New multiscale finite elements for high-contrast elliptic interface problems

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Warwick, January 2009

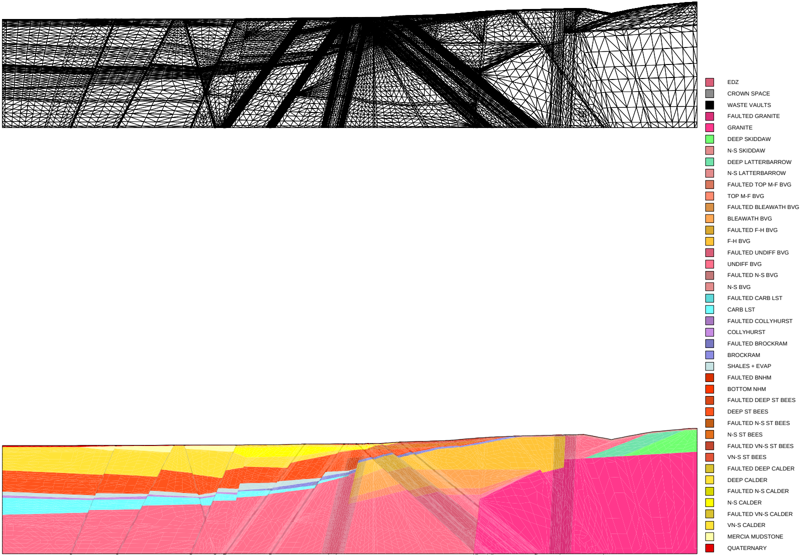
- **Motivation for problem:** flow in heterogeneous porous media
- **Motivation for methods:** recent theory of “multiscale coarsening” in domain decomposition
- **Model Problem:** Elliptic interface problems (**jumping coefficients**)
- **MSFE:** Solve local homogeneous PDEs for basis functions
- **New result:** methods with **optimal convergence** independent of the **contrast** even with “naive meshing”.
- Method involves **new boundary conditions on element edges** for basis functions.
- Theory involves **new regularity results** for elliptic interface problems
- Method is a **generalisation** of the P1-continuous Galerkin method. (Theory presented in 2D).

Flow in porous medium

Find $u \in H_0^1(\Omega)$:

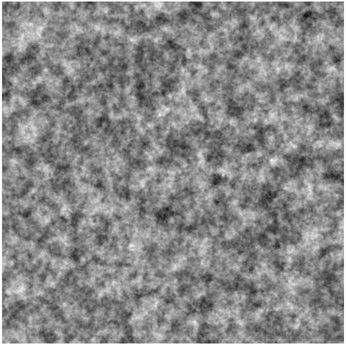
$$\int_\Omega \mathcal{A}(x) \nabla u(x) . \nabla v(x) dx \ = \ \int_\Omega F(x) v(x) dx \ , \quad v \in H_0^1(\Omega) \ ,$$

where $\mathcal{A}$ exhibits a **high degree of heterogeneity**.



Example from R. Scheichl's PhD thesis (2000) ©UK Nirex

Gaussian Random Field



Lengthscale λ, **variance** σ^2
In this picture $h = 2^{-8}$, $\lambda = 4h$, $\sigma^2 = 8$.

$$\max_{x,y \in \Omega} \frac{\alpha(x)}{\alpha(y)} \approx 10^{10} .$$

Multiscale methods

Special basis functions to capture local features, feed into variational formulation.

"Subgrid modelling", e.g. LES in turbulence models, modelling convective storms in NWF, etc..

Hughes 1995... Variational Multiscale Method, RFB's

Hou and Wu, JCP 1997:

$$-\nabla.a(x/\epsilon)\nabla u = f \quad \text{with } a \text{ periodic, smooth}$$

Many related papers, **Abdulle and E, 03, E & Engquist 04, Efendiev, Hou and Wu, 00, Arbogast & Boyd 06...**
Proofs of accuracy by homgenization arguments

A different use of the same idea: **preconditioning**

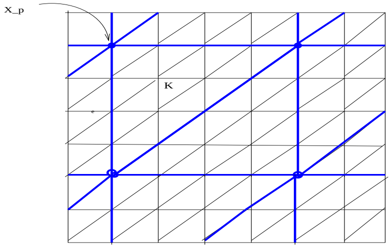
A diversion: Preconditioning and Robustness

IGG, Lechner, Scheichl (Numer Math 2007):

Suppose the discretisation resolves the heterogeneity.
DD Preconditioner P: local solves plus global coarse solve on $\mathrm{span}\,\{\Phi_p\}$, then **(under some conditions)**

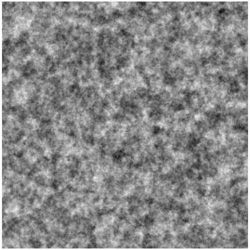
$$\kappa(P^{-1}A) \quad \lesssim \quad \max_p H_p^{2-d} |\Phi_p|_{H^1(\Omega),\alpha} \ .$$

Robustness indicator



Motivates local problems : for coarse basis

$\Phi_p \in \mathcal{S}^h(K)$:
$\int_K \alpha(x) \nabla \Phi_p . \nabla v_h = 0$
for all $v_h \in \mathcal{S}_0^h(K)$.



Gaussian Random Field

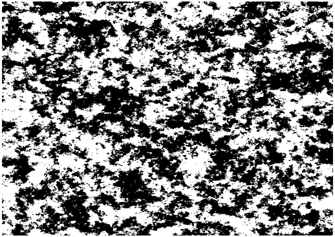
$h = 2^{-8}$, $\lambda = 4h$, $\sigma^2 = 8$.

Average **CG Iterates** and **(CPU times)** over 100 realisations :

σ^2	Linear	MS, Oscil.
0	**17 (1.66)**	**17 (1.71)**
4	**47 (3.57)**	**30 (2.55)**
8	**88 (6.19)**	**41 (3.23)**
16	**222 (14.8)**	**64 (4.74)**
20	**324 (21.2)**	**77 (5.57)**

“Aggregation coarsening” is also energy minimising:

Scheichl, Vainikko, Computing, 2006



CG–iterations $h = 2^{-8}$ and $\lambda = 4h$, clipped random fields.

$\max_{\tau,\tau'} \frac{\alpha_\tau}{\alpha_{\tau'}}$	AGGREGATION DD	CLASSICAL DD
$1.5 * 10^1$	24	32
$2.2 * 10^2$	27	89
$3.3 * 10^3$	29	296
$4.9 * 10^4$	26	498
$7.4 * 10^5$	26	724

High contrast diffusion

Robust solvers and a posteriori error estimates

- **DD and multigrid: IGG and Hagger 99, Vuik et. al 00, Xu and Zhu, 07, Aksoylu, IGG, Klie, Scheichl, 08, Pechstein and Scheichl 08, Van lent, Scheichl & IGG, 08**
- **Robustness of a posteriori error estimators: Bernardi and Verfürth 00, Ainsworth 05, Vohralik 08.**

A priori accuracy of underlying methods ??

[Plum & Wieners, 03]

Remark: The theory for MS DD coarsening does not require any homogenisation structure.

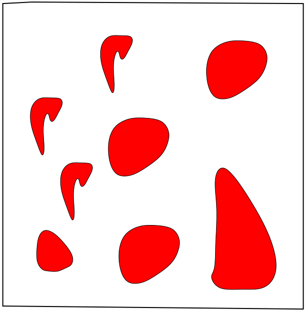
Question: Can the same tools be used to analyse **accuracy** for MSFE approximation?

Model Problem:

Find $u \in H_0^1(\Omega)$:

$$\int_\Omega \mathcal{A}(x)\nabla u(x).\nabla v(x)dx \;=\; \int_\Omega F(x)v(x)dx \;, \quad v \in H_0^1(\Omega) \;,$$

“High contrast” piecewise constant coefficient $\mathcal{A}$:



Inclusions: $\Omega_1, \ldots, \Omega_m$ $\quad \Omega_0 = \Omega \backslash \cup_{i=1}^m \Omega_i$. $\quad$ Interface Γ.

Problem Scaling

Scale by $\mathcal{A}_{\min} = \min_x \mathcal{A}(x)$: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) \;:=\; \int_\Omega \alpha(x)\nabla u(x).\nabla v(x)dx \;=\; (f,v)_{L_2(\Omega)}\;, \quad v \in H_0^1(\Omega)\;,$$

with

$$\alpha(x) = \frac{1}{\mathcal{A}_{\min}}\mathcal{A}(x)\;, \quad f(x) = \frac{1}{\mathcal{A}_{\min}}F(x)\;.$$

Then $\alpha(x) \;\geq\; 1$ and the difficulty is characterised by the **contrast**, a **large parameter**

$$\hat{\alpha} := \frac{\max_x \mathcal{A}(x)}{\min_x \mathcal{A}(x)} \geq 1\;.$$

Asymptotic cases

Case I: $\widehat{\alpha} := \min_{i=1,\ldots,m} \alpha_i \to \infty \;, \quad \alpha_0 = 1$

Highly permeable inclusions in hardly permeable matrix

Case II: $\widehat{\alpha} := \alpha_0 \to \infty \;, \quad \max_{i=1,\ldots,m} \alpha_i \;\leq\; \mathrm{Const.}$

Hardly permeable inclusions in highly permeable matrix.

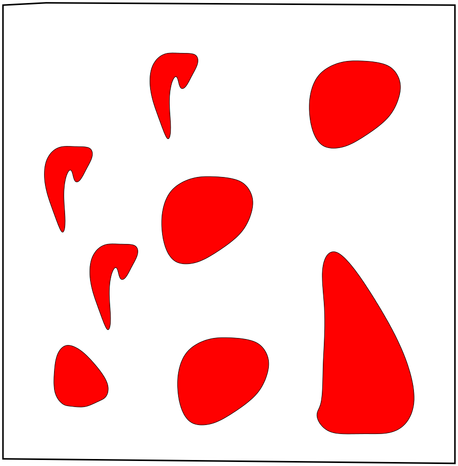
Regularity of solution:

Across an interface Γ separating Ω_- and Ω_+:

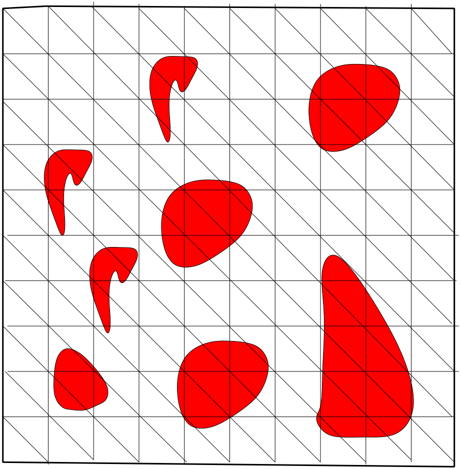
$$\alpha_- \frac{\partial u_-}{\partial n} = \alpha_+ \frac{\partial u_+}{\partial n}$$

Hence $u \in H^{3/2-\epsilon}(\Omega)$. For smooth problems $u \in H^2(\Omega)$

Naive meshing



Naive meshing



Accuracy of standard FEM suboptimal. Many methods: Barrett and Elliott, 87 (UFEM), Composite FEM, XFEM, IIM, IFEM.....

Dependence on $\hat{\alpha}$?

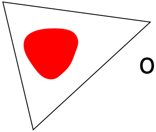
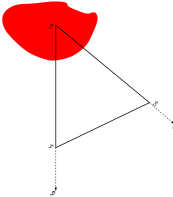
“Multiscale” Finite Element Methods

Special finite element space: $\mathcal{V}^{\mathrm{MS}} = \mathrm{span}\{\Phi_p^{\mathrm{MS}}\}$

Nodal basis: $\Phi_p^{\mathrm{MS}}(x_q) = \delta_{p,q}$

$\Phi_p^{\mathrm{MS}}|_\tau$ is linear, $\tau \cap \Gamma = \emptyset$,

$\Phi_p^{\mathrm{MS}}|_\tau$ solves (*), $\tau \cap \Gamma \neq \emptyset$,

e.g.  or 

Local Homogeneous Problems for the basis functions:

$$\int_\tau \alpha \nabla \Phi_p^{\mathrm{MS}} . \nabla v \; = \; 0 \, , \quad \text{for all } v \in H_0^1(\tau) \qquad (*)$$

Need **Boundary conditions** and **subgrid approximation** .

MSFEM: seek $u_h^{\mathrm{MS}} \in \mathcal{V}^{\mathrm{MS}}$:

$$a(u_h^{\mathrm{MS}}, v_h^{\mathrm{MS}}) = (f, v_h^{\mathrm{MS}})_{L^2(\Omega)} \, , \quad v_h^{\mathrm{MS}} \in \mathcal{V}^{\mathrm{MS}} \, .$$

The main result

Theorem **Assume**

- Ω is a convex polygon or smooth.
- the interface Γ is sufficiently smooth.
- $f \in H^{1/2}(\Omega)$.
- mesh sequence is quasiuniform

Then **there exists** a choice of **boundary condition** for each Φ_p^{MS} such that

$$(i) \quad |u - u_h^{\mathrm{MS}}|_{H^1(\Omega),\alpha} \lesssim h \left[h|f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2},$$

$$(ii) \quad \|u - u_h^{\mathrm{MS}}\|_{L_2(\Omega)} \lesssim h^2 \left[h|f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2}.$$

Hidden constants are independent of h and $\widehat{\alpha}$.

There are several technical assumptions! .

Analysis of MSFE: The Main Idea

Optimality: MS interpolant $\mathcal{I}_h^{\mathrm{MS}}$

$$|u - u_h^{\mathrm{MS}}|_{H^1(\Omega),\alpha} \;\leq\; |E_h^{\mathrm{MS}}|_{H^1(\Omega),\alpha} \;, \qquad E_h^{\mathrm{MS}} := u - \mathcal{I}_h^{\mathrm{MS}} u \;.$$

By **definition** of basis functions, for **any** element τ ,

$$a_\tau(E_h^{\mathrm{MS}}, v) \;=\; a_\tau(u,v) \;=\; (f,v)_{L^2(\tau)} \;, \quad \text{for all} \quad v \in H_0^1(\tau) \;.$$

Simple energy argument:

$$|E_h^{\mathrm{MS}}|_{H^1(\tau),\alpha} \;\lesssim\; |\widetilde{E}_h^{\mathrm{MS}}|_{H^1(\tau),\alpha} \;+\; h_\tau \|f\|_{L_2(\tau)} \;,$$

for any $\widetilde{E}_h^{\mathrm{MS}}$ with $\widetilde{E}_h^{\mathrm{MS}} = E_h^{\mathrm{MS}}$ on $\partial\tau$. **Then**

$$|E_h^{\mathrm{MS}}|^2_{H^1(\Omega),\alpha} \;\lesssim\; h^2 \left[h^{-2} \sum_\tau |\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} + \|f\|^2_{L_2(\Omega)} \right] \;.$$

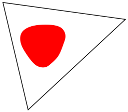
Seek BC on each $\partial\tau$ s.t. there exists $\widetilde{E}_h^{\mathrm{MS}}$ with

$$h^{-2} \sum_\tau |\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} \;\lesssim\; h|f|^2_{H^{1/2}(\Omega)} \;+\; \|f\|^2_{L_2(\Omega)} \;.$$

A simple application: Inclusion inside element

Example: $\hat{\alpha}$ **in interior**

1 in exterior (Ω_0)



Linear BC's and define $\widetilde{E}_h^{\mathrm{MS}} = \begin{cases} E_h^{\mathrm{MS}} & \text{on} \quad \partial\tau \\ 0 & \text{on} \quad \text{inclusion} \end{cases}$.

Inverse Trace (Extension) theorem :

$$|\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} \lesssim h^{-1}\|E_h^{\mathrm{MS}}\|^2_{L_2(\partial\tau)} + h\,|E_h^{\mathrm{MS}}|^2_{H^1(\partial\tau)}$$

$$\lesssim h^3\,\|D_t^2 u\|^2_{L^2(\partial\tau)}$$ **tangential derivative**

Forward Trace theorem :

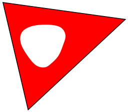
$$|\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} \lesssim h^3\left\{|u|^2_{H^{5/2}(\tau\cap\Omega_0)} + h^{-1}|u|^2_{H^2(\tau\cap\Omega_0)}\right\}$$

$$h^{-2}\sum_\tau |\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} \lesssim h|u|^2_{H^{5/2}(\Omega_0)} + |u|^2_{H^2(\Omega_0)}$$

" $\widehat{\alpha}$-Explicit" Regularity $\lesssim h\,|f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)}$

A simple application: Inclusion inside element

Example: $\hat{\alpha}$ **in exterior** (Ω_0)
1 in interior



Linear BC's and define $\widetilde{E}_h^{\mathrm{MS}} = \begin{cases} E_h^{\mathrm{MS}} & \text{on } \partial\tau \\ 0 & \text{on inclusion} \end{cases}$.

Inverse Trace (Extension) theorem :

$$\begin{aligned} |\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} &\lesssim h^{-1}\hat{\alpha}\|E_h^{\mathrm{MS}}\|^2_{L_2(\partial\tau)} + h\,\hat{\alpha}|E_h^{\mathrm{MS}}|^2_{H^1(\partial\tau)} \\ &\lesssim h^3\ \hat{\alpha}\ \|D_t^2 u\|^2_{L^2(\partial\tau)} \end{aligned}$$

tangential derivative

Forward Trace theorem :

$$|\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} \lesssim h^3 \left\{ \hat{\alpha}\ |u|^2_{H^{5/2}(\tau\cap\Omega_0)} + h^{-1}\hat{\alpha}|u|^2_{H^2(\tau\cap\Omega_0)} \right\}$$

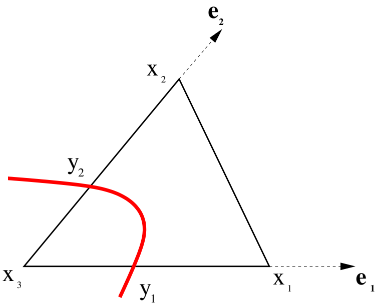
$$h^{-2}\sum_{\tau} |\widetilde{E}_h^{\mathrm{MS}}|^2_{H^1(\tau),\alpha} \lesssim h\ \hat{\alpha}\ |u|^2_{H^{5/2}(\Omega_0)} + \hat{\alpha}\ |u|^2_{H^2(\Omega_0)}$$

“ $\widehat{\alpha}$-Explicit” Regularity $\lesssim h\ \hat{\alpha}^{-1}\ |f|^2_{H^{1/2}(\Omega)} + \hat{\alpha}^{-1}\ \|f\|^2_{L_2(\Omega)}$

Bad parameter $\mathrm{dist}\{\partial\tau, \Gamma\}$ **!**

Much more complicated: "cutting through"

Generic case:



Look for **piecewise linear** boundary condition for basis functions.

Taylor expansion of true solution u on edges e_i, $i = 1, 2$

Continuity of u across interface

$$r_i^-(D_{e_i}u^-)(y_i) + r_i^+(D_{e_i}u^+)(y_i) \quad = \quad u(x_1) - u(x_3) + \mathcal{O}(h^2) \; .$$

Two equations in four unknowns

Extended system

True solution u satisfies

$$M_{\widehat{\alpha},\theta_1,\theta_2,\beta}\,\mathbf{d}(u) \;=\; \mathbf{c}(u) \;+\; \text{“small”}$$

where $\mathbf{c}(u)$ depends only on nodal values of u,

$$\begin{aligned}\mathbf{d}(u) : \;=\; & [(D_{e_1}u^-)(y_1), (D_{e_1}u^+)(y_1), (D_{e_2}u^-)(y_2), (D_{e_2}u^+)(y_2), \ldots \\ & (D_{n_1}u^-)(y_1), (D_{t_1}u^-)(y_1)]^T \;,\end{aligned}$$

and

$$M_{\widehat{\alpha},\theta_1,\theta_2,\beta} := \begin{bmatrix} -I & 0 & A_{\widehat{\alpha},\theta_1} \\ 0 & -I & A_{\widehat{\alpha},\theta_2} R_{\theta_2-\theta_1-\beta} \\ \mathcal{R}_1 & \mathcal{R}_2 & 0 \end{bmatrix} \;,$$

$$\mathcal{R}_1 = \begin{bmatrix} r_1^- & r_1^+ \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{R}_2 = \begin{bmatrix} 0 & 0 \\ r_2^- & r_2^+ \end{bmatrix} .$$

Neglecting “small”: Get BC for each basis function.

interface cutting through

- If Γ orthogonal to edges, system reduces to **two independent conditions** cf. Hou and Wu 1997.
- The recipe leads to **non-conforming** elements, but **averaging** returns conformity without loss of convergence.
- In conforming case $\operatorname{supp}(\Phi_p^{\mathrm{MS}})$ can grow with one extra layer of triangles
- Convergence theorem as before:

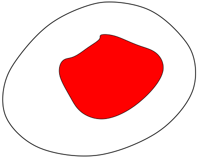
$$(i) \quad |u - u_h^{\mathrm{MS}}|_{H^1(\Omega),\alpha} \lesssim h \left[h |f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2},$$

$$(ii) \quad \|u - u_h^{\mathrm{MS}}\|_{L_2(\Omega)} \lesssim h^2 \left[h |f|^2_{H^{1/2}(\Omega)} + \|f\|^2_{L_2(\Omega)} \right]^{1/2}.$$

Subject to technical assumptions...

Regularity theory

Particular case, Ω_0 exterior, Ω_1 interior:



$$\begin{aligned} -\nabla.\alpha\nabla u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Theorem

$$|u|_{H^{2+s}(\Omega_0)} \lesssim \frac{1}{\hat{\alpha}}\|f\|_{H^s(\Omega)} \quad s \geq 0 \qquad (1)$$

$$|u|_{H^{2+s}(\Omega_1)} \lesssim \|f\|_{H^s(\Omega)} \quad s \geq 0 \qquad (2)$$

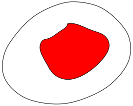
Thanks: N. Babych, I.V. Kamotski and V.P. Smyshlyaev

Idea of proof: Introduce $\widehat{u}$ solution of

$$-\nabla.\alpha_i\nabla\widehat{u} = f_i \ , \quad \text{on} \quad \Omega_i, \ i = 0, 1 \ , \quad \hat{u} = 0 \quad \text{on} \quad \partial\Omega, \Gamma$$

decoupled problems, $\widehat{u}$ satisfies estimates!

Consider remainder: $\widetilde{u} := u - \widehat{u}$:



$$-\Delta \widetilde{u}_i \;=\; 0 \quad \text{on } \Omega_1 \text{ and } \Omega_0 \quad \text{and} \quad \widetilde{u} = 0 \quad \text{on} \quad \partial\Omega$$

Jump condition on interface $\Gamma = \partial\Omega_1$:

$$\hat{\alpha}\frac{\partial \widetilde{u}_0}{\partial n} - \frac{\partial \widetilde{u}_1}{\partial n} \;=\; F \quad := \quad \frac{\partial \widehat{u}_1}{\partial n} - \hat{\alpha}\frac{\partial \widehat{u}_0}{\partial n} \qquad (\dagger)$$

Let $\widetilde{v} := \widetilde{u}|_\Gamma$ and introduce **Dirichlet to Neumann maps** $\mathcal{N}_i$

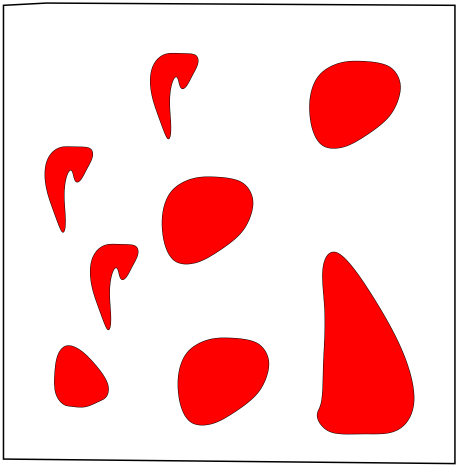
$$(\dagger) \iff (\hat{\alpha}\mathcal{N}_0 - \mathcal{N}_1)\widetilde{v} \;=\; F$$
$$\iff (I - \hat{\alpha}^{-1}\mathcal{N}_0^{-1}\mathcal{N}_1)\widetilde{v} \;=\; \hat{\alpha}^{-1}\mathcal{N}_0^{-1}F$$

Contraction mapping ($\hat{\alpha}^{-1} \to 0$):

$$\|\widetilde{v}\|_{H^{s+3/2}(\Gamma)} \;\lesssim\; \hat{\alpha}^{-1}\|\mathcal{N}_0^{-1}F\|_{H^{s+3/2}(\Gamma)} \lesssim \hat{\alpha}^{-1}\|F\|_{H^{s+1/2}(\Gamma)}$$
$$\lesssim\; \hat{\alpha}^{-1}\|\widehat{u}_1\|_{H^{s+2}(\Omega_1)} + \|\widehat{u}_0\|_{H^{s+2}(\Omega_0)} \lesssim \hat{\alpha}^{-1}\|f\|_{H^s(\Omega)}$$

(In this case $\|\tilde{u}\|_{H^{2+s}(\Omega_0)} \;=\; \mathcal{O}(\hat{\alpha}^{-1})$)

Slightly harder case:



Dirichlet to Neumann maps not invertible on “floating” domains.
Seminorm decays **but not norm** as $\widehat{\alpha} \to \infty$.

Numerical Results

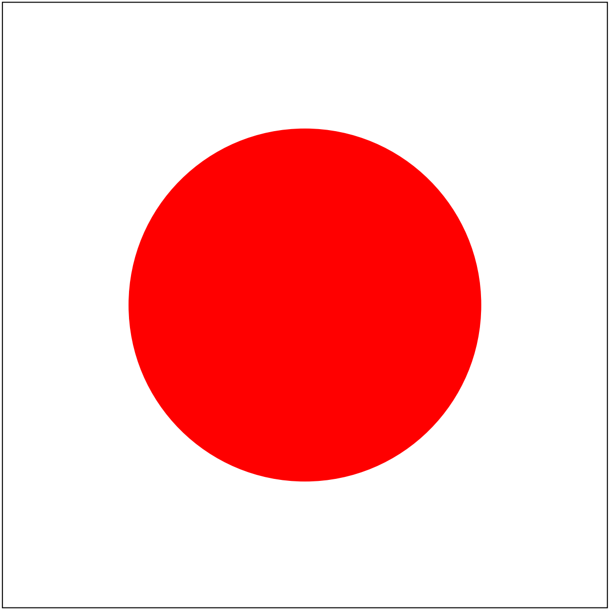
$$\begin{aligned} -\nabla.\alpha\nabla u &= f \quad \text{on} \quad \Omega := [0,1]^2, \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

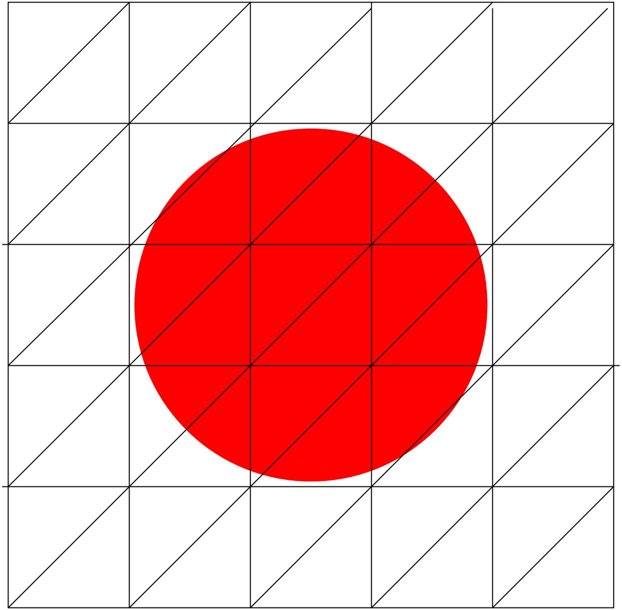
Interface is a circle of radius r_0,

$$\alpha(x) = \begin{cases} \alpha_1, & r < r_0 \\ \alpha_0, & r \geq r_0 \end{cases}$$

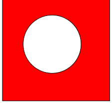
Exact solution:

$$u(x) = u(r,\theta) = \begin{cases} \frac{r^3}{\alpha_1} & r < r_0 \\ \frac{r^3}{\alpha_0} + \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right) r_0^3 & r \geq r_0 \end{cases}$$





$\alpha_1 = 1, \quad \alpha_0 = \hat{\alpha} \to \infty$



(Impermeable inclusion in high permeable matrix)

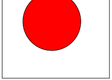
H^1 seminorm errors:

h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	2.55e-1	2.51e-1	2.54e-1
1/16	1.33e-1	1.24e-1	1.24e-1
1/32	6.22e-2	6.15-2	6.14e-2
1/64	3.26e-2	3.15e-2	3.07e-2
rate	1.0	1.0	1.0

L_2 errors:

h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	2.27e-2	2.27e-2	2.29e-2
1/16	5.75e-3	5.76e-3	5.78e-3
1/32	1.45e-3	1.45e-3	1.45e-3
1/64	3.73e-4	3.67e-4	3.63e-4
rate	1.98	1.98	1.99

$\alpha_0 = 1, \quad \alpha_1 = \hat{\alpha} \to \infty$
(Highly permeable inclusion in impermeable matrix)



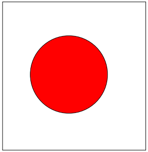
H^1 seminorm errors:

h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	1.09e-1	5.81e-2	5.90e-2
1/16	4.57e-2	2.75e-2	2.77e-2
1/32	1.43e-2	1.30e-2	1.27e-2
1/64	1.01e-2	6.52e-3	6.10e-3
rate	1.11	1.00	1.09

L_2 errors:

h	$\hat{\alpha} = 10$	$\hat{\alpha} = 10^3$	$\hat{\alpha} = 10^5$
1/8	4.83e-3	3.89-3	3.89e-3
1/16	1.32e-3	1.10e-3	1.10e-3
1/32	3.32e-4	2.91e-4	2.91e-4
1/64	8.73e-5	7.56e-5	7.53e-5
rate	1.92	1.88	1.88

Solution of subgrid problems



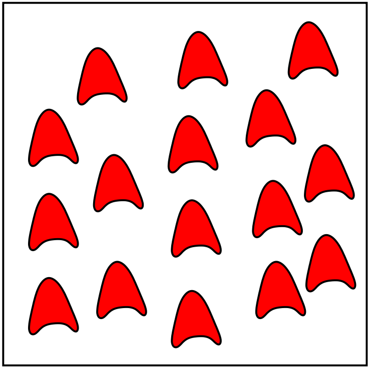
Subgrid problems solved by **Immersed finite element method** (Li, Lin, Wu (2003)).

L^2 errors, $\widehat{\alpha} = 10^4$, M = # of subgrid elements

h	M = 16	M = 64	M = 256	M = 1024
1/4	9.8226e-2	9.1744e-2	8.9859e-2	8.9489e-2
1/8	3.1606e-2	2.2946e-2	2.2903e-2	2.2891e-2
1/16	5.9537e-3	5.8252e-3	5.7816e-3	5.7824e-3
1/32	1.4916e-3	1.4511e-3	1.4512e-3	1.4517e-3
1/64	3.6856e-4	3.6374e-4	3.6359e-4	3.6369e-4

Extensions under construction

Distance between inclusions and distance of inclusions from the boundary are “bad parameters” in general.



With I. Kamotski and V.P. Smyshlyaev (Bath): inclusions separated by $\mathcal{O}(\epsilon)$ and diameter $\mathcal{O}(\epsilon)$. Working conjecture: same regularity estimate independent of ϵ.

Conclusion: Summary of results

- Elliptic interface problems with complicated interfaces have **irregular solutions** depending on contrast and interface
- Application of standard FE technoology will require **complicated mesh adaptivity** to resolve difficulties
- MSFE can **resolve** these difficulties on **"naive" meshes**.
- The **extra cost** is the solution of subgrid problems on some elements
- Analysis helps explain success of MSFE outside the homogenization framework.
- Regularity theory also helps with analysis of **standard methods**.
- Possibility to use $\mathcal{H}$**-matrix techniques** to approximate optimal basis functions without artificial boundary conditions. **Work in Progress with W. Hackbusch and S.A. Sauter**