rceil \tag{7}$$

holds, as $M \to \infty$ for **polynomial subspace** $\mathcal{P}(\Lambda^M_{\mu,\nu}) \otimes H^1_0(D)$
i. ($\mathcal{P}$): ex. $b, \hat{c} > 0$ s.t.

$$\inf_{v \in \mathcal{P}(\Lambda^M_{\mu,\nu}) \otimes H^1_0(D)} \|\tilde{u}_M - v\|_{L^\infty(I^M; H^1_0(D))} \lesssim \exp(-bM^{1/d})$$

$$N_\Omega := \dim \mathcal{P}(\Lambda^M_{\mu,\nu}) \lesssim \exp(\hat{c} M^{1/(d+1)} \log(M)) \tag{8}$$

ii. sGFEM converges w. **spectral rate**:

$$\forall s > 0 : \quad ex. C(s) \quad s.t. \inf_{v \in \mathcal{P}(\Lambda^M_{\mu,\nu}) \otimes H^1_0(D)} \|\tilde{u}_M - v\|_{L^\infty(I^M; H^1_0(D))} \leq C(s) N_\Omega^{-s}$$

iii. ($\mathcal{B}$): In L^2_ρ-ONbasis of $\mathcal{P}(\Lambda^M_{\mu,\nu})$ the stiffness matrix of $(sBVP)_M$ in I^M is well-conditioned and sparse (at most $O(M)$ nontrivial "entries" / row)

High dimensional dBVP - nonlinear approximation

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Recall: $C_a \in H^{t,t}_{pw}(D \times D) \Rightarrow a(x,\omega) = \mathbb{E}[a](x) + \sum_{m\geq 1} \psi_m(x) Y_m(\omega)$

$$\lambda_m \lesssim m^{-t/d}, \qquad \|\psi_m\|_{L^\infty(D)} = \lambda_m^{1/2} \|\varphi_m\|_{L^\infty(D)} \lesssim m^{-t/2d-\delta} \qquad m = 1, 2, ...$$

KL - convergence rate: if $t > 2d$ then ex. $M_0 > 0$ such that

$$\|u - u_M\|_{L^2(\Omega; H^1_0(D))} \lesssim \|a - a_M\|_{L^\infty(\Omega; L^\infty(D))} \leq CM^{-s} \quad \forall M \geq M_0, \quad 0 < s < t/2d - 1.$$

- Convergence rate of sGFEM of $O(N_\Omega^{-s'})$ possible?
- Which Λ?
- Which $s'(t) > 1/2$?

High dimensional dBVP - nonlinear approximation

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Benchmark:
find $M \leq \infty$ and $\Lambda \subset \mathbb{N}_0^{\mathbb{N}}$ by *nonlinear, best N_Ω-term approximation*.

Notation:

- $\Lambda = \{$all sequences $\alpha = (\alpha_m)_{m=1}^{\infty} \subset \mathbb{N}_0$ w. finite support $\} \subset \mathbb{N}_0^{\mathbb{N}}$

- Λ countable: define

$$\Lambda(M) := \{\alpha : \operatorname{supp} \alpha \subset \{1, \dots, M\}\} \subset \Lambda, \quad M = 1, 2, \dots$$

  Then $\Lambda(M)$ countable and $\Lambda = \bigcup_{M=1}^{\infty} \Lambda(M)$, hence countable.

- $$\alpha \in \Lambda \quad \Rightarrow \quad |\alpha|_0 := \# \operatorname{supp} \alpha < \infty, \quad |\alpha|_1 := \sum_{m \geq 1} \alpha_m < \infty.$$

High dimensional dBVP - nonlinear approximation

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Let

$$y \in U := B_1(\ell_\infty) = (-1,1)^\infty \qquad V = H^1_0(D).$$

Then

$$u = \sum_{\alpha \in \Lambda} u_\alpha y^\alpha = \sum_{\alpha \in \Lambda} c_\alpha \mathcal{L}_\alpha(y) \in L^\infty(U,V)$$

where, for any $\alpha \in \Lambda$,

$$u_\alpha := \frac{1}{\alpha!}(D^\alpha_y u)(0) \in V, \quad c_\alpha := \int_U u(x,y)\mathcal{L}_\alpha(y)\rho(y)dy \in V.$$

High dimensional dBVP - nonlinear approximation

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

For any $\Lambda_0 \subset \Lambda$,

$$\inf_{v \in \mathcal{L}(\Lambda_0) \otimes V} \|u - v\|^2_{L^2_\rho(U,V)} \leq \|u - \sum_{\alpha \in \Lambda_0} c_\alpha \mathcal{L}_\alpha\|^2_{L^2_\rho(U,V)} \leq \sum_{\alpha \notin \Lambda_0^c} \|c_\alpha\|^2_V,$$

$$\inf_{v \in \mathcal{P}(\Lambda_0) \otimes V} \|u - v\|_{L^\infty(U,V)} \leq \|u - \sum_{\alpha \in \Lambda_0} u_\alpha y^\alpha\|_{L^\infty(U,V)} \leq \sum_{\alpha \notin \Lambda_0^c} \|u_\alpha\|_V$$

ℓ^τ**-summability of** $\{\|u_\alpha\|_V : \alpha \in \Lambda\}$**,** $\quad \{\|c_\alpha\|_V : \alpha \in \Lambda\}$ **?**

A-priori estimates of $\|u_\alpha\|_V$, $\|c_\alpha\|_V$ for $\alpha \in \Lambda$.

High dimensional dBVP - Regularity I

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

$$a(x,y) = a_0(x) + \sum_{m=1}^{\infty} y_m \psi_m(x), \qquad y = (y_1, y_2, ...) \in U$$

$$\text{essinf}_{x \in D}\{a_0(x)\} \geq a_{0,min} > 0, \quad \psi_m := \lambda_m^{1/2} \varphi_m(x), \quad m = 1, 2, ...$$

Recall: if $C_a \in H^{t,t}_{pw}(D \times D)$, then for any $0 < s < t/2d$,

$$b_m := \frac{\|\psi_m\|_{L^\infty(D)}}{a_{0,min}}, \quad \tilde{b}_m := \frac{\|\psi_m\|_{L^\infty(D)}}{2a_{min}} \lesssim \lambda_m^{1/2} \|\varphi_m\|_{L^\infty(D)} \lesssim m^{-s}, \quad m = 1, 2, ...$$

Then $\{b_m\}_{m \geq 1} \in \ell^\tau$ for any $1/\tau < t/(2d)$ and,

$$\forall \alpha \in \Lambda : \qquad b^\alpha = \prod_{m \geq 1} b_m^{\alpha_m} < \infty.$$

High dimensional dBVP - Regularity II

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Theorem 7
Let $\{b_m\}$, $\{\tilde{b}_m\}$ be as above and denote

$$\beta_m := b_m m^{1+\delta}, \quad \tilde{\beta}_m := \tilde{b}_m m^{1+\delta}, \quad \delta > 0.$$

Then, for every $\alpha \in \Lambda$,

$$\|u_\alpha\|_V \lesssim \|f\|_{V^*} \begin{cases} \frac{|\alpha|!}{\alpha!} b^\alpha, \\ \\ \beta^\alpha \end{cases} \quad \text{and} \quad \|c_\alpha\|_V \lesssim \|f\|_{V^*} \begin{cases} \frac{|\alpha|!}{\alpha!} \tilde{b}^\alpha \\ \\ \tilde{\beta}^\alpha \end{cases} \quad \alpha \in \Lambda.$$

High dimensional dBVP - Regularity III

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Idea of Proof: consider

$$b = \{b_m\}_{m=1}^{\infty} \in \ell_1, \quad \|b\|_{\ell_1} \leq \gamma < 1,$$

$$y = (y_1, y_2, y_3, ...) \in B_1(\ell_\infty)$$

Then for

$$a(y) = 1 + b_1 y_1 + ... = 1 + \sum_{m=1}^{\infty} b_m y_m$$

we have

$$a(y)u(y) = 1 \quad \Longleftrightarrow u(y) = \frac{1}{a(y)} = \frac{1}{1 + b_1 y_1 + b_2 y_2 + ...}, \quad y \in B_1(\ell_\infty).$$

$a(y)$ is *linear in each* y_i and

$u(y)$ is *continuous in* $y \in B_1(\ell_\infty)$ and *analytic in each* y_i.

Moreover, for every $\alpha \in \Lambda$,

$$D_y^\alpha u(y) = D_y^\alpha [1 + b_1 y_1 + b_2 y_2 + ...]^{-1} = (-1)^{|\alpha|} [a(y)]^{-|\alpha|} |\alpha|! \prod_{m \geq 1} b_m^{\alpha_m}$$

First bound: 'real variable' bootstrap argument.
(Todor and Schwab, IMA Journ. Numer. Anal. 2007)

Second bound: 'several complex variables' argument.

Hartogs' Theorem and Cauchy's inequalities for functions $u(\cdot, z)$ which are *separately analytic w.r. to each* z_i *in the Bernstein ellipses* $\mathcal{E}_{\rho_i}$, $i = 1, 2, ...$, imply $u(\cdot, z)$ are
jointly analytic in $z = (z_1, z_2, ...)$ *in the polycylinders*

$$[-1, 1]^M \subset \mathcal{E}_{\rho_1} \times .... \times \mathcal{E}_{\rho_M} \subset \mathbb{C}^M \qquad \textit{with } \rho_m \sim 1/b_m \to \infty \textit{ as } m \to \infty$$

for *any* M.

High dimensional dBVP - convergence rate of sGFEM I

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Best N_Ω-term approximation:
find finite sets $\Lambda_0 \subset \Lambda$ of monomials y^ν / Legendre polynomials $\mathcal{L}_\nu(y)$, $\nu \in \Lambda_0$, of cardinality $N_\Omega = \#\Lambda_0 \to \infty$, and "coefficients" $c_\mu \in V$ such that

$$\|u - \sum_{\nu \in \Lambda_0} c_\nu \mathcal{L}_\nu\|_{L^2_\rho(U,V)} \leq C(\#\Lambda_0)^{-\sigma(t)},$$

with, hopefully, $\sigma(t) > 1/2$ (MCM).

ok for $\sigma = \frac{1}{\tau} - \frac{1}{2} < \frac{t}{2d} - \frac{1}{2}$ **provided** $\{\|c_\alpha\|_V : \alpha \in \Lambda\} \in \ell^\tau$!

When are

$$\left\{\tilde{\beta}^\nu : \nu \in \Lambda\right\}, \quad \left\{\frac{|\nu|!}{\nu!}\tilde{b}^\nu : \nu \in \Lambda\right\} \in \ell^\tau?$$

High dimensional dBVP - convergence rate of sGFEM II

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Consider for $b = \{b_m\}$ the conditions

$$\|b\|_{\ell^1} = |b_1| + |b_2| + ... < 1, \tag{9}$$

$$\|b\|_{\ell^\infty} = \sup_m \{|b_m|\} < 1, \tag{10}$$

and

$$b \in \ell^\tau \qquad \text{for some} \quad \tau < 1. \tag{11}$$

Proposition 8 (Cohen, DeVore & CS '08)
(10) and (11) $\rightarrow$

$$\{b^\alpha : \alpha \in \Lambda\} \in \ell^\tau,$$

(9) and (11) $\rightarrow$

$$\left\{\frac{|\alpha|!}{\alpha!} b^\alpha : \alpha \in \Lambda\right\} \in \ell^\tau. \tag{12}$$

High dimensional dBVP - convergence rate of sGFEM III

$$C_a \in H^{t,t}_{pw}(D \times D), \quad t < \infty$$

Theorem 9
If

$$C_a \in H^{t,t}_{pw}(D \times D), \quad 2d < t < \infty$$

then the solution $u(x,y) \in L^2(U,V)$ admits the Legendre expansion

$$u(x,y) = \sum_{\alpha \in \Lambda} c_\alpha(x) \mathcal{L}_\alpha(y)$$

and the 'coefficient sequence' $\{c_\alpha : \alpha \in \Lambda\} \subset V$ satisfies

$$\{\|c_\alpha\|_V : \alpha \in \Lambda\} \in \ell^\tau \quad \text{for} \quad 1 < \frac{1}{\tau} < \frac{t}{2d}.$$

There exists a sequence $\{\Lambda_\ell\}_{\ell=1}^\infty$ of finitely supported index sets such that the sGFEM based on $\mathcal{P}(\Lambda_\ell) \otimes V$ converges with rate

$$\|u - u_{\Lambda_\ell}\|_{L^2_\rho(U,V)} \lesssim (\#\Lambda_\ell)^{-\sigma}, \quad 0 < \sigma < \frac{t}{2d} - \frac{1}{2}.$$

Number Theory (finding Λ_ℓ) - I

Problem: Theorem 9 does *not* give a constructive way of finding sets

$$\Lambda_1 \subset \Lambda_2 \subset ... \subset \Lambda_\ell \subset ...\Lambda$$

Two strategies: 1. Adaptive sGFEM, 2. A-priori selection of the Λ_ℓ.

Consider 2:
given $\mu = (\mu_1, \mu_2, ...) > 0$ s.t. $1 > \mu_1 \geq \mu_2 \geq ... \geq \mu_m \to 0$ and $\varepsilon > 0$, define

$$\Lambda_\varepsilon(\mu) := \left\{\alpha \in \mathbb{N}_0^{\mathbb{N}} \quad | \quad \mu^\alpha \geq \varepsilon\right\} \quad = \quad \left\{\alpha \in \mathbb{N}_0^{\mathbb{N}} \quad | \quad \mu^{-\alpha} \leq 1/\varepsilon\right\}$$

Based on [sharp] bounds on coefficients $\|u_\alpha\|_V$, $\|c_\alpha\|_V$,
$\Lambda_\varepsilon(b)$ *will contain essentially the* $\#\Lambda_\varepsilon(b)$ *'largest coefficients'* $\|u_\alpha\|_V$, $\|c_\alpha\|_V$.

Monotonicity: for any $\mu, \bar{\mu} \in \Lambda$ as above and any $\bar{\varepsilon}, \varepsilon > 0$

1. $M_\varepsilon(\mu) := \max_{m\in\mathbb{N}}\{\mu_m \geq \varepsilon\} < \infty$,

2. $\forall \alpha \in \Lambda_\varepsilon(\mu) : \mathrm{supp}(\alpha) \subset \{1, 2, \dots, M_\varepsilon(\mu)\}$,

3. $\bar{\varepsilon} \geq \varepsilon$ implies $\Lambda_{\bar{\varepsilon}}(\mu) \subseteq \Lambda_\varepsilon(\mu)$,

4. $\mu \preceq \bar{\mu}$ implies $\Lambda_\varepsilon(\mu) \subseteq \Lambda_\varepsilon(\bar{\mu})$.

Number Theory (finding Λ_ℓ) - II

Consider *comparison sequences* $\nu_\sigma \in \ell^\infty$ such that

$$b \preceq \nu_\sigma := \{(m+1)^{-\sigma} : m = 1, 2, ...\}, \qquad \sigma > 0.$$

Note:

$$b \preceq \nu_\sigma \quad \Rightarrow \quad \forall \alpha \in \Lambda : \qquad b^\alpha \leq \nu_\sigma^\alpha$$

Scaling: given $\sigma > 0$, find

$$\Lambda_\varepsilon(\nu_\sigma) = \Lambda_{\varepsilon^{1/\sigma}}(\nu_1) = \left\{ \alpha \in \mathbb{N}_0^{\mathbb{N}} : \prod_{m\in\mathbb{N}} (m+1)^{\alpha_m} \leq \varepsilon^{-1/\sigma} \right\}, \quad \varepsilon \to 0.$$

$\alpha \in \Lambda_\varepsilon(\nu_\sigma)$ **iff** α **multiplicative partition of some integer** $x < \varepsilon^{-1/\sigma}$.

Complexity (finding Λ_ℓ)

Proposition 10 (R. Andreev '08)
1. $\Lambda_\varepsilon(\nu_\sigma)$ can be localized in work and memory growing log-linearly in $\#\Lambda_\varepsilon(\nu_\sigma)$.

2. As $\varepsilon \to 0$,

$$\#\Lambda_\varepsilon(\nu_\sigma) \sim x \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}} \quad \text{with} \quad x = \varepsilon^{-1/\sigma},$$

- E.R. Canfield, Paul Erdös and C. Pomerance:
  On a Problem of Oppenheim concerning "Factorisatio Numerorum"
  J. Number Theory **17** (1983) 1 ff.,

- G. Szekeres and P. Turán:
  Über das zweite Hauptproblem der "Factorisatio Numerorum"
  Acta Litt. Sci. Szeged **6** (1933) 143-154.

Sparse Tensor sGFEM - I

Issues:

- Selection of polynomial basis:
  gPC from 1-d polyn. orthog. w.r.to *separable* (prior) $\rho(y)$.

- So far, only semidiscrete approximation:
  Fully Discrete sGFEM needs (A)FEM in D: $c_\mu \to c_\mu^L \in V_L \subset V$.

Sparse Tensor sGFEM - II

Regularity of deterministic problem:

- Smoothness Scale of approximation spaces:

$$V = \mathcal{A}^0 \supset \mathcal{A}^1 \supset \mathcal{A}^2 \supset \mathcal{A}^s ... \supset C^\infty(\overline{D})$$

  Examples (Dirichlet Problem w. random coefficient):

  1. (Isotropic Sobolev Scale, h-FEM in D on quasiuniform meshes)
  $$\mathcal{A}^0 = H_0^1(D), \quad \mathcal{A}^s = H^s(D) \cap H_0^1(D), \quad s > 1.$$
  2. (Weighted Kondrat'ev Scale, h-FEM in D on graded meshes)
  $$\mathcal{A}^0 = H_0^1(D), \quad \mathcal{A}^s = V_\beta^s(D) \cap H_0^1(D), \quad s > 1 \quad \text{integer.}$$
  3. (Besov Scale, h-AFEM in D)
  $$\mathcal{A}^0 = H_0^1(D), \quad \mathcal{A}^s = B_{2,\infty}^s(D) \cap H_0^1(D), \quad s > 1.$$

- Spatial regularity at order $s' > 1$:

$$u(y) = \sum_{\alpha \in \Lambda} y^\alpha u_\alpha, \quad y \in U, \quad u_\alpha \in \mathcal{A}^{s'} \subset V. \tag{13}$$

Sparse Tensor sGFEM - III

Proposition 11 (sGFEM /full tensor approximation):

Assume

1. spatial regularity (13) of order $s' > 1$: $\psi_\mu \in \mathcal{A}^{s'}$,
2. Covariance regularity of order t:
$$C_a \in H^{t,t}_{pw}(D \times D), \quad t > 2d,$$
3. multilevel spatial approximation scale:
$$V_0 \subset V_1 \subset V_2 \subset ... \subset V$$
with uniformly bounded, V-stable and quasioptimal projectors $P_\ell : V \to V_\ell$ and
$$N_{D,\ell} := \text{dim} V_\ell = O(2^{\ell d}), \qquad \ell \to \infty.$$

Then ex. $\{\Lambda_k\}_{k=1}^{\infty}$ with $\Lambda_k \subset \Lambda$, $\#\Lambda_k \sim N_\Omega(k) \to \infty$ and

$$\|u - \sum_{\alpha\in\Lambda_k} y^\alpha P_\ell \psi_\alpha\|_{L^\infty(U;V)} \lesssim N_\Omega^{-r(t)} + N_D^{-s'/d}$$

with *total 'number of DOF'*

$$N_{total} = N_\Omega N_D$$

Note: for Monte Carlo we have $r = 1/2$.

Sparse Tensorization (not possible for MCM):

$$u = \sum_{\alpha\in\Lambda} c_\alpha(x)\mathcal{L}_\alpha(y), \quad u_{\Lambda_\ell} = \sum_{\alpha\in\Lambda_\ell} c_\alpha(x)\mathcal{L}_\alpha(y).$$

Sparse Tensor discretization: approximate c_α by $c_{\ell(\alpha)} \in V_{\ell(\alpha)} \subset V$, $\alpha \in \Lambda_k$.

Error:

$$\|u - \sum_{\alpha\in\Lambda_k} c_{\ell(\alpha)}\mathcal{L}_\alpha\|^2_{L^2(U,V)} \le \|u - \sum_{\alpha\in\Lambda_k} c_\alpha\mathcal{L}_\alpha\|^2_{L^2(U,V)} + \sum_{\alpha\in\Lambda_k} \|c_\alpha - P_{\ell(\alpha)} c_\alpha\|_V^2$$

If $\{\|c_\alpha\|_{\mathcal{A}^s} : \alpha \in \Lambda\} \in \ell^\tau(t)$, proper selection of Λ_ℓ and $\ell(\alpha)$ gives rate

$$N_\Omega^{-r} + N_D^{-s'/d} \quad \text{with} \quad N_{total} = N_\Omega \log N_D + N_D \log N_\Omega.$$

Conclusion

- Elliptic PDE with stochastic coefficients:
  Variational Formulation, Existence, Uniqueness

- Karhúnen - Loève Expansion of Random Input Data:
  Exponential pointwise convergence for p.w. analytic C_a,
  Algebraic pointwise convergence for $C_a \in H^{t,t}_{pw}$,

- Fast Computation of KL-expansion in general domains $D \subset \mathbb{R}^3$ by gFMM (Rokhlin and Greengard), $\mathcal{H}$-Matrix techniques (Hackbusch etal),...

- Transform SPDE into parametric, deterministic PDE on $U = (-1,1)^\infty$

- Truncation to $M < \infty$ dimensions; conditional expectation; error estimates.

- Convergence Rates of h-, p- type sGFEM for sparse tensor approximations of

$$B_1(\ell_\infty) \ni y \to u(y,\cdot) \in V$$

- Stochastic Regularity of random solution = domain of analyticity of parametric, deterministic problem

- Finite Smoothness Covariance: algebraic Conv. of h-sGFEM and Spectral Conv. of p-sGFEM as

$$h \to 0, \quad p \to \infty, \quad M \to \infty.$$

- Sparse collocation on input-KL adapted 'lattices' of integration points in $(-1,1)^M$ (Schwab and Todor IMAJNA (2007))

- Sparse tensorization of D and Ω Galerkin discretizations:

$$N_{total} \simeq N_\Omega \log N_D + N_D \log N_\Omega$$

- Diffusion problems in physical dimension $d = 2$, with $M = 1500$.

- requires *mixed regularity* in *both*, D and U: if

$$u = \sum_{\alpha \in \Lambda} c_\alpha(x) \mathcal{L}_\alpha(y), \quad \text{then} \quad \{\|c_\alpha\|_{\mathcal{A}^s} : \alpha \in \Lambda\} \in \ell^\tau$$

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