Optimal Control of Stationary Variational Inequalities

K. Kunisch

Department of Mathematics and Computational Sciences
University of Graz, Austria

jointly with

D. Wachsmuth,
Radon Institute, Linz, Austria

Problem Statement

$$(P)\begin{cases} \text{Minimize} \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \\ a(y, \phi - y) \geq (u, \phi - y), \quad y \in K, \text{ for all } \phi \in K, \end{cases}$$

$$\nu_1 |v|^2_{H^1_0} \leq a(v,v), \text{ and } a(v,w) \leq \nu_2 |v|_{H^1_0} |w|_{H^1_0},$$

$$K = \{v \in H^1_0(\Omega) : v \leq \psi\}.$$

$$a(v_1, v_2) = \langle A v_1, v_2 \rangle, \text{ for } v_1, v_2 \in H^1_0(\Omega),$$

here: A second order, regular differential operator,
$A\psi \in L^2, \ \psi | \partial\Omega \geq 0$

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optimal control of complementarity problem

$$a(y, \phi - y) \geq (u, \phi - y)_{L^2}, \quad y \in K, \text{ for all } \phi \in K,$$

$\Leftrightarrow$

$$\min \tfrac{1}{2}\, a(y, y) - (y, u)_{L^2} \text{ over } y \in K,$$

$\Leftrightarrow$

$$Ay + \lambda = u, \quad y \leq \psi, \ \lambda \geq 0, \ \langle \lambda, y - \psi \rangle = 0,$$

$\Leftrightarrow$

$$Ay + \lambda = u, \quad \lambda = \max(0, \lambda + c(y - \psi)), \text{ for any } c$$

approximation

$$Ay_c + \max{}_c\,(0, \bar{\lambda} + c(y_c - \psi)) = u,$$

$\bar{\lambda}$ governs feasibility

Approximating problem

$$(P_c) \quad \begin{cases} \text{Minimize} \quad J(y,u) = g(y) + j(u) \\ \text{over } u \in L^2(\Omega) \text{ subject to} \\ Ay + \max_c(0, \bar{\lambda} + c(y-\psi)) = u, \end{cases}$$

$$\max_c(0,x) = \begin{cases} x, & \text{for } x \geq \frac{1}{2c} \\ \frac{c}{2}(x + \frac{1}{2c})^2, & \text{for } |x| \leq \frac{1}{2c} \\ 0, & \text{for } x \leq -\frac{1}{2c}. \end{cases}$$

$$(OS_c) \quad \begin{cases} Ay_c + \max{}_c(0, \bar{\lambda} + c(y_c - \psi)) = u_c, \\ Ap_c + \max_c'(\bar{\lambda} + c(y_c - \psi))\, p_c + g'(y_c) = 0, \\ j'(u_c) - p_c = 0, \end{cases}$$

Keywords

- Optimality system of unregularized problem: two techniques
- Sufficient optimality conditions
- $c \to \infty$: L^∞ – *estimates*
- feasibility for properly chosen $\bar{\lambda} \geq 0$
- Semi-smooth Newton methods: well-posedness
- Geometric properties of value function of (P_c)

Towards first order optimality

Lemma

$u \mapsto (y(u), \lambda(u)) : L^2 \to H_0^1 \times L^2$ is directionally differentiable.

$$a(y', \phi - y') \geq (h, \phi - y'), \quad y' = y'(u;h) \in S(y),\ \forall \phi \in S(y),$$

where

$$S(y) = \{\phi \in H_0^1(\Omega) : \ \phi = 0 \text{ on } \lambda > 0, \quad \phi \geq 0 \text{ on } B\}.$$

$$B = \{\lambda = y - \psi = 0\}$$

$$Ay' + \lambda'(u;h) = h,\ \lambda'(u;h)(y - \psi) = 0,\ y'\,\lambda = 0,$$

and

$$y' \geq 0,\ \lambda' \geq 0,\ y'\,\lambda' = 0 \text{ on } B.$$

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$$y' \geq 0, \ \lambda' \geq 0, \ y' \lambda' = 0 \text{ on } B.$$

Theorem

(y^*, u^*) be a locally optimal for (P). $\exists$ uniquely determined $p \in H_0^1(\Omega)$ and $\mu \in H^{-1}(\Omega) \cap (L^\infty(\Omega))^*$:

$$Ap + \mu + g'(y^*) = 0, \quad p \geq 0 \text{ on } y^* = \psi, \tag{1}$$

$$\langle \mu, y^* - \psi \rangle = 0, \qquad \lambda p = 0 \text{ a.e.}, \qquad \text{and } \langle \mu, p \rangle \geq 0 \tag{2}$$

$$j'(u^*) - p = 0. \tag{3}$$

$$\langle \mu, \phi \rangle \geq 0, \quad \forall \phi \in H_0^1(\Omega), \phi \geq 0 \text{ on } \{y^* = \psi\}, \langle \lambda, \phi \rangle = 0, \tag{4}$$

Sign condition for μ on B:

$$\langle \mu, \phi \rangle = 0, \quad \forall \phi \in H_0^1(\Omega), \phi \geq 0 \text{ on } B, \phi = 0 \text{ on } \Omega \setminus B, \tag{5}$$

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Theorem

(y^*, u^*) be a locally optimal for (P). $\exists$ *uniquely determined $p \in H_0^1(\Omega)$ and $\mu \in H^{-1}(\Omega) \cap (L^\infty(\Omega))^*$:*

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$$\langle \mu, \phi \rangle \geq 0, \quad \forall \phi \in H_0^1(\Omega), \phi \geq 0 \text{ on } \{y^* = \psi\}, \ \langle \lambda, \phi \rangle = 0, \tag{9}$$

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Mignot-Puel 1984, Ito-K 2000. Barbu, Bergounioux. ”strongly stationary”.

Theorem

$\mu = \mu_r + \mu_{finad}$

$\mu_r = -g'(y^*)$ *on* $\{\lambda > 0\}$, supp $\mu_{finad} \subset \overline{\{\lambda = 0\}} \cap \overline{\{y = \psi\}}$.

Proposition

$$u_c \rightharpoonup u^* \text{ in } L^2, \quad y_c(u_c) \to y^* \text{ in } H_0^1,$$

$$\lambda_{c_n} = \max{}_c(0, \bar{\lambda} + c(y_c - \psi)) \rightharpoonup \lambda(y^*) \text{ in } H^{-1}$$

In the feasible case with $y_c \leq \psi$*,*

$\{(p_{c_n}, \mu_{c_n})\} \rightharpoonup (p, \mu) \in H_0^1(\Omega) \times L^\infty(\Omega)^*$ *satisfying (1)-(3)*

Proposition

$\bar{\lambda} \geq \max(0,\ u_c - A\psi) + \kappa(|p_c|_{L^\infty}) \quad \Rightarrow \quad y_c \leq \psi.$

Proposition

(a) feasible case ($\bar{\lambda} \gg 0$)

$$\|y^* - y_c\|_{L^\infty} \leq \frac{\|\bar{\lambda}\|_{L^\infty}}{c} + \frac{1}{2c^2}.$$

(b) infeasible case ($\bar{\lambda} = 0$)

$$\mathcal{A} = \{x \in \Omega \colon y^*(x) = \psi(x)\}, \quad \mathcal{A}_c = \{x \in \Omega \colon y_c(x) = \psi(x)\}.$$

$\partial\mathcal{A}$ is $C^{1,1}$ and $\partial\mathcal{A}_c$ is Lipschitzian

$$\|y^* - y_c\|_{L^\infty(\Omega)} \leq \frac{1}{c}\|f - A\psi\|_{L^\infty(\Omega)}.$$

Theorem

$\mu = \mu_r + \mu_{finad}$

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Assumption (H1)

$$g''(y^*)(y', y') + j''(u^*)(h, h) > 0$$

for all $h \in L^2(\Omega) \setminus \{0\}$ and $y' = y'(u^*; h)$ with

$$g'(y^*)y' + j'(u)h = 0.$$

There exists $\tau > 0$ such that

$$p \geq 0, \text{ on } \{x : \psi - \tau < y^* < \psi\}$$

and

$$\langle \mu, \phi \rangle \geq 0, \quad \forall \phi \in H_0^1(\Omega), \ \phi \geq 0.$$

Second order sufficient optimality

Theorem

Let $(y^*, u^*, \lambda, p, \mu)$ satisfy the first-order optimality system, (H1). Then there exists $\alpha > 0, r > 0$ such that

$$J(y^*, u^*) + \alpha\|v - u^*\|_{L^2}^2 \leq J(y(v), v), \quad \forall v \in L^2(\Omega) : \|u - v\|_{L^2} < r.$$

$u_c \to u^*$

Semi-smooth Newton method

$$j(u) = \frac{\beta}{2}|u|^2, \qquad (g''(y)\delta y,\, \delta y)_{L^2} \geq \alpha\,|\delta y|^2.$$

$$\begin{cases} Ay_c + \max_c(0,\, \bar\lambda + c(y_c - \psi)) - \frac{1}{\beta}p_c = 0 \\ Ap_c + \max_c'(\bar\lambda + c(y_c - \psi))\, p_c + g'(y_c) = 0. \end{cases}$$

Let $F : (H^2 \cap H_0^1) \times (H^2 \cap H_0^1) \to L^2(\Omega) \times L^2(\Omega)$ be given by

$$F(y,p) = \begin{pmatrix} Ay + \max_c(0, \bar\lambda + c(y - \psi)) - \frac{1}{\beta}p \\ Ap + \max_c'(\bar\lambda + c(y - \psi))\, p + g'(y) \end{pmatrix}. \tag{11}$$

Definition

$F : D \subset X \to Z$ is called Newton differentiable in $U \subset D$, if there exist $G : U \to \mathcal{L}(X, Z)$:

$$(A)\ lim_{h\to 0} \frac{1}{\mid h \mid} \mid F(x+h) - F(x) - G(x+h)h \mid = 0, \text{ for all } x \in U.$$

Example

$F : L^p(\Omega) \to L^q(\Omega),\ F(\varphi) = \max(0, \varphi)$ is Newton differentiable if $q < p$, and

$$G_{\max}(\varphi)(x) = \begin{cases} 1 & \text{if } \varphi(x) > 0 \\ 0 & \text{if } \varphi(x) < 0 \\ \delta & \text{if } \varphi(x) = 0,\ \delta \in \mathbb{R} \text{ arbitrary.} \end{cases}$$

Example

$p = q$ ... (A) is not satisfied.

Theorem

Let $F(x^*) = 0$, F Newton differentiable in $U(x^*)$, and $\{||G(x)^{-1}||_{\mathcal{L}(X,Z)} : x \in U(x^*)\}$ bounded.

Then the Newton iteration converges locally superlinearly.

Rate of convergence

Ref.: Chen-Nashed, Hintermüller-Ito-K, Kummer, M. Ulbrich.

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Let $F(x^*) = 0$*, F Newton differentiable in* $U(x^*)$*, and* $\{||G(x)^{-1}||_{\mathcal{L}(X,Z)} : x \in U(x^*)\}$ *bounded.*

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Theorem

If (H2), holds and c is sufficiently large, then the semi-smooth Newton algorithm applied to $F(y,p)=0$ is welldefined and converges locally super-linearly.

Assumption (H2)

$p_c \geq 0$ on $\{|\frac{\bar{\lambda}}{c} + y_c - \psi| \leq \rho\}, \quad c \geq C_0$

e.g. $g(y) = |y - y_d|^2_{L^2}$ with $\psi \leq y_d$

then: $p_c \geq 0, \quad p \geq 0, \quad \mu_c \geq 0, \quad \mu \geq 0$

how to choose and/or update c ?

Value Function $V(c) = g(y_c) + j(u_c)$

Proposition

Infeasible case: $\dot{V}(c) = O(\frac{1}{c})$ *for* $c \to \infty$

Feasible case: $\dot{V}(c) = o(\frac{1}{c})$ *for* $c \to \infty.$

Proposition

(Monotonicity of V). Assume that

$$p_c \geq 0 \text{ on } \mathcal{A}_c^s = \{x : \bar{\lambda} + c(y_c - \psi) \geq \frac{1}{2c}\}$$

Infeasible case: $\dot{V}(c) \geq -\frac{5}{8c^2}\,|p_c|_{L^1(\mathcal{N}_c)}$, *on* $(0, \infty)$.

Feasible case: $\dot{V}(c) \leq 0$ *for all c sufficiently large.*

$$\mathcal{N}_c = \{x : \; |\bar{\lambda} + c(y_c - \psi)| \; < \frac{1}{2c}\}, \}.$$

Proposition

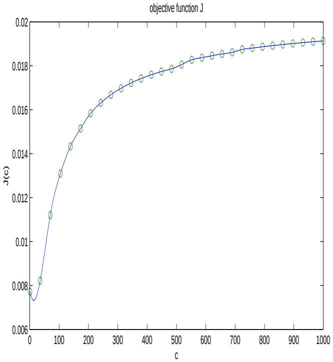
('Concavity' of V, infeasible case)

$$\ddot{V}(c) \le -(g''(y_c)\dot{y}_c, \dot{y}_c) - c^2 \, |p_c \dot{y}_c^2|_{L^1(\mathcal{N}_c)} + \frac{5}{4c^3} |p_c|_{L^1(\mathcal{N}_c)},$$

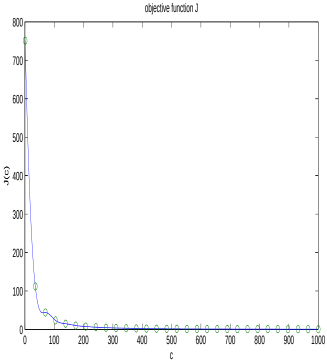
for all $c \gg 0$.

('Convexity' of V, feasible case).

$$\ddot{V}(c) \ge \beta \, |\dot{u}_c|^2 + g''(y_c)(\dot{y}_c, \dot{y}_c) + (-\frac{1}{c^3} - \frac{\bar{\lambda}}{c^2}) \, |p_c|_{L^1(\mathcal{N}_c)}.$$



Infeasible Case



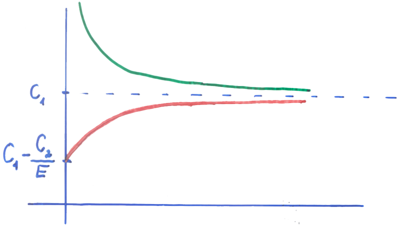
Feasible Case

Model Function.

$$m(c) = C_1 - \frac{C_2}{(E+c)}.$$

m has monotonicity / concavity properties of V.

$$\dot{m} \sim \dot{V} = (p_c, \frac{\partial}{\partial c} \max_c (\bar{\lambda} + (c(y_c - \psi))_{L^2}$$



exact path following

$$|V^* - V(c_{k+1})| \leq \tau_k \, |V^* - V(c_k)|$$

$$|C_{1,k} - m(c_{k+1})| \leq \tau_k \, |C_{1,k} - V(c_k)| =: \alpha_k$$

$$c_{k+1} = \left(\frac{C_{2,k}}{\alpha_k}\right)^{1/r} - E_k.$$

Theorem

(exact path following)

$$\lim_{k\to\infty} (y_{c_k}, u_{c_k}, \lambda_{c_k}) \to (y^*, u^*, \lambda^*).$$

Hintermüller & K.

Remark

- PDA with regularization may lead to cycling,
- Calibration of Black Scholes with American options is related, but.
- Time dependent case, $V \subset H \subset V^*$, V not compact in H, motivated by e.g. Ω unbounded domain, Ito &K.