Approximations of the The Mumford-Shah Functional for Unit Vector Fields

Jonas Haehnle

University of Tübingen

Warwick, Jan 2009

Vectorial Mumford-Shah Functional

(sphere valued functions)

For $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, find

$$(\mathbf{v}, \Gamma) := \operatorname*{argmin}_{\substack{K \subset \Omega \text{ closed}, \\ \mathbf{u} \in H^1(\Omega \setminus K, \mathbb{S}^2)}} E(\mathbf{u}, K),$$

$$E(\mathbf{u}, K) := \frac{\alpha}{2} \int_{\Omega \setminus K} |\nabla \mathbf{u}|^2 \mathrm{d}\mathbf{x} + \frac{\gamma}{2} \int_{\Omega} |\mathbf{u} - \mathbf{g}|^2 \mathrm{d}\mathbf{x} + \beta \mathscr{H}^{d-1}(K),$$

$\mathbf{g} \in L^\infty(\Omega, \mathbb{S}^2)$: original image, K: edge set (closed).

Vectorial Mumford-Shah Functional

(sphere valued functions)

For $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, find

$$(\mathbf{v}, \Gamma) := \underset{\substack{K \subset \Omega \text{ closed}, \\ \mathbf{u} \in H^1(\Omega \setminus K, \mathbb{S}^2)}}{\operatorname{argmin}} E(\mathbf{u}, K),$$

$$E(\mathbf{u}, K) := \frac{\alpha}{2} \int_{\Omega \setminus K} |\nabla \mathbf{u}|^2 \mathrm{d}\mathbf{x} + \frac{\gamma}{2} \int_{\Omega} |\mathbf{u} - \mathbf{g}|^2 \mathrm{d}\mathbf{x} + \beta \mathscr{H}^{d-1}(K),$$

$\mathbf{g} \in L^\infty(\Omega, \mathbb{S}^2)$: original image, K: edge set (closed).

3 Terms: smoothing, fidelity, edge length.

Weak Formulation

(sphere valued functions)

For $\mathbf{g} \in L^\infty(\Omega; \mathbb{S}^2)$, $\mathbf{u} \in SBV(\Omega, \mathbb{S}^2)$, minimise

$$\overline{E}(\mathbf{u}) := \frac{\alpha}{2}\int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}\mathbf{x} + \frac{\gamma}{2}\int_\Omega |\mathbf{u}-\mathbf{g}|^2 \mathrm{d}\mathbf{x} + \beta \mathscr{H}^{d-1}(S_{\mathbf{u}} \cap \Omega).$$

Weak Formulation

(sphere valued functions)

For $\mathbf{g} \in L^\infty(\Omega; \mathbb{S}^2)$, $\mathbf{u} \in SBV(\Omega, \mathbb{S}^2)$, minimise

$$\overline{E}(\mathbf{u}) := \frac{\alpha}{2}\int_\Omega |\nabla \mathbf{u}|^2 d\mathbf{x} + \frac{\gamma}{2}\int_\Omega |\mathbf{u} - \mathbf{g}|^2 d\mathbf{x} + \beta \mathscr{H}^{d-1}(S_\mathbf{u} \cap \Omega).$$

Thm: (Carriero-Leaci, 91): $\exists$ solution $\mathbf{u}$, and $(\mathbf{u}, \overline{S_\mathbf{u}})$ solves original problem.

Vectorial Ambrosio-Tortorelli Functional

(sphere valued functions)

For $\mathbf{u}, \mathbf{g} \in H^1(\Omega; \mathbb{S}^2)$, $s \in H^1(\Omega, [0,1])$, minimise

$$AT_\varepsilon(\mathbf{u}, s) := \frac{\alpha}{2}\int_\Omega (s^2 + k_\varepsilon)|\nabla \mathbf{u}|^2 d\mathbf{x} + \frac{\gamma}{2}\int_\Omega |\mathbf{u} - \mathbf{g}|^2 d\mathbf{x} + \beta \int_\Omega \varepsilon|\nabla s|^2 + \frac{1}{4\varepsilon}(1-s)^2 d\mathbf{x}.$$

Vectorial Ambrosio-Tortorelli Functional

(sphere valued functions)

For $\mathbf{u}, \mathbf{g} \in H^1(\Omega; \mathbb{S}^2)$, $s \in H^1(\Omega, [0,1])$, minimise

$$AT_\varepsilon(\mathbf{u}, s) := \frac{\alpha}{2}\int_\Omega (s^2 + k_\varepsilon)|\nabla \mathbf{u}|^2 d\mathbf{x} + \frac{\gamma}{2}\int_\Omega |\mathbf{u} - \mathbf{g}|^2 d\mathbf{x} + \beta \int_\Omega \varepsilon |\nabla s|^2 + \frac{1}{4\varepsilon}(1-s)^2 d\mathbf{x}.$$

s : "phase function", i.e. $K \approx \{s \approx 0\}$, $\Omega \setminus K \approx \{s \approx 1\}$.

Vectorial Ambrosio-Tortorelli Functional

(sphere valued functions)

For $\mathbf{u}, \mathbf{g} \in H^1(\Omega; \mathbb{S}^2)$, $s \in H^1(\Omega, [0,1])$, minimise

$$AT_\varepsilon(\mathbf{u}, s) := \frac{\alpha}{2}\int_\Omega (s^2 + k_\varepsilon)|\nabla \mathbf{u}|^2 \mathrm{d}\mathbf{x} + \frac{\gamma}{2}\int_\Omega |\mathbf{u} - \mathbf{g}|^2 \mathrm{d}\mathbf{x} + \beta \int_\Omega \varepsilon |\nabla s|^2 + \frac{1}{4\varepsilon}(1-s)^2 \mathrm{d}\mathbf{x}.$$

s : "phase function", i.e. $K \approx \{s \approx 0\}$, $\Omega \setminus K \approx \{s \approx 1\}$.

Thm (Ambrosio-Tortorelli, 90/92):
$0 < k_\varepsilon = o(\varepsilon) \Longrightarrow AT_\varepsilon(\mathbf{u}, s) \xrightarrow{\Gamma} \overline{E}(\mathbf{u}, K)$.

Prototype Problem

- Non-convex Functional (Mumford-Shah).

Prototype Problem

- Non-convex Functional (Mumford-Shah).
- Non-convex constraint ($\mathbf{u} \in \mathbb{S}^2$).

Prototype Problem

- Non-convex Functional (Mumford-Shah).
- Non-convex constraint ($\mathbf{u} \in \mathbb{S}^2$).
- With AT: Strong nonlinear coupling between **u** and s.

Prototype Problem

- Non-convex Functional (Mumford-Shah).
- Non-convex constraint ($\mathbf{u} \in \mathbb{S}^2$).
- With AT: Strong nonlinear coupling between $\mathbf{u}$ and s.
- Extending work on harmonic maps to $\mathbb{S}^2$
  (convex functional, non-convex constraint):
  Alouges (97), Bartels (05), Bartels-Prohl (07).

Colour Image Segmentation

- Chromaticity & Brightness (CB) Colour Model:

Colour Image Segmentation

- Chromaticity & Brightness (CB) Colour Model:
  RGB image $\mathbf{g} : \Omega \to [0,255]^3 \subset \mathbb{N}^3$:
  - $B := |\mathbf{g}|$, and
  - $\mathbf{C} := \mathbf{g}/|\mathbf{g}| = \mathbf{g}/B : \Omega \to \mathbb{S}^2$.

Colour Image Segmentation

- Chromaticity & Brightness (CB) Colour Model:
  RGB image $\mathbf{g} : \Omega \to [0, 255]^3 \subset \mathbb{N}^3$:
  - $B := |\mathbf{g}|$, and
  - $\mathbf{C} := \mathbf{g}/|\mathbf{g}| = \mathbf{g}/B : \Omega \to \mathbb{S}^2$.
- Sources: Mumford-Shah (89)
  Chan-Kang-Shen (01): TV, CB
  Tang-Sapiro-Caselles (01): Harmonic flow, CB
  Osher-Vese (04): *p*-harmonic flow, CB
  Bartels-Prohl (06): PM, CB
  Bellettini-Coscia (94), Bourdin (99): MS, grayscale

Nematic Liquid Crystals

- Simplified Ericksen's energy

$$\int_\Omega \frac{1}{2} s^2 |\nabla \mathbf{n}|^2 + |\nabla s|^2 + W_0(s)\,\mathrm{d}\mathbf{x},$$

Nematic Liquid Crystals

- Simplified Ericksen's energy

$$\int_\Omega \frac{1}{2} s^2 |\nabla \mathbf{n}|^2 + |\nabla s|^2 + W_0(s)\,\mathrm{d}\mathbf{x},$$

- $s \in [-1/2, 1]$: degree of orientation (often $s \geq 0$),
  $\mathbf{n} \in \mathbb{S}^2$: director.

Nematic Liquid Crystals

- Simplified Ericksen's energy

$$\int_{\Omega} \frac{1}{2} s^2 |\nabla \mathbf{n}|^2 + |\nabla s|^2 + W_0(s)\, d\mathbf{x},$$

- $s \in [-1/2, 1]$: degree of orientation (often $s \geq 0$),
  $\mathbf{n} \in \mathbb{S}^2$: director.
- Sources: Lin (89), Lin-Luskin (89)
  Virga (94), Alouges (97), Bartels-Prohl (07)

Algorithm 1: Splitting & Projection

$$\boxed{\text{FE Discr.}} \xrightarrow[h\to 0]{} \boxed{\text{Alouges-Splitting}} \xrightarrow[n\to\infty]{?}$$

$$\boxed{\text{AT on } \mathbb{S}^2} \xrightarrow[\varepsilon\to 0]{\Gamma} \boxed{\text{MS on } \mathbb{S}^2}$$

Algorithm 1: Splitting & Projection

$$\boxed{\text{FE Discr.}} \xrightarrow[h\to 0]{} \boxed{\text{Alouges-Splitting}} \xrightarrow[n\to\infty]{?}$$

$$\boxed{\text{AT on } \mathbb{S}^2} \xrightarrow[\varepsilon\to 0]{\Gamma} \boxed{\text{MS on } \mathbb{S}^2}$$

Easy to solve numerically.
Exact sphere constraint.

Algorithm 1: Splitting & Projection

- Splitting: In every iteration, minimise first for $\mathbf{u}$, then for s.

Algorithm 1: Splitting & Projection

- Splitting: In every iteration, minimise first for **u**, then for *s*.
- 1^{st} step, based on Alouges (97):
  Look for update **w** s.t. $AT_\varepsilon(\mathbf{u} - \mathbf{w}, s) \leq AT_\varepsilon(\mathbf{u}, s)$.

Algorithm 1: Splitting & Projection

- Splitting: In every iteration, minimise first for **u**, then for *s*.
- 1^{st} step, based on Alouges (97):
  Look for update **w** s.t. $AT_\varepsilon(\mathbf{u}-\mathbf{w},s) \le AT_\varepsilon(\mathbf{u},s)$.
- Only take **w** s.t. $\mathbf{w} \perp \mathbf{u}$ a.e.

Algorithm 1: Splitting & Projection

- Splitting: In every iteration, minimise first for $\mathbf{u}$, then for s.
- 1^{st} step, based on Alouges (97):
  Look for update $\mathbf{w}$ s.t. $AT_\varepsilon(\mathbf{u}-\mathbf{w},s) \leq AT_\varepsilon(\mathbf{u},s)$.
- Only take $\mathbf{w}$ s.t. $\mathbf{w} \perp \mathbf{u}$ a.e. $\implies$
  - projecting $\mathbf{u}-\mathbf{w}$ never increases energy
  - existence ($\mathbf{u}\cdot\mathbf{w}=0$ is linear constraint)

Algorithm 1: Splitting & Projection

- Splitting: In every iteration, minimise first for $\mathbf{u}$, then for s.
- 1^{st} step, based on Alouges (97):
  Look for update $\mathbf{w}$ s.t. $AT_\varepsilon(\mathbf{u} - \mathbf{w}, s) \leq AT_\varepsilon(\mathbf{u}, s)$.
- Only take $\mathbf{w}$ s.t. $\mathbf{w} \perp \mathbf{u}$ a.e. $\implies$
  - projecting $\mathbf{u} - \mathbf{w}$ never increases energy
  - existence ($\mathbf{u} \cdot \mathbf{w} = 0$ is linear constraint)
- 2^{nd} step: $\exists$ by convexity & coercivity, $0 \leq s \leq 1$ by max-principle (or testing with cutoff functions).

Properties

- Energy decreasing.

Properties

- Energy decreasing.
- $(\mathbf{u}_n, s_n) \rightharpoonup (\mathbf{u}, s)$ in H^1 (up to subseq).

Properties

- Energy decreasing.
- $(\mathbf{u}_n, s_n) \rightharpoonup (\mathbf{u}, s)$ in H^1 (up to subseq).
- $\mathbf{u}_n \to \mathbf{u}$ in L^2, and $\mathbf{w}_n \to 0$ in H^1.

Properties

- Energy decreasing.
- $(\mathbf{u}_n, s_n) \rightharpoonup (\mathbf{u}, s)$ in H^1 (up to subseq).
- $\mathbf{u}_n \to \mathbf{u}$ in L^2, and $\mathbf{w}_n \to 0$ in H^1.
- Limit fulfils constraints.

Properties

- Energy decreasing.
- $(\mathbf{u}_n, s_n) \rightharpoonup (\mathbf{u}, s)$ in H^1 (up to subseq).
- $\mathbf{u}_n \to \mathbf{u}$ in L^2, and $\mathbf{w}_n \to 0$ in H^1.
- Limit fulfils constraints.
- Problem: Cannot identify limit

$$\lim_{n\to\infty} \left(|\nabla \mathbf{u}_{n+1}|^2 s_n, \varphi \right),$$

so cannot show $(\mathbf{u}, s)$ stationary point of $AT_\varepsilon(\cdot,\cdot)$.

Discretisation

- Polygonal Lipschitz domain, uniform triangulation, continuous, piecewise affine FE.

Discretisation

- Polygonal Lipschitz domain, uniform triangulation, continuous, piecewise affine FE.
- Only linear eqns in every iteration.

Discretisation

- Polygonal Lipschitz domain, uniform triangulation, continuous, piecewise affine FE.
- Only linear eqns in every iteration.
- Energy decreasing, if all int. angles $\leq \pi/2$ (AC).

Discretisation

- Polygonal Lipschitz domain, uniform triangulation, continuous, piecewise affine FE.
- Only linear eqns in every iteration.
- Energy decreasing, if all int. angles $\leq \pi/2$ (AC).
- Sphere constraint exactly fulfilled on nodal points.

Discretisation

- Polygonal Lipschitz domain, uniform triangulation, continuous, piecewise affine FE.
- Only linear eqns in every iteration.
- Energy decreasing, if all int. angles $\leq \pi/2$ (AC).
- Sphere constraint exactly fulfilled on nodal points.
- Limit fulfils constraints.

Discretisation

- Polygonal Lipschitz domain, uniform triangulation, continuous, piecewise affine FE.
- Only linear eqns in every iteration.
- Energy decreasing, if all int. angles $\leq \pi/2$ (AC).
- Sphere constraint exactly fulfilled on nodal points.
- Limit fulfils constraints.
- Problem: As before.

Algorithm 2: Splitting & Penalisation

$$\boxed{\text{FE Discr.}} \xrightarrow[h \to 0]{} \boxed{\text{Splitting}} \xrightarrow[n \to \infty]{}$$

$$\boxed{\text{Reg. GL-AT}} \xrightarrow[\varepsilon \to 0]{\Gamma} \boxed{\text{MS on } \mathbb{S}^2}$$

Algorithm 2: Splitting & Penalisation

$$\boxed{\text{FE Discr.}} \xrightarrow[h\to 0]{} \boxed{\text{Splitting}} \xrightarrow[n\to\infty]{}$$

$$\boxed{\text{Reg. GL-AT}} \xrightarrow[\varepsilon\to 0]{\Gamma} \boxed{\text{MS on } \mathbb{S}^2}$$

Harder to solve numerically.
Approx. sphere constraint.

Algorithm 2: Splitting & Penalisation

- Splitting again.

Algorithm 2: Splitting & Penalisation

- Splitting again.
- No projection, but Ginzburg-Landau penalisation: add

$$\frac{1}{\delta_\varepsilon}\int_\Omega \left(|\mathbf{u}|^2 - 1\right)^2 d\mathbf{x}.$$

Algorithm 2: Splitting & Penalisation

- Splitting again.
- No projection, but Ginzburg-Landau penalisation: add

$$\frac{1}{\delta_\varepsilon}\int_\Omega \left(|\mathbf{u}|^2 - 1\right)^2 \mathrm{d}\mathbf{x}.$$

- Identifying limits needs higher regularity of s, so add

$$\frac{\eta_\varepsilon}{p}\int_\Omega |\triangle s|^p \mathrm{d}\mathbf{x}.$$

Properties

- Γ-convergence for proper scalings of δ_ε and η_ε (δ_ε slower than ε, η_ε much faster).

Properties

- Γ-convergence for proper scalings of δ_ε and η_ε (δ_ε slower than ε, η_ε much faster).
- Solutions exist in each step.

Properties

- Γ-convergence for proper scalings of δ_ε and η_ε (δ_ε slower than ε, η_ε much faster).
- Solutions exist in each step.
- Energy decreasing.

Properties

- Γ-convergence for proper scalings of δ_ε and η_ε (δ_ε slower than ε, η_ε much faster).
- Solutions exist in each step.
- Energy decreasing.
- $(\mathbf{u}_n, s_n) \rightharpoonup (\mathbf{u}, s)$ in $H^2 \times W^{2,p}$ (up to subseq).

Properties

- Γ-convergence for proper scalings of δ_ε and η_ε (δ_ε slower than ε, η_ε much faster).
- Solutions exist in each step.
- Energy decreasing.
- $(\mathbf{u}_n, s_n) \rightharpoonup (\mathbf{u}, s)$ in $H^2 \times W^{2,p}$ (up to subseq).
- For $p > d$, $\mathbf{u}_n \to \mathbf{u}$ in H^1, and $s_n \to s$ in L^∞, so identifying limits works.

Discretisation/Disadvantages

- s_n now needs hermite elements (but...).

Discretisation/Disadvantages

- s_n now needs hermite elements (but...).
- Nonlinear eqns need to be solved.

Discretisation/Disadvantages

- s_n now needs hermite elements (but...).
- Nonlinear eqns need to be solved.
- Sphere constraint only approximated.

Discretisation/Disadvantages

- s_n now needs hermite elements (but...).
- Nonlinear eqns need to be solved.
- Sphere constraint only approximated.
- Two additional parameters.

Algorithm 3: Penalisation without Splitting

$$\boxed{\text{FE Discr.}} \xrightarrow[h\to 0]{\Gamma} \boxed{\text{MS on } \mathbb{S}^2}$$

Algorithm 3: Penalisation without Splitting

$$\boxed{\text{FE Discr.}} \xrightarrow[h\to 0]{\Gamma} \boxed{\text{MS on } \mathbb{S}^2}$$

Hard to solve numerically.
Approx. sphere constraint.

Penalisation without Splitting

- No splitting.

Penalisation without Splitting

- No splitting.
- No projection, but Ginzburg-Landau penalisation.

Penalisation without Splitting

- No splitting.
- No projection, but Ginzburg-Landau penalisation.
- Direct Γ-convergence of 1^{st}-order FE-functional (as in Bellettini-Coscia, 94).

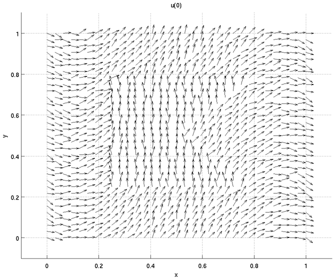
Penalisation without Splitting

- No splitting.
- No projection, but Ginzburg-Landau penalisation.
- Direct Γ-convergence of 1^{st}-order FE-functional (as in Bellettini-Coscia, 94).
- Advantage: No additional regularisation term needed.

Penalisation without Splitting

- No splitting.
- No projection, but Ginzburg-Landau penalisation.
- Direct Γ-convergence of 1^{st}-order FE-functional (as in Bellettini-Coscia, 94).
- Advantage: No additional regularisation term needed.
- Disadvantage: No convergent algorithm known.

Projection



Projection

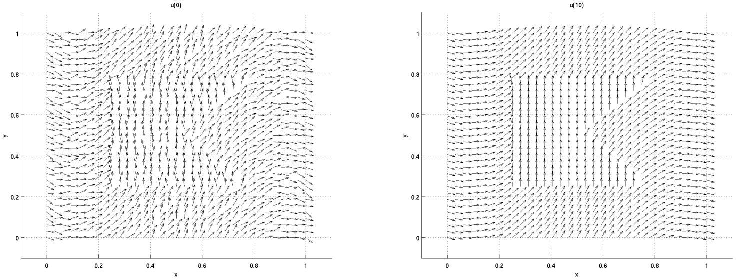
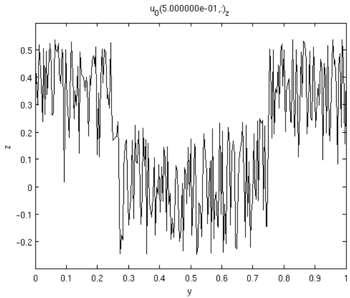


Figure: $\mathbf{U}_0 \equiv \mathbf{g}$ (left) and $\mathbf{U}_{10}$ (right).

Projection


u0(5.000000e-01,:)z
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
z
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
y

Projection

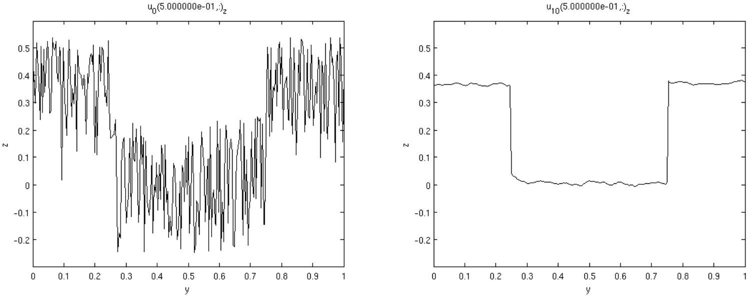
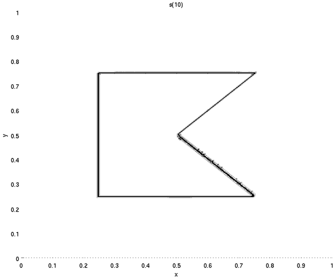
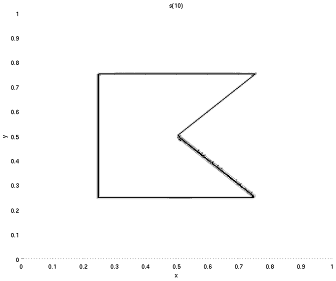


Figure: Section at $x = 0.5$ through $\mathbf{U}_0 \equiv \mathbf{g}$ and $\mathbf{U}_{10}$.

Projection vs. Penalisation



Projection vs. Penalisation



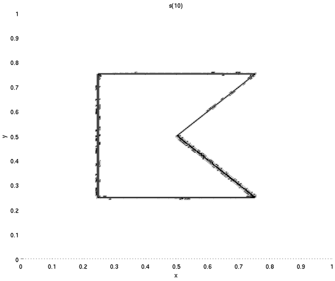
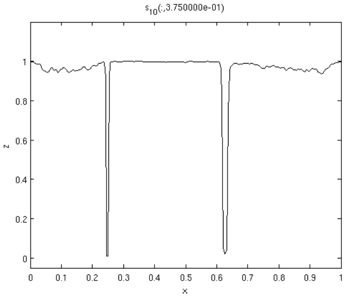


Figure: S_{10}, projection (left) and penalisation (right).

Projection vs. Penalisation



Projection vs. Penalisation

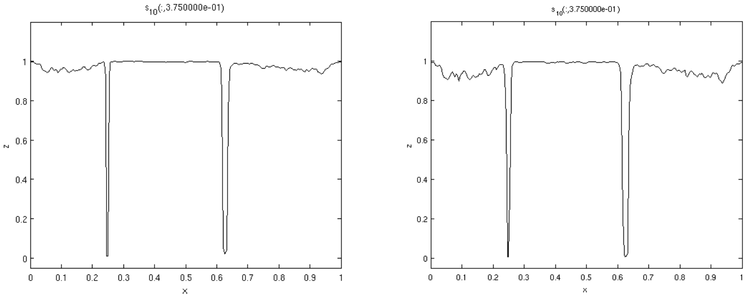


Figure: Section at $y = 0.375$ through S_{10}, projection (left) and penalisation (right).

Penalisation

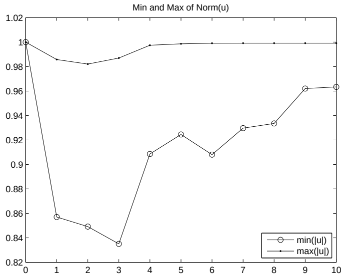


Figure: Min and max of $\|\mathbf{U}\|$.

CB Model

$$\begin{aligned} \mathrm{AT}_{\varepsilon}(\mathbf{u}, v, s) \quad := \quad & \frac{\alpha}{2}\int_{\Omega}(s^2 + k_{\varepsilon})|\nabla \mathbf{u}|^2 \mathrm{d}\mathbf{x} + \frac{\gamma}{2}\int_{\Omega}|\mathbf{u} - \mathbf{g}|^2 \mathrm{d}\mathbf{x} \\ & + \frac{\alpha_1}{2}\int_{\Omega}(s^2 + k_{\varepsilon})|\nabla v|^2 \mathrm{d}\mathbf{x} + \frac{\gamma_1}{2}\int_{\Omega}|v - b|^2 \mathrm{d}\mathbf{x} \\ & + \beta\int_{\Omega}\varepsilon|\nabla s|^2 + \frac{1}{4\varepsilon}(1 - s)^2 \mathrm{d}\mathbf{x}, \end{aligned}$$

CB Model

$$\begin{aligned} AT_{\varepsilon}(\mathbf{u}, v, s) \quad := \quad & \frac{\alpha}{2}\int_{\Omega}(s^2 + k_{\varepsilon})|\nabla \mathbf{u}|^2 d\mathbf{x} + \frac{\gamma}{2}\int_{\Omega}|\mathbf{u} - \mathbf{g}|^2 d\mathbf{x} \\ & + \frac{\alpha_1}{2}\int_{\Omega}(s^2 + k_{\varepsilon})|\nabla v|^2 d\mathbf{x} + \frac{\gamma_1}{2}\int_{\Omega}|v - b|^2 d\mathbf{x} \\ & + \beta\int_{\Omega}\varepsilon|\nabla s|^2 + \frac{1}{4\varepsilon}(1 - s)^2 d\mathbf{x}, \end{aligned}$$

- $\mathbf{g}, \mathbf{u}$: Original and iterate chromaticity,
- b, v: Original and iterate brightness,
- s: Joint edge-indicator.

CB

CB

Figure: $\mathbf{U}_0 \equiv \mathbf{G}$ (left) and $\mathbf{U}_{10}$ (right).

CB vs. RGB

CB vs. RGB

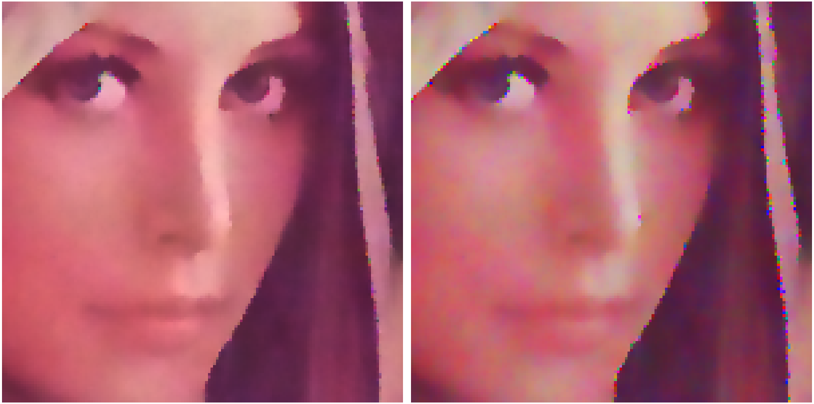


Figure: Crop of $\mathbf{U}_{10}$, CB (left) and RGB (right).

CB vs. RGB

CB vs. RGB

Figure: S_{10}, CB (left) and RGB (right).

Thank You