Optimization with PDEs in the presence of constraints – tailored discrete concepts and error analysis

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(joint with Klaus Deckelnick and Andreas Günther)

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Topic of this talk

Optimal control of pdes with pointwise constraints:

$$\min_{u \in U_{ad}, y \in Y_{ad}} J(y, u) \text{ s.t. } PDE(y) = B(u)$$

Analysis: Casas 85,93 (pointwise state constraints), Casas & Fernandez 93 (pointwise constraints on gradient)

Numerical analysis (pointwise state constraints):

- **A priori:**
  - **Original problem: Casas & Mateos; Deckelnick & H.; Meyer;....**
  - **Ralaxation: Group of Rösch; Group of Tröltzsch; Hintermüller & H.; H. & Meyer; H. & Schiela; ...**
- **A posteriori: Benedix, Vexler & Wollner; Günther & H.; Hintermüller, Hoppe & Kieweg.**

Numerical analysis (pointwise constraints on gradient): Deckelnick, Günther, & H.

State and control constraints

Model problem

$$\min_{u \in U_{ad}} J(u) = \frac{1}{2}\int_D |y - z|^2 + \frac{\alpha}{2}\|u\|_U^2$$
$$\text{subject to } y = \mathcal{G}(Bu) \text{ and } y \le b \text{ in } D.$$

Here, $U_{ad} \subseteq U$ closed and convex, $\alpha > 0$, z, b, sufficiently smooth, and $y = \mathcal{G}(Bu)$ iff

$$Ay = Bu \text{ in } D, \text{ plus b.c. (plus i.c.)}$$

- elliptic case: $D = \Omega$ and $Ay := -\sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^{d} b_i y_{x_i} + cy$ uniformly elliptic operator,
- parabolic case: $D = (0, T] \times \Omega$ and $Ay := y_t - \sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^{d} b_i y_{x_i} + cy$ with strongly elliptic leading part.

Slater condition: $\exists \tilde{u} \in U_{ad}$ such that $\mathcal{G}(B\tilde{u}) < b$ in $\bar{D}$.

Optimality conditions (Casas 86,93)

Let $u \in U_{ad}$ denote the unique optimal control and $y = \mathcal{G}(Bu)$. Then there exist $\mu \in \mathcal{M}(\bar{D})$ and some p such that there holds

$$\int_D pAv = \int_D (y - z)v + \int_{\bar{D}} v d\mu \qquad \forall v \in X,$$

$$\langle B^* p + \alpha u, v - u\rangle_{U^*,U} \geq 0 \qquad \forall v \in U_{ad},$$

$$\mu \geq 0,\ y \leq b \text{ in } D \text{ and } \int_{\bar{D}} (b - y) d\mu = 0,$$

where

- elliptic case: $p \in W^{1,s}(\Omega)$ for all $s < d/(d-1)$ and $X = H^2(\Omega)$ with $\sum_{i,j=1}^{d} a_{ij} v_{x_i} \nu_j = 0$ on $\partial\Omega$,
- parabolic case: $p \in L^s(W^{1,\sigma})$ for all $s, \sigma \in [1, 2)$ with $2/s + d/\sigma > d + 1$ and $X = \{v \in C^0(\bar{Q}); v(0, \cdot) = 0\} \cap \{v \in L^2(H^2), v_t \in L^2(H^1)\}$.

Discretization – a variational concept

Discrete optimal control problem:

$$\min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\int_D |y_h - z|^2 + \frac{\alpha}{2}\|u\|_U^2$$
$$\text{subject to } \ y_h = \mathcal{G}_h(Bu) \text{ and } y_h \leq I_h b.$$

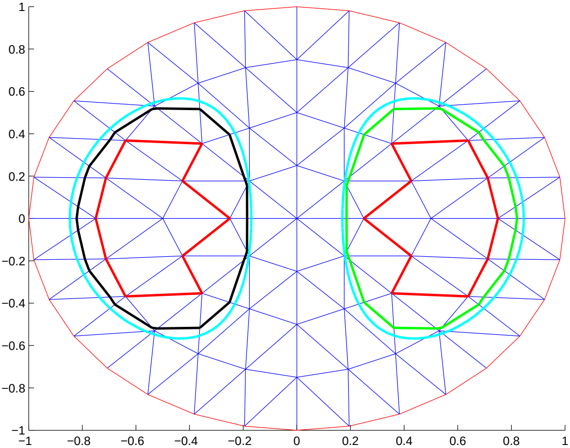
Here, $y_h(u) = \mathcal{G}_h(Bu)$ denotes the

- **p.l. and continuous fe approximation to $y(u)$ (elliptic case),**
- **dg(0) in time and p.l. and continuous fe in space approximation to $y(u)$ (parabolic case), i.e.**

$$a(y_h, v_h) = \langle Bu, v_h\rangle \text{ for all } v_h \in X_h.$$

We do not discretize the control!

Variational versus conventional discretization



Movie time dependent problems

Discrete optimality conditions

Let $u_h \in U_{ad}$ denote the unique variational–discrete optimal control, $y_h = \mathcal{G}(Bu_h)$. There exist $\mu \in \mathbb{R}^k$ and $p_h \in X_h$ such that with

- $\mu_h = \sum_{j=1}^{nv} \mu_j \delta_{x_j}$ **(elliptic case, x_i fe nodes, $k = nv$),**
- $\mu_h = \sum_{i=1}^{m} \sum_{j=1}^{nv} \mu_{ij} \delta_{x_j} \circ \frac{1}{|I_i|} \int_{I_i} \bullet dt$ **(parabolic case, x_i fe nodes, I_i dg intervals, $k = nv + m$),**

we have

$$a(v_h, p_h) = \int_D (y_h - z) v_h + \int_{\bar{D}} v_h d\mu_h \qquad \forall v_h \in X_h,$$

$$\langle B^* p_h + \alpha u_h, v - u_h \rangle_{U^*,U} \geq 0 \qquad \forall v \in U_{ad},$$

$$\mu_j \geq 0,\ y_h \leq I_h b, \text{ and } \int_{\bar{D}} \big(I_h b - y_h \big) d\mu_h = 0.$$

Here, δ_x denotes the Dirac measure concentrated at x and I_h is the usual Lagrange interpolation operator.

Results

Let $u_h \in U_{ad}$ denote the variational–discrete optimal solution with corresponding state $y_h \in X_h$ and $\mu_h \in \mathcal{M}(\bar{D})$. Then for h small enough

$$\|y_h\|,\ \|u_h\|_U,\ \|\mu_h\|_{\mathcal{M}(\bar{D})} \leq C.$$

For the proof a discrete counterpart to the Slater condition is needed, which is deduced from uniform convergence of the discrete states associated to the Slater point $B\tilde{u}$.

Results, cont.

Let u denote the solution of the continuous problem and u_h the variational discrete optimal control. Then

$$\alpha\|u-u_h\|^2+\|y-y_h\|^2\le$$
$$\le C(\|\mu\|_{\mathcal{M}(\bar{D})},\|\mu_h\|_{\mathcal{M}(\bar{D})})\left\{\|y-y_h(u)\|_\infty+\|y^h(u_h)-y_h\|_\infty\right\}+$$
$$+C(\|u\|,\|u_h\|)\left\{\|y-y_h(u)\|+\|y^h(u_h)-y_h\|\right\}.$$

Here, $y_h(u)=\mathcal{G}_h(Bu), y^h(u_h)=\mathcal{G}(Bu_h)$.

We need uniform estimates for discrete approximations.

Error estimates, parabolic case

Controls $u \in L^2(0,T)^m$, and $f_i \in H^1(\Omega)$ given actuations.

$$Bu := \sum_{i=1}^{m} u_i(t) f_i(x), \quad y_0 \in H^2(\Omega).$$

Then $y = \mathcal{G}(Bu) \in \{v \in L^\infty(H^2), v_t \in L^2(H^1)\}$ and we have with $y_h = \mathcal{G}_h(Bu)$ and time stepping $\delta t \sim h^2$

$$\|y - y_h\|_\infty \leq C \begin{cases} h\sqrt{|\log h|}, & (d=2) \\ \sqrt{h}, & (d=3) \end{cases}$$

This is not an off-the-shelf result! It yields

$$\alpha\|u - u_h\|^2 + \|y - y_h\|^2 \leq C \begin{cases} h\sqrt{|\log h|}, & (d=2) \\ \sqrt{h}, & (d=3). \end{cases}$$

Error estimates, elliptic case

Deckelnick, H. (SINUM 2007, ENUMATH 2007):

- $Bu \in L^2(\Omega)$:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} = \begin{cases} O(h^{\frac{1}{2}}), & \text{if } d = 2, \\ O(h^{\frac{1}{4}}), & \text{if } d = 3, \end{cases}$$

- $Bu \in W^{1,s}(\Omega)$:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} \leq Ch^{\frac{3}{2} - \frac{d}{2s}} \sqrt{|\log h|}.$$

- $Bu \in L^\infty(\Omega)$:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} \leq Ch|\log h|.$$

- $U = L^2(\Omega), U_{ad} = \{u \leq d\}$, u_h p.c.:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} \leq C \begin{cases} h|\log h|, & \text{if } d = 2, \\ \sqrt{h}, & \text{if } d = 3. \end{cases}$$

Similar results obtained by C. Meyer for discrete controls.

Numerical experiment 1

$\Omega := B_1(0)$, $\alpha > 0$,

$$z(x) := 4 + \frac{1}{\pi} - \frac{1}{4\pi}|x|^2 + \frac{1}{2\pi}\log|x|,\ b(x) := |x|^2 + 4,$$

and $u_0(x) := 4 + \frac{1}{4\alpha\pi}|x|^2 - \frac{1}{2\alpha\pi}\log|x|$.

$$J(u) := \frac{1}{2}\int_\Omega |y - z|^2 + \frac{\alpha}{2}\int_\Omega |u - u_0|^2,$$

where $y = \mathcal{G}(u)$.
Unique solution $u \equiv 4$ with corresponding state $y \equiv 4$ and multipliers

$$p(x) = \frac{1}{4\pi}|x|^2 - \frac{1}{2\pi}\log|x| \ \text{ and }\ \mu = \delta_0.$$

Experimental order of convergence

RL	$\|u - u_h\|$	$\|y - y_h\|$
1	0.788985	0.536461
2	0.759556	1.147861
3	0.919917	1.389378
4	0.966078	1.518381
5	0.986686	1.598421

Relaxing constraints – Lavrentiev (H., Meyer COAP 2008)

Lavrentiev Regularization: relax $y \le b$ to $\lambda u + y \le b$ ($\lambda > 0$). Numerical analysis yields

- **$Bu^\lambda \in L^2(\Omega)$ uniformly:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h^{1-d/4},$$

- **$Bu^\lambda \in W^{1,s}(\Omega)$ uniformly for all $s \in (1, \frac{d}{d-1})$:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h^{2-d/2-\epsilon},$$

- **$Bu^\lambda \in L^\infty(\Omega), Bu_h^\lambda \in L^\infty(\Omega)$ uniformly:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h|\log h|.$$

Relaxing constraints – penalization (Hintermüller, H.)

Relax $y \le b$ with $\frac{\gamma}{2}\int_\Omega |(y-b)^+|^2 dx$ in cost functional.

- **$Bu^\gamma \in L^2(\Omega)$ uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h^{1-d/4},$$

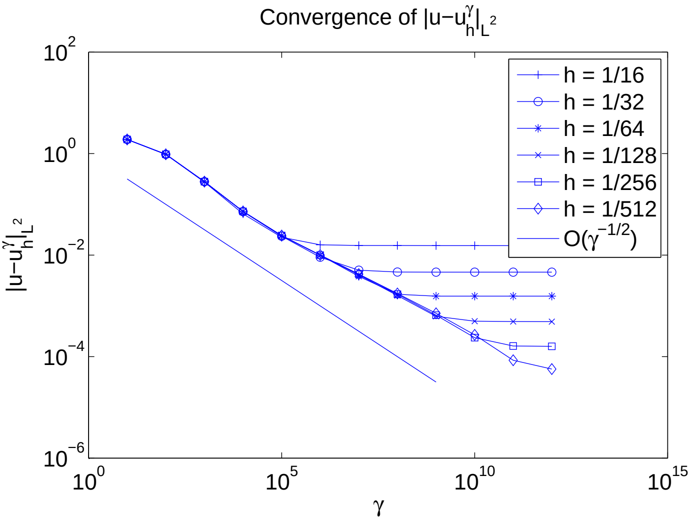
- **$Bu^\gamma \in W^{1,s}(\Omega)$ for all $s \in (1, \frac{d}{d-1})$ uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h^{2-d/2-\epsilon},$$

- **$Bu^\gamma \in L^\infty(\Omega), Bu_h^\gamma \in L^\infty(\Omega)$ uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h|\log h|.$$

Relaxing constraints – penalization, numerical results



Relaxing constraints – barriers

Barriers: relax $y \leq b$ by adding $-\mu \int_\Omega \log(b-y)dx$ to cost functional ($\mu > 0$). Numerical analysis yields

- **$Bu^\mu \in L^2(\Omega)$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h^{1-d/4},$$

- **$Bu^\mu \in W^{1,s}(\Omega)$ for all $s \in (1, \frac{d}{d-1})$ uniformly:**

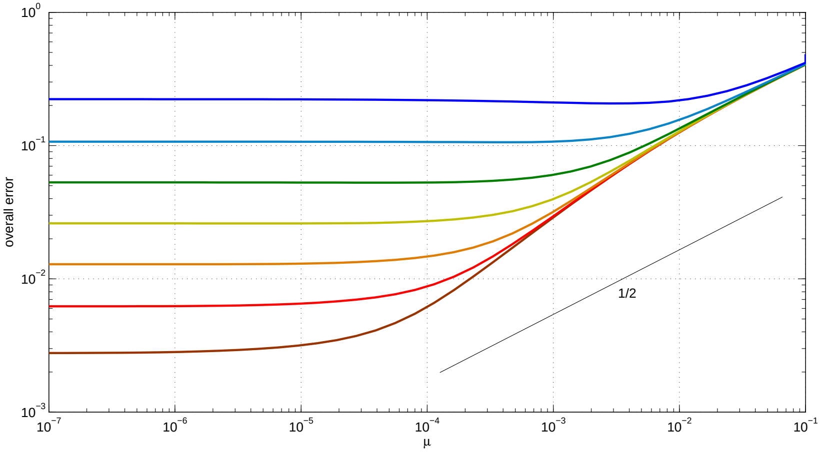
$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h^{2-d/2-\epsilon},$$

- **$Bu^\mu \in L^\infty(\Omega), Bu_h^\mu \in L^\infty(\Omega)$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h|\log h|.$$

This is work in progress with Anton Schiela.

Relaxing constraints – barriers, numerical results



Consequence: Grid size h and parameters (λ, γ, μ) should be coupled;

Lavrentiev: $\sqrt{\lambda} \sim h^{2-d/2}$,
Barriers: $\sqrt{\mu} \sim h^{2-d/2}$,
Penalization $(p = \infty)$: $\frac{1}{\sqrt{\gamma}} \sim h^{1+d/2}$ **(optimal ?).**

Constraints on the gradient

Consider

$$\min_{u \in U_{ad}} J(u) = \frac{1}{2}\int_\Omega |y - z|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \left(+ \frac{\alpha}{2}\int_\Omega |u|^2 \right)$$

where $y = \mathcal{G}(u)$, **i.e. solves the pde, and** $\nabla y \in Y_{ad}$.
Here

$$Y_{ad} = \{z \in C^0(\bar{\Omega})^d \,|\, |z(x)| \leq \delta, x \in \bar{\Omega}\},$$

and

$$\begin{aligned} r = 2: &\qquad U_{ad} = \{u \in L^2(\Omega) \,|\, a \leq u \leq b \text{ a.e. in } \Omega\}(a, b \in L^\infty), \\ r > d: &\qquad U_{ad} = L^r(\Omega). \end{aligned}$$

Then $U_{ad} \subset L^r(\Omega)$ **for** $r > d \Rightarrow \nabla y \in C^0(\bar{\Omega})^d$.
Slater condition:

$\exists \hat{u} \in U_{ad} \,|\nabla \hat{y}(x)| < \delta,\ x \in \bar{\Omega},$ **where** $\hat{y}$ **solves the pde with** $u = \hat{u}$.

Optimality conditions (Casas & Fernandez)

An element $u \in U_{ad}$ is a solution if and only if there exist $\vec{\mu} \in \mathcal{M}(\bar{\Omega})^d$ and $p \in L^t(\Omega)$ $(t < \frac{d}{d-1})$ such that

$$\begin{aligned}
\int_\Omega p\mathcal{A}z - \int_\Omega (y-z)z &= \int_{\bar{\Omega}} \nabla z \cdot d\vec{\mu} && \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega) \\
\int_{\bar{\Omega}} (z - \nabla y) \cdot d\vec{\mu} &\le 0 && \forall z \in Y_{ad}, \\
\int_\Omega (p + \alpha u)(\tilde{u} - u) &\ge 0 && \forall \tilde{u} \in U_{ad} \text{ for } r = 2, \text{ or} \\
p + \alpha((u+)|u|^{r-2}u) &= 0 && \text{in } \Omega \text{ for } r > d.
\end{aligned}$$

Structure of multiplier: $\vec{\mu} = \frac{1}{\delta}\nabla y\, \mu$, where $\mu \in \mathcal{M}(\bar{\Omega}) \ge 0$ is concentrated on $\{x \in \bar{\Omega} \,|\, |\nabla y(x)| = \delta\}$.

FE discretization, conventional

Piecewise linear, continuous Ansatz for the state $y_h = \mathcal{G}_h(u) \in X_h$.

The discrete control problem reads

$$\min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \left(+\frac{\alpha}{2}\int_\Omega |u|^2\right)$$
$$\text{subject to} \;\; y_h = \mathcal{G}_h(u) \text{ and } \left(\frac{1}{|T|}\int_T \nabla y_h\right)_{T \in \mathcal{T}_h} \in Y_{ad}^h,$$

where

$$Y_{ad}^h := \{c_h : \bar{\Omega} \to \mathbb{R}^d \,|\, c_{h|T} \text{ is constant and } |c_{h|T}| \le \delta,\ T \in \mathcal{T}_h\}.$$

FE discretization, conventional, optimality conditions

The variational discrete problem has a unique solution $u_h \in U_{ad}$. There exist $\mu_T \in \mathbb{R}^d$, $T \in \mathcal{T}_{h,X}$ and $p_h \in X_h$ such that with $y_h = \mathcal{G}_h(u_h)$ we have

$$a(v_h, p_h) = \int_\Omega (y_h - z)v_h + \sum_{T \in \mathcal{T}_{h,X}} |T| \, \nabla v_{h|T} \cdot \mu_T \qquad \forall v_h \in X_h,$$

$$\sum_{T \in \mathcal{T}_{h,X}} |T| \big(c_{h_T} - \nabla y_{h|T} \big) \cdot \mu_T \leq 0 \quad \forall c_h \in C_h,$$

$$p_h + \alpha((u_h +)|u_h|^{r-2}u_h) = 0 \quad \text{in } \Omega.$$

Structure of the multiplier: $\vec{\mu}_T = \mu_T \, \frac{1}{\delta} \nabla y_{h_T}$, where $\mu_T \in \mathbb{R}$. Furthermore, $\mu_T \geq 0$ and $\mu_T > 0$ only if $|\nabla y_{h_T}| = \delta$.

Results

Deckelnick, Günther, H. (Oberwolfach Report 2008): Let $u_h \in U_{ad}$ be the variational discrete optimal solution with corresponding state $y_h \in X_h$ and adjoint variables $p_h \in X_h$, $\vec{\mu}_T (T \in \mathcal{T}_h)$.

Then for h small enough

- $\|y_h\|$, $\|u_h\|_{L^r}$, $\|p_h\|_{L^{\frac{r}{r-1}}}$, $\sum_{T \in \mathcal{T}_{h,X}} |T|\, |\mu_T| \le C$,
- $\|y - y_h\| \le Ch^{\frac{1}{2}(1-\frac{d}{r})}$, $\|u - u_h\|_{L^r} \le Ch^{\frac{1}{r}(1-\frac{d}{r})}$, **and** $\|u - u_h\|_{L^2} \le Ch^{\frac{1}{2}(1-\frac{d}{r})}$.

These results are also valid for a piecewise constant Ansatz of the control.

FE discretization, Raviart Thomas

Mixed fe approximation of the state with lowest order Raviart–Thomas element, i.e.

$$(y_h, v_h) = \mathcal{G}_h(u) \in Y_h \times V_h$$

denotes the solution of

$$\begin{aligned} \int_\Omega A^{-1} v_h \cdot w_h + \int_\Omega y_h \, \mathrm{div} w_h &= 0 \qquad \forall w_h \in V_h \\ \int_\Omega z_h \, \mathrm{div} v_h - \int_\Omega a_0 y_h \, z_h + \int_\Omega u \, z_h &= 0 \qquad \forall z_h \in Y_h. \end{aligned}$$

FE discretization, cont.

The discrete control problem reads

$$\min_{u\in U_{ad}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{2}\int_\Omega |u|^2$$
$$\text{subject to } (y_h, v_h) = \mathcal{G}_h(u) \text{ and } \left(\frac{1}{|T|}\int_T A^{-1}v_h\right)_{T\in\mathcal{T}_h} \in Y^h_{ad},$$

where

$$Y^h_{ad} := \{c_h : \bar{\Omega} \to \mathbb{R}^d \,|\, c_{h|T} \text{ is constant and } |c_{h|T}| \le \delta,\ T \in \mathcal{T}_h\}.$$

FE discretization, optimality conditions

The discrete problem has a unique solution $u_h \in U_{ad}$. Furthermore, there are $\vec{\mu}_T \in \mathbb{R}^d$ and $(p_h, \chi_h) \in Y_h \times V_h$ such that with $(y_h, v_h) = \mathcal{G}_h(u_h)$ we have

$$\begin{aligned}
\int_\Omega A^{-1}\chi_h \cdot w_h + \int_\Omega p_h \, \mathrm{div} w_h + \sum_{T \in \mathcal{T}_h} \vec{\mu}_T \cdot \fint_T A^{-1} w_h &= 0 \forall w_h \in V_h \\
\int_\Omega z_h \, \mathrm{div}\chi_h - \int_\Omega a_0 p_h \, z_h + \int_\Omega (y_h - z)\, z_h &= 0 \forall z_h \in Y_h. \\
\int_\Omega (p_h + \alpha u_h)(\tilde{u} - u_h) &\geq 0 \forall \tilde{u} \in U_{ad} \\
\sum_{T \in \mathcal{T}_h} \vec{\mu}_T \cdot \left(c_{h|T} - \fint_T A^{-1} v_h \right) &\leq 0 \, \forall c_h \in Y^h_{ad}
\end{aligned}$$

Structure of the multiplier: $\vec{\mu}_T = \mu_T \, \frac{1}{\delta} \fint_T A^{-1} v_h$, where $\mu_T \in \mathbb{R}$. Furthermore, $\mu_T \geq 0$ and $\mu_T > 0$ only if $\left| \fint_T A^{-1} v_h \right| = \delta$.

Results

Deckelnick, Günther, H. (Numer. Math 2008): Let $u_h \in U_{ad}$ be the optimal solution of the discrete problem with corresponding state $(y_h, v_h) \in Y_h \times V_h$ and adjoint variables $(p_h, \chi_h) \in Y_h \times V_h$, $\vec{\mu}_T, T \in \mathcal{T}_h$.

Then for h small enough

- $\|y_h\|, \sum_{T \in \mathcal{T}_h} |\vec{\mu}_T| \leq C,$ **and**
- $\|u - u_h\| + \|y - y_h\| \leq Ch^{\frac{1}{2}} |\log h|^{\frac{1}{2}}.$

Constraints on the gradient, example

We take $\Omega = B_2(0)$ and consider

$$\min J(u) = \frac{1}{2}\|y - z\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}$$

with pointwise bounds on the constraints, i.e. $\{a \le u \le b\}$, where $a, b \in L^\infty(\Omega)$, and pointwise bounds on the gradient, i.e. $|\nabla y(x)| \le \delta := 1/2$. State and control satisfy

$$-\Delta y = f + u \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega.$$

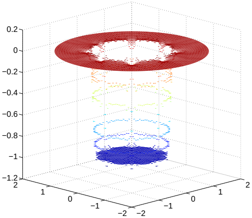
Data:

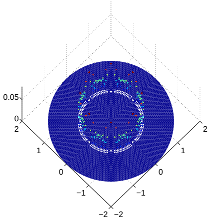
$$z(x) := \begin{cases} \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4}|x|^2 & , 0 \le |x| \le 1 \\ \frac{1}{2}\ln 2 - \frac{1}{2}\ln|x| & , 1 < |x| \le 2 \end{cases} \qquad f(x) := \begin{cases} 2 \\ 0 \end{cases}$$

Solution:

$$y(x) \equiv z(x) \text{ and } u(x) = \begin{cases} -1 & , 0 \le |x| \le 1 \\ 0 & , 1 < |x| \le 2 \end{cases}$$

Numerical experiment, piecewise constant control Ansatz



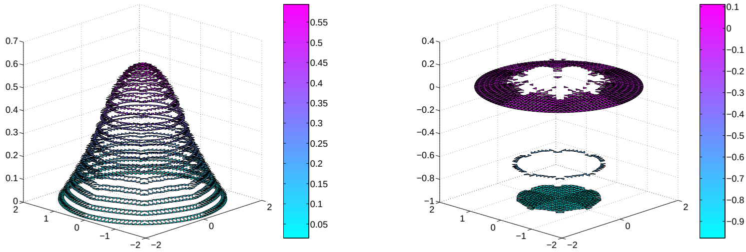


Experimental order of convergence

RL	$\|u - u_h\|_{L^4}$	$\|u - u_h\|$	$\|y - y_h\|$
1	0.76678	0.72339	1.90217
2	0.33044	0.64248	1.25741
3	0.27542	0.54054	1.23233
4	0.28570	0.53442	1.16576

Results show the predicted behaviour, since $r = \infty$.

Numerical solution, mixed finite elements



Experimental order of convergence

RL	$\|u - u_h\|$	$\|y - y_h\|$	$\|y^P - y_h^P\|$
1	0.98576	1.06726	1.08949
2	0.51814	1.02547	1.09918
3	0.50034	1.01442	1.08141

Superscript P denotes post–processed piecewise linear state. It attains the same order of convergence but yields significantly smaller approximation error.

Thank you very much for your attention

Goal oriented adaptivity

Let $a_{ij} = \delta_{ij}, b_i = 0$, and $c = 1$. Further let

$$U_{ad} = \{c \le u \le d\}, \quad Y_{ad} = \{a \le y \le b\},$$

and, for example,

$$J(y,u) = \frac{1}{2}\int_\Omega |y - z|^2 + \frac{\alpha}{2}\|u - u_0\|_U^2,$$

$$J_h(y_h,u_h) = \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{2}\|u_h - u_0\|_U^2.$$

Aim: Extend DWR method of Becker, Kapp, Rannacher,...) to construct ideal meshes w.r.t. the error $J(y_h,u_h) - J(y,u)$.

Goal oriented adaptivity, error representation

Let

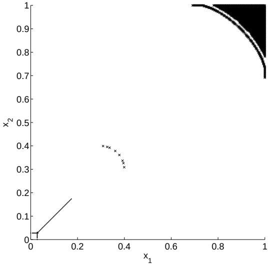
$$\begin{aligned}\rho^p(\cdot) &:= \mathrm{J_y}(\mathrm{y_h},\mathrm{u_h})(\cdot) - \mathrm{a}(\cdot,\mathrm{p_h}) + \langle\mu_h,\cdot\rangle,\\ \rho^u(\cdot) &:= \mathrm{J_u}(\mathrm{y_h},\mathrm{u_h})(\cdot) - (\cdot,\mathrm{p_h}) \text{ and}\\ \rho^y(\cdot) &:= -\mathrm{a}(\mathrm{y_h},\cdot) + (\mathrm{u_h},\cdot).\end{aligned}$$

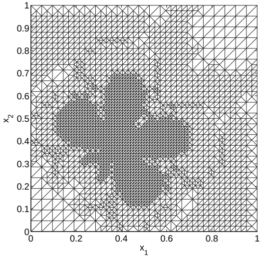
Then for $b = I_h b$ (Günther, H. (J. Numer. Math. 2008))

$$\begin{aligned}J(y,u) - J(y_h,u_h) = \frac{1}{2}\rho^p(y - i_h y) + \frac{1}{2}\rho^y(p - i_h p) + \\ + \frac{1}{2}\left\{\langle\mu + \mu_h, y_h - y\rangle + \langle\lambda + \lambda_h, u_h - u\rangle\right\}\end{aligned}$$

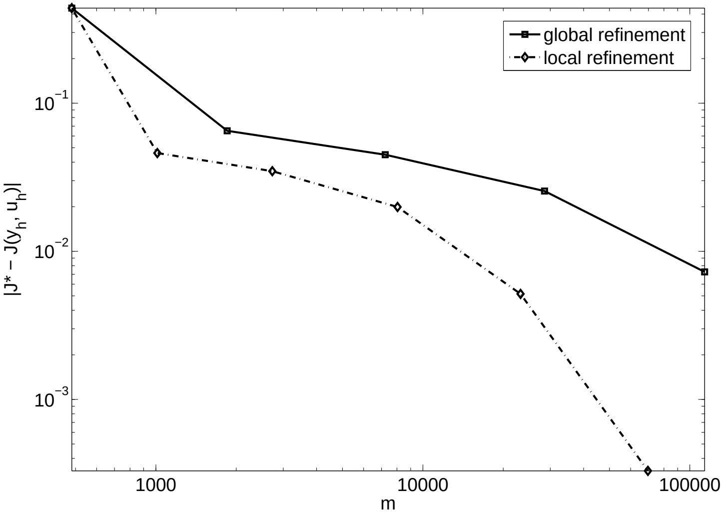
- $\rho^u(\cdot)$ **does not appear in this representation.**
- **No differences of the multipliers $\mu, \mu_h, \lambda, \lambda_h$ appear in this representation.**
- **Constraints on gradient:**
  $\rho^p(\cdot) = J_y(y_h,u_h)(\cdot) - a(\cdot,p_h) - \langle \mathrm{div}\vec{\mu_h},\cdot\rangle$ **and**
  $\langle\mu + \mu_h, y_h - y\rangle \longrightarrow \langle -\mathrm{div}(\vec{\mu} + \vec{\mu_h}), y_h - y\rangle$

Goal oriented adaptivity, multiplier support and mesh





Goal oriented adaptivity, error



Goal oriented adaptivity, efficiency

m	h	h_{min}	$\lvert J^* - J(y_h, u_h) \rvert$	I_{eff}
484	0.0673	0.0476	0.43855	2.0
1013	0.0673	0.0238	0.04606	0.5
2730	0.0673	0.0119	0.03477	1.2
8038	0.0673	0.0060	0.01992	1.9
23216	0.0673	0.0030	0.00516	1.4
69645	0.0673	0.0015	0.00033	0.3

Work in progress, collaboration with O. Benedix and B. Vexler

Thank you very much for your attention