Nonstandard a posteriori error bounds in Euler-Galerkin schemes for parabolic problems

with elliptic reconstruction techniques

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New Directions in Computational Partial Differential Equations

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Aim

Derive **a posteriori error estimates** for

Problem (general linear diffusion)

Find $u : \Omega \times [0,T] \to \mathbb{R}$ *satisfying*

(linear parabolic diffusion PDE) $\partial_t u + \mathscr{A} u = f$ *in* $\Omega \times (0,T] \subset \mathbb{R}^d \times \mathbb{R}$,

(with initial Cauchy condition) $u(\cdot, 0) = u_0$,

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with elliptic operator $\mathscr{A}(t) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}^{-1}(\Omega)$ *satisfying*

$$\langle \mathscr{A}(t) v \,|\, \phi \rangle := a(v, \phi) := \int_\Omega \nabla\phi(x)^\intercal \boldsymbol{a}(x,t) \nabla v(x) \ \forall\, \phi \in \mathrm{H}_0^1(\Omega),$$

$$\alpha_\flat(t) |\boldsymbol{w}|^2 \le \boldsymbol{w}^\intercal \boldsymbol{a}(t) \boldsymbol{w} \le \alpha_\sharp(t) |\boldsymbol{w}|^2 \ \forall\, \boldsymbol{w} \in \mathbb{R}^d$$

(*e.g.*, $\mathscr{A}(t) = -\Delta$ *and* 0-*Dirichlet BC*) .

Discrete Model Problem

weak form and spatial semidiscretization

Problem (weak formulation)

Find $u : [0, T] \to \mathrm{H}^1_0(\Omega)$ *such that*

$$\langle \partial_t u, \phi \rangle + a\,(u, \phi) = \langle f, \phi \rangle \;\forall\, \phi \in \mathrm{H}^1_0(\Omega) \quad \textit{and} \quad u(0) = u_0 \in \mathrm{L}_2(\Omega).$$

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Find $U : [0, T] \to \mathbb{V}_h$ *such that*

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- **consistent** method $\Rightarrow B\left(V, \Phi\right) = a\left(V, \Phi\right)$,

Discrete Model Problem

full discretization

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Problem (Fully discrete implicit Euler-Galërkin FEM)

$(\mathbb{V}^n)_{n=0,\dots,N}$ *a sequence of FE spaces, find* $(U^n)_{n=0,\dots,N}$ *such that*

$$U^0 = \Pi^0 u_0 \quad \textit{and} \quad \forall\, n \in [1:N]:$$

$$\left\langle \frac{U^n - U^{n-1}}{\tau_n}, \Phi \right\rangle + B(U^n, \Phi) = \langle f^n, \Phi \rangle, \ \forall\, \Phi \in \mathbb{V}^n.$$

Finite element spaces $\mathbb{V}^0, \mathbb{V}^1, \ldots, \mathbb{V}^N$

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- example of (conforming) finite elment space

$$p \in \mathbb{N} \quad \text{and} \quad \mathbb{V}^n := \{\Phi \in C(\Omega) : \ \Phi|_K \text{ poly of deg } p\}.$$

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a posteriori estimates in general

first used in linear algebra 1960's

Exact problem

Given f find $u \in \mathscr{V}$ $(\dim \mathscr{V} = \infty)$ such that $\lambda[u] = f$.

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Let $F \approx f, \Lambda \approx \lambda$ find $U \in \mathbb{V}$ $(\dim \mathbb{V} < \infty)$ s.t. $\Lambda[U] = F$.

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"Model" theorem

Error bound: There exists a **computable estimator functional** $\mathscr{E}$ such that

(upper bound) $\|U - u\| \leq \mathscr{E}[U; f, F; \lambda, \Lambda]$

(optimal order) and $\mathscr{E}[U; f, F, \lambda, \Lambda] = \mathrm{O}(\|U - u\|)$.

Main point: "estimator" $\mathscr{E}$ is **independent of exact solution** u.

1. Subtract exact

$$\langle \partial_t u, \phi \rangle + a\,(u, \phi) = \langle f, \phi \rangle \;\forall\, \phi \in \mathrm{H}^1_0(\Omega)$$

from residual, i.e., apply exact weak PDO on discrete solution,

$$\langle \partial_t U, \phi \rangle + a\,(U, \phi)\,.$$

Direct (violent) FEM a posteriori analysis
Error-Residual PDE

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2. Obtain error ($e = U - u$) relation

$$\underbrace{\langle \partial_t e, \phi \rangle + a(e, \phi)}_{\text{(weak) PDO on error}} = \langle \partial_t U - f, \phi \rangle + a(U, \phi) =: \underbrace{\langle r \,|\, \phi \rangle}_{\text{residual}} .$$

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3. Briefly, error-residual PDE (generalized sense)

$$\partial_t e + \mathscr{A} e = r.$$

Direct (violent) FEM a posteriori analysis (continued)

Galërkin orthogonality of residual

Remark (Galërkin orthogonality of residual)

Key property in analysis:

$$\langle r \,|\, \Phi \rangle = \langle \partial_t U - f, \Phi \rangle + a\,(U, \Phi) = 0 \quad \forall\, \Phi \in \mathbb{V}_h.$$

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Combined with error-residual PDE ($\partial_t e + \mathscr{A}\, e = r$)

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$$\left\| (\phi - \Pi_h \phi)/h \right\| \leq C_1 \left| \phi \right|_a \quad \text{and} \quad \left\| (\phi - \Pi_h \phi)/\sqrt{h} \right\|_\Sigma \leq C_2 \left| \phi \right|_a,$$

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- $\left|\phi\right|_a = \left\| \nabla \phi \right\|$.

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Heat energy estimate

- Test with $\phi = e$ relation

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- Obtain

$$\begin{aligned}\frac{1}{2}\,\mathrm{d}_t \left\| e \right\|^2 + \left| e \right|_a^2 &= \langle r \,|\, e - \Pi_h e \rangle \\ &= \langle R, e - \Pi_h e \rangle + \langle J, e - \Pi_h e \rangle_\Sigma\,, \\ &= \langle Rh, (e - \Pi_h e)/h \rangle + \langle J\sqrt{}h, (e - \Pi_h e)/\sqrt{}h \rangle_\Sigma\,,\end{aligned}$$

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- Residual decomposition $r := J|_\Sigma + R$ where

$R = R[U; f, \Pi^h f, \mathscr{A}]$ internal (regular) part of distribution r

$J = J[U; \mathscr{A}]$ jump (singular Σ-concentrated) part of r

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- Energy-residual, interpolation and CBS inequalities yield

$$\frac{1}{2}\,\mathrm{d}_t \left\|e\right\|^2 + \left|e\right|_a^2 \le \underbrace{\left(C_1 \left\|Rh\right\| + C_2 \left\|J\sqrt{h}\right\|_\Sigma\right)}_{=:\,\mathscr{E}[U;f,\Pi^h f,\mathscr{A},\Sigma] = \mathscr{E}[U]} \left|e\right|_a .$$

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- Integrate in time to obtain final a posteriori error estimate

$$\left(\|e(t)\|^2 + \int_0^t |e|_a^2\right)^{1/2} \leq C\int_0^t \mathscr{E}[U].$$

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Elliptic reconstruction's purpose:

Get rid of cons (and retain pros :-)

Explicit a posteriori estimates for the heat equation

A one-slide (incomplete!) history

Duality techniques

- [Eriksson and Johnson, 1991] and many others,
- optimal order in $\mathrm{L}_\infty(0,T;\mathrm{L}_2(\Omega))$,
- serious mesh restrictions in many cases,
- no estimates for gradients.

Energy techniques

- [Picasso, 1998], [Chen and Jia, 2004], [Verfürth, 2003], [Bergam et al., 2005], [Bernardi and Süli, 2005] ...
- optimal order in $\mathrm{L}_2(0,T;\mathrm{H}^1_0)$,
- suboptimal order in $\mathrm{L}_2(\Omega)$ spaces,
- mesh change effects either absent or implicit (i.e., hidden in constants).

Other possible techniques

Anything that works for the PDE stability analysis is worth trying (especially for nonlinear problems) in time:

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2. Continuous dependence, useful in nonlinear problems [Feng and Wu, 2005]
3. Semigroup techniques, e.g., [Whalbin et al]
4. Monotonicity estimates, e.g., [Nochetto et al., 2000]

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- Ultimate goal of a posteriori error estimates are: error control via mesh-adaptive methods.

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- Elliptic convergence thoroughly understood for linear problems [Nochetto, 2008, Binev et al., 2004, **?**] and some nonlinear problems [Veeser 2007].

Mesh adaption and a posteriori

- Ultimate goal of a posteriori error estimates are: error control via mesh-adaptive methods.
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  - Convergence for wavelets on tensor meshes in space-time formulation [Schwab and Stevenson, 2008].

A world without reconstruction is possible,

but one with it is better.

- Most work on parabolic a posteriori estimates is based on elliptic estimates in one way or another.
- Each elliptic technique (mainly residuals) is rederived at each attempt to derive parabolic estimates.
- Estimators are divided into "elliptic", "time" and "mixed".
- Fully discrete estimates can be very complicated.
- Why re-invent the wheel when elliptic a posteriori estimates can be read off the book, e.g., [Ainsworth and Oden, 2000]?

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What can Reconstruction buy us,

that the direct approach wouldn't?

Energy Optimal-rate $\mathrm{L}_2(\Omega)$-norm estimates via energy techniques [Makridakis and Nochetto, 2003], [Lakkis and Makridakis, 2006]. (Application is the use of L_2 estimates for Allen–Cahn simulations.)

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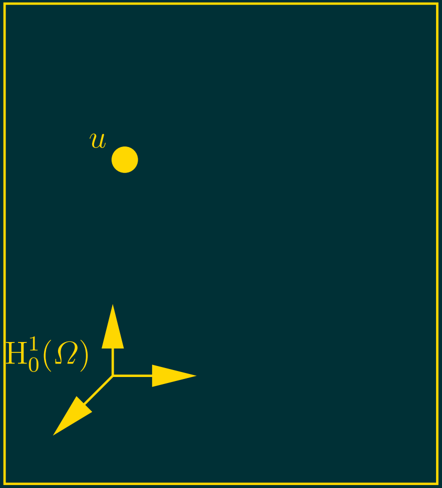
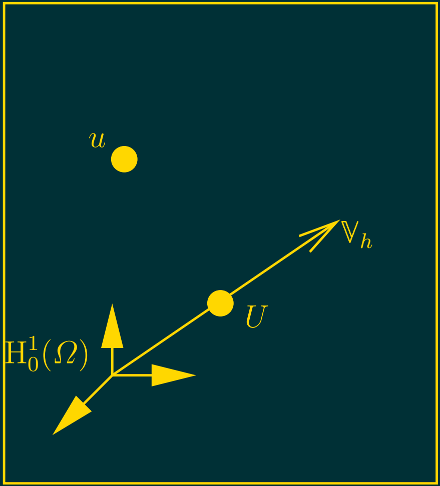
Non-conforming FEMs New simple estimates in non-conforming methods, e.g., [Georgoulis and Lakkis, vorg] for fully discrete



- $u \in \mathrm{H}_0^1(\Omega)$ exact solution



- $u \in \mathrm{H}_0^1(\Omega)$ exact solution
- $U \in \mathbb{V}_h$ parabolic $\mathbb{V}_h$-FE approximation of u.

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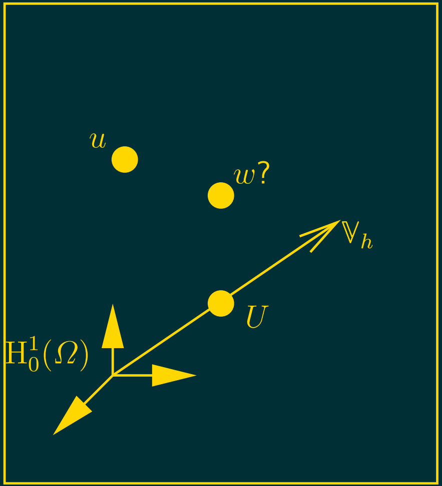
A User's Guide to the Elliptic Reconstruction

[Makridakis and Nochetto, 2003] and [Lakkis and Makridakis, 2006]



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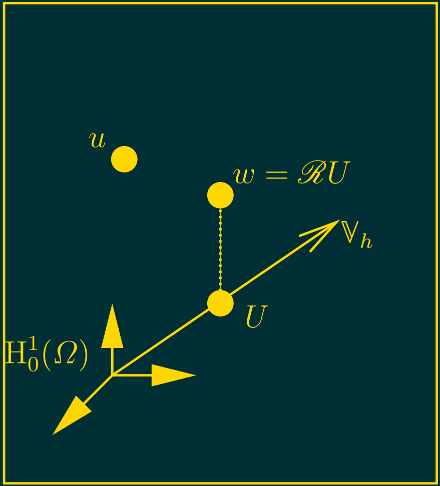

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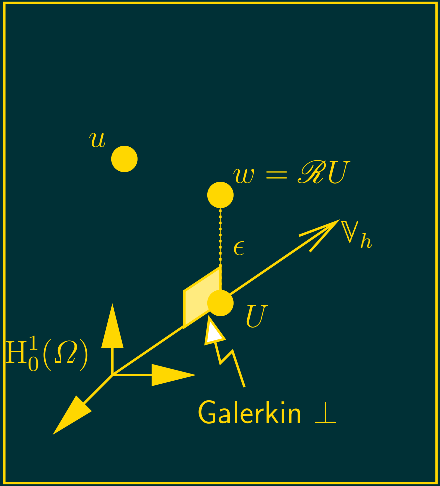

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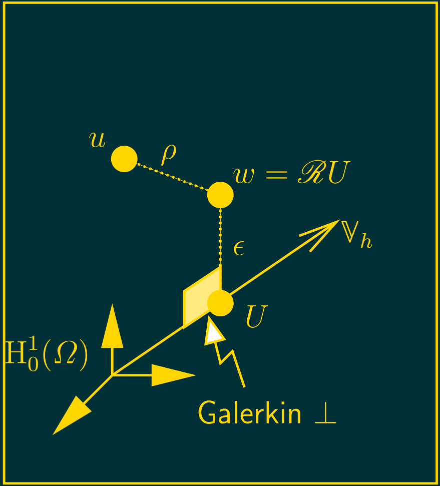

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- Galerkin orthogonality $\Rightarrow$ a posteriori bounds on $\|\epsilon\|$ are available off the shelf,
- w not computable,
- but parabolic error $\rho = w - U$ satifies parabolic equation with computable data.

Crucial parabolic-elliptic ρ-ϵ error relation

Lemma (elliptic-parabolic error relation)

For each time-slab I_n, $n \in [1 : N]$, and $\phi \in \mathrm{H}^1_0(\Omega)$,

$$\langle \partial_t \rho \,|\, \phi \rangle + a(\rho, \phi) = \langle \partial_t \epsilon, \phi \rangle + a((w - w^n), \phi) + \langle \Pi^n f^n - f, \phi \rangle + \tau_n^{-1} \langle \Pi^n U^{n-1} - U^{n-1}, \phi \rangle$$

$$\Leftrightarrow \partial_t \rho + \mathscr{A} \rho = \underbrace{\partial_t \epsilon}_{\text{space disc.n}} - \underbrace{\mathscr{A}(w - w^n)}_{\text{time disc.n}} + \underbrace{(\Pi^n f^n - f)}_{\text{data approx.n}} + \underbrace{\frac{\Pi^n U^{n-1} - U^{n-1}}{\tau_n}}_{\text{mesh change}}$$

$w^n := \mathscr{R}^n U$ *elliptic reconstruction,* $w(t)$ *p.w.linear extension,*

$$e := U - u = \text{ total error } = \begin{cases} \rho & := w - u \text{ parabolic error,} \\ -\epsilon & := U - w \text{ elliptic error,} \end{cases}$$

$\Pi^n := \mathrm{L}_2(\Omega)$*-projection onto* $\mathring{\mathbb{V}}^n$.

Elliptic reconstruction: a user's guide (controlling ρ)

$\partial_t \rho + \mathscr{A} \rho = \partial_t \epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_t \epsilon$:

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- $\partial_t \epsilon = \partial_t w - \partial_t U = \mathscr{R}\partial_t U - \partial_t U$.

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- $\partial_t U$ elliptic $\mathbb{V}_h$-FE solution with exact $\mathscr{R} \partial_t U = \partial_t w \in \mathrm{H}_0^1(\Omega)$.

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- $\Rightarrow \|\partial_t \epsilon\|$ elliptic error controlled a posteriori by estimator $\mathscr{E}[\partial_t U, \partial_t f, \mathbb{V}_h]$.

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control of the time error $\mathscr{A}(w - w^n)$:

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control of the time error $\mathscr{A}(w - w^n)$:Variety of methods depending on parabolic technique used.

Elliptic reconstruction: a user's guide (controlling ρ)

$\partial_t\rho + \mathscr{A}\rho = \partial_t\epsilon + \mathscr{A}(w - w^n) +$ controlled terms

control of the spatial error $\partial_t\epsilon$:

- Use PDE for ρ with $\partial_t\epsilon$ as data to obtain bound on $\|\rho\| < C\,\|\partial_t\epsilon\|$.
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control of the time error $\mathscr{A}(w - w^n)$:Variety of methods depending on parabolic technique used.Example: use relation

$$\partial_t U + \mathscr{A} w^n = \Pi^n f^n$$

leads to explicit a posteriori representation

$$\mathscr{A} w^n := \Pi^n f^n - \partial_t U.$$

Towards an Energy estimate

- If interested only in energy estimates.
- Semidiscrete case

$$\partial_t e + \mathscr{A}\rho = 0.$$

- Fully discrete case

$$\partial_t e + \mathscr{A}\rho = \underbrace{\left(I^n U^{n-1} - U^{n-1}\right)/\tau_n + f^n - f}_{\text{data approximation/interpolation error}} + \underbrace{\mathscr{A}w - \mathscr{A}^n w^n}_{\text{time, operator \& mesh}} .$$

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Comparison with direct approach

Error relation via reconstruction

$$\langle \partial_t \rho \,|\, \phi \rangle + a(\rho, \phi) = \langle \partial_t \epsilon, \phi \rangle + a(w - w^n, \phi) + \langle \Pi^n f^n - f, \phi \rangle + \tau_n^{-1} \left\langle \Pi^n U^{n-1} - U^{n-1}, \phi \right\rangle$$

Direct (reconstructionless) error relation

$$\langle \partial_t e, \phi \rangle + a(e, \phi) = \langle \partial_t U, \phi \rangle + a(U^n, \phi) + a(U - U^n, \phi) + \langle \Pi^n f^n - f, \phi \rangle + \tau_n^{-1} \left\langle \Pi^n U^{n-1} - U^{n-1}, \phi \right\rangle .$$

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A useful analogy with the a priori

- ER is to a posteriori what Ritz/elliptic projection is to a priori analysis.
- Optimal-order yielding properties of ER can interpreted as an a posteriori analog of the similar phenomena of superconvergence observed in the a priori analysis with the Ritz projection.
- In spatially discrete (or very small timestep) case ρ converges with a higher order error than ϵ.
- However, ρ plays an important role when time error is comparable to spatial error.

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Energy-residual estimates in $\mathrm{L}_2(\mathrm{H}^1)$ and $\mathrm{L}_\infty(\mathrm{L}_2)$

Lemma (optimal-order elliptic residual a posteriori estimates)

For all $V \in \mathbb{V}^n$ we have

$$|\mathscr{R}V - V|_{\mathrm{H}^1(\Omega)} \leq \mathscr{E}[V, \mathrm{H}^1(\Omega)] = \mathrm{O}(h_n) \text{ Residual-type estimator}$$

$$\|\mathscr{R}V - V\|_{\mathrm{L}_2(\Omega)} \leq \mathscr{E}[V, \mathrm{L}_2(\Omega)]\,\mathrm{O}(h_n^2) \text{ Residual-type estimator on convex } \Omega$$

Lemma (parabolic energy a posteriori estimate)

There are estimators $\mathscr{E}_1$ and $\mathscr{E}_2$ such that

$$\left(\max_{[0,t_m]} \|\rho(t)\|^2_{\mathrm{L}_2(\Omega)} + 2\int_0^{t_m} |\rho(t)|_a^2 \,\mathrm{d}t\right)^{1/2} \leq \left\|\rho^0\right\| + C\,\underbrace{\left(\mathscr{E}_{1,m} + \mathscr{E}_{2,m}\right)}_{\mathrm{O}(h_n)^2 \text{ (small } \tau_n)}.$$

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Remarks on ρ's order of convergence

The space estimate of ρ derives from estimating the term

$$\langle \partial_t \epsilon, \rho \rangle \le \|\partial_t \epsilon\| \, \|\rho\|$$

$$\text{or } \left(\|\partial_t \epsilon\|_{\mathrm{H}^{-1}(\Omega)} \, |\rho|_a \right)$$

Remark (superconvergence of ρ)

The fact that energy norm $|\rho|_a = \mathrm{O}(h_n^2)$ leads to optimal-order estimates for norms $\|e\|_X$ $X = \mathrm{L}_2(\Omega), \mathrm{L}_\infty(\Omega)$
[Makridakis and Nochetto, 2003, Lakkis and Makridakis, 2006, Lakkis and Makridakis, 2007, Demlow et al., vorg].

Similar elliptic results, but higher norm (test by $\partial_t \rho$) yield

Lemma (higher-order parabolic energy a posteriori estimate)

There are estimators $\tilde{\mathscr{E}}_i$, $i = 1, 2$, *such that*

$$\left(\max_{t \in [0,t_m]} |\rho(t)|_a^2 + 2 \int_0^{t_m} \|\partial_t \rho\|^2 \right)^{1/2} \leq \left|\rho^0\right|_a + 4 \left(\tilde{\mathscr{E}}_{1,m}^2 + \tilde{\mathscr{E}}_{2,m}^2 \right)^{1/2},$$

Lead to higher order norms optimal-order energy estimates.
[Lakkis and Makridakis, 2006].

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Duality estimates

Definition (Error estimators)

Suppose an a posteriori elliptic error estimator function $\mathscr{E}[\cdot,\cdot,\cdot]$ is available. Let

$$\varepsilon_n := \mathscr{E}[U^n, \mathbb{V}^n, \mathrm{L}_2(\Omega)],$$
$$\eta_n := \mathscr{E}[\partial_t U^n, \mathbb{V}^n \cap \mathbb{V}^{n-1}, \mathrm{L}_2(\Omega)],$$
$$\theta_n := \begin{cases} \frac{1}{2}\left\|\Pi^1 f^1 - \overline{\partial} U^1 - A^0 U^0\right\| & \text{for } n = 1, \\ \frac{1}{2}\left\|\partial\left(\Pi^n f^n - \overline{\partial} U^n\right)\right\| \tau_n & \text{for } n \in [2:N], \end{cases}$$

Duality estimates

Definition (Logarithmic factors)

$$b_n = \begin{cases} \frac{1}{4}\log\left(\frac{T-t_{n-1}}{T-t_n}\right) & \text{for } n \in [1:N-1]\,, \\ \frac{1}{8} & \text{for } n = N; \end{cases}$$

$$a_n = \begin{cases} \lambda\left(\frac{\tau_n}{T-t_n}\right) - \lambda\left(-\frac{\tau_{n+1}}{T-t_n}\right), \ \text{for } n \in [0:n-2]\,, \\ \lambda\left(\frac{\tau_{N-1}}{\tau_N}\right) - 1, \ \text{for } n = N-1, \end{cases}$$

where

$$\lambda(x) := \begin{cases} (1+1/x)\log(1+x) & \text{for } |x| \in (0,1), \\ 1 & \text{for } x = 0. \end{cases}$$

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Duality estimates

Theorem (General explicit duality-based a posteriori error estimates)

$$
\begin{aligned}
\|\rho(T)\| \leq \|e(0)\| &+ \left(\sum_{n=0}^{N-1} a_n \varepsilon_n^2\right)^{1/2} + \eta_N \\
&+ \left(\sum_{n=1}^{N} b_n \left(\left\|U^{n-1} - U^n\right\| + \eta_n\right)^2\right)^{1/2} + \sqrt{\frac{\tau_N}{2}}\, \|\delta_N h_N\| \\
&+ \left(\sum_{n=1}^{N-1} b_n \left\|\delta_n h_n^2\right\|^2\right)^{1/2} + \sum_{n=1}^{N} \tau_n \|\partial_t f\|_{\mathrm{L}_1(I_n;\mathrm{L}_2(\Omega))}
\end{aligned}
$$

[Lakkis and Makridakis, 2007]

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Estimates in the $\mathrm{L}_\infty(\Omega)$ norm

[Demlow et al., vorg]

- We take $\mathscr{A} = -\Delta$ thus study:

$$\partial_t u - \Delta u = f,\; u(0) = u_0 \quad \text{and} \quad u|_{\Omega} = 0.$$

- Direct approach is messy and hard, especially for fully discrete case [Boman 2000, cf.].
- Our result based on elliptic a posteriori estimates [Nochetto, 1995, Nochetto et al., 2006]

$$\|V - \mathscr{R}V\| \leq C\,(\log h_n)^2\,\mathscr{E}_{\infty,0}[V, g, \mathbb{V}^n] = \mathrm{O}\left((\log h_n h_n)^2\right)$$

for residual-based a posteriori estimator functional $\mathscr{E}_{\infty,0}$.

L_∞ estimates

Main idea [Demlow et al., vorg]

Theorem (semidiscrete estimates)

Recalling $e = U - u$, $\rho = \mathscr{R}U - u$ *and* $\epsilon = \mathscr{R}U - U$*:*

$$\|e(t)\|_{\mathrm{L}_\infty(\Omega)} \leq \|\epsilon(0)\|_{\mathrm{L}_\infty(\Omega)} + \|\epsilon(t)\|_{\mathrm{L}_\infty(\Omega)} + \int_0^t \|\partial_t \epsilon(s)\| \,\mathrm{d}s,$$

$$\|e(t)\|_{\mathrm{L}_\infty(\Omega)} \leq \|\epsilon(0)\|_{\mathrm{L}_\infty(\Omega)} + \|\epsilon(t)\|_{\mathrm{L}_\infty(\Omega)} + C_{p,q}(t) \|\partial_t \epsilon\|_{\mathrm{L}_q(0,t;\mathrm{W}_p^{-1}(\Omega))}$$

for $p, q \in (2, \infty]$ *and* $d/p + 2/q < 1$.

Proof's idea.

Use heat kernel estimates to bound $\rho(= e + \epsilon)$ in terms of ϵ knowing

$$\partial_t \rho - \Delta \rho = \partial_t \epsilon.$$

Negative Sobolev useful for convex domains and $\mathbb{P}^2$ or higher elements.

L_∞ estimates
[Demlow et al., vorg]

Theorem (fully discrete estimates)

$$\|e(t_n)\|_{L_\infty(\Omega)} \leq \|e(0)\|_{L_\infty(\Omega)} + \textit{data} + \sum_{n=1}^{N} \tau_n \left\|g^n - g^{n-1}\right\|_{L_\infty(\Omega)}$$
$$+ C \log(t_n/\tau_n) \left(\log h_n\right)^2 \max_{1\leq n\leq N} \mathscr{E}_{\infty,0}[U^n, g^n, \mathbb{V}^n]$$
$$g^n := \left(U^n - U^{n-1}\right)/\tau_n - f^n = A^n - A^{n-1} + f^n - \Pi^n f^n.$$

Elliptic error estimators [Nochetto, 1995]

$$\mathscr{E}_{\infty,0}[U^n, g^n, \mathbb{V}^n] := \max_{K\in\mathscr{T}_n} \left(\left\| h_n^2 \left(g^n - \Delta U^n\right)\right\|_{L_\infty(K)} + \left\| h_n \left[\!\left[\nabla U^n\right]\!\right]\right\|_{L_\infty(\partial K)} \right)$$

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Application to the Allen–Cahn equation

Known results by [Kessler et al., 2004] and [Feng and Wu, 2005] on the Allen-Cahn equation

$$\partial_t u - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0,$$

yield a posteriori error estimates in energy norm with a polynomial (instead of exponential!) dependence on $1/\varepsilon$ exploiting special spectral properties of the linearised operator. (Un)fortunately rate is optimal only for energy norm in both results. Using the ER we provide optimal-order estimates for the $\mathrm{L}_\infty(0, T; \mathrm{L}_2(\Omega))$ norm based on identity

$$\frac{1}{2} \mathrm{d}_t \|\rho\|^2 - \lambda_0 \|\rho\|^2 = \langle \partial_t \epsilon, \rho \rangle + \frac{1}{\varepsilon^2} \langle f(w) - f(U), \rho \rangle$$

Obtain an ε-free error.

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Towards an Energy estimate

- If interested only in energy estimates.
- Semidiscrete case

$$\partial_t e + \mathscr{A}\rho = 0.$$

- Fully discrete case

$$\partial_t e + \mathscr{A}\rho = \underbrace{\left(I^n U^{n-1} - U^{n-1}\right)/\tau_n + f^n - f}_{\text{data approximation/interpolation error}} + \underbrace{\mathscr{A}w - \mathscr{A}^n w^n}_{\text{time, operator \& mesh}} .$$

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1. The parabolic-elliptic error relation $\partial_t \rho + \mathscr{A} \rho = \partial_t \epsilon$ is sometime more useful written as
$$\partial_t e + \mathscr{A} \rho = 0.$$
2. Testing with e we obtain
$$\frac{1}{2} \|e\|^2 + \|\rho\|_a^2 = a(\rho, \epsilon).$$
Continuity of a and integration in time yield estimate
$$\int_0^t \|\rho\|_a^2 \leq C_a \int_0^t \|\epsilon\|_a^2.$$
3. Note that superconvergence properties of $\|\rho\|_a$ are lost (but not needed) in this context.
4. These simple observations and some elliptic technical work, lead to interesting nonstandard results: recovery estimators and nonconforming methods.

Recovery-based estimates

[Lakkis and Pryer, 2008]

Theorem (Zienkiewicz-Zhou)

Let G be the Zienkiewicz–Zhou gradient recovery operator. Then

$$\|V - \mathscr{R}V\|_{\mathrm{H}^1(\Omega)} \leq C \|\nabla V - GV\| .$$

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Lemma ((semidiscrete) energy norm parabolic-elliptic estimate)

$$\|e\|^2 + \int_0^t \|\rho\|_a^2 \leq C \left(\|e(0)\|_a^2 + \int_0^t \|\epsilon\|_a^2 \right) .$$

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In combination lead to recovery based estimates for parabolic equations. (Related results by [Leykekhman and Wahlbin, 2006].)

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Recovery estimates

Fully discrete estimators

- elliptic (gradient recovery error) estimator $\varepsilon_n := \|\nabla U^n - G^n[U^n]\|$,

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Lemma (gradient recovery a posteriori error estimates)

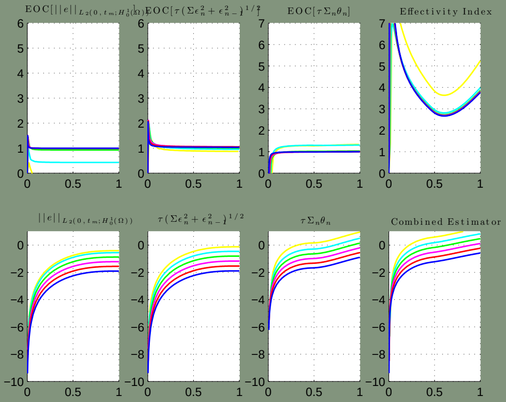
$$\left(\max_{t\in[0,T]} \|e(t)\|^2 + 2 \int_0^T \|\rho(t)\|_a^2 \, \mathrm{d}t \right)^{1/2}$$

$$\leq \|e(0)\| \, C \left(\sum_{n=1}^{N} (\beta_n \tau_n + \theta_n \tau_n + \gamma_n \tau_n) + 8 \int_0^T \|\epsilon\|_a^2 \right)^{1/2}.$$

Convergence tests for ZZ-type estimators

[Lakkis and Pryer, 2008]

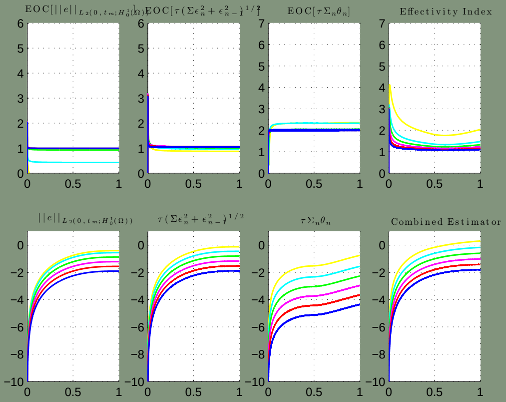
$\mathbb{P}^1$ elements $\tau = h$



Convergence tests for ZZ-type estimators

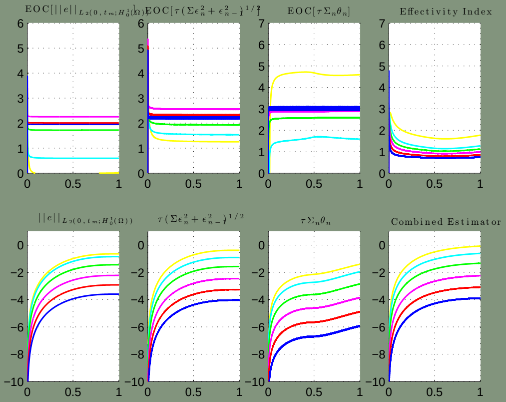
[Lakkis and Pryer, 2008]

$\mathbb{P}^1$ elements $\tau = h^2$



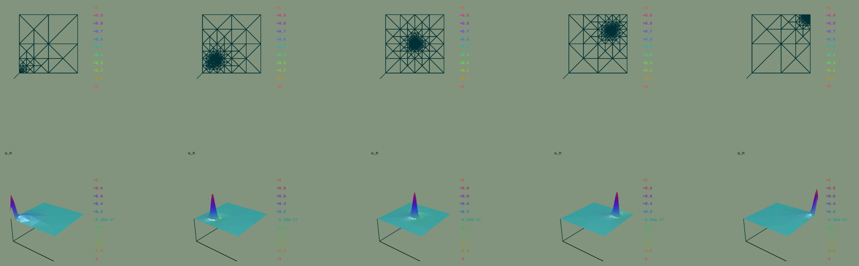
Convergence tests for ZZ-type estimators
[Lakkis and Pryer, 2008]

$\mathbb{P}^2$ elements $\tau = h^3$



ZZ Adaptive Mesh Refinement

[Lakkis and Pryer, 2008]



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Recovery adaptive method

'Easy" (regular) problem

	Uniform		Adaptive					
			refine % $= 0.65$		refine % $= 0.70$		refine % $= 0.75$	
Tolerance	DOF's	CPU	DOF's	CPU	DOF's	CPU	DOF's	CPU
0.573	232,290	3	24,080	4	22,792	5	22,240	4
0.295	3,489,090	49	42,042	8	39,414	8	38,630	6
0.149	54,097,020	598	82,172	15	77,932	15	76,452	16
0.0625	OOM	OOM	206,709	39	195,810	37	191,650	37

Space-error dominated problem

	Uniform		Adaptive					
			refine % $= 0.65$		refine % $= 0.7$		refine % $= 0.75$	
Tolerance	DOF's	CPU	DOF's	CPU	DOF's	CPU	DOF's	CPU
0.296	3,489,090	47	12,092	5	11,430	5	11,498	5
0.21	13,940,289	196	17,038	7	16,140	8	16,201	7
0.104	54,097,020	602	106,188	32	100,058	29	22,597	10
0.03125	OOM	OOM	513,694	120	460,637	118	449,568	115

Recovery adaptive method

Time-error dominated problem (implicit strategy)

	Uniform		Adaptive			
			refine % $= 0.7$		refine % $= 0.75$	
Tolerance	DOF's	CPU	DOF's	CPU	DOF's	CPU
1.000	925,809	12	159,070	43	127,610	58
0.569	3,489,090	49	237,960	142	204,376	180
0.295	54,097,020	605	471,733	755	471,542	920
0.149	OOM	OOM	940,618	1410	940,138	1850

Time-error dominated problem (explicit strategy)

	Uniform		Adaptive			
			refine % $= 0.7$		refine % $= 0.75$	
Tolerance	DOF's	CPU	DOF's	CPU	DOF's	CPU
1.000	925,809	12	135,788	5	127,004	4
0.569	3,489,090	49	198,628	7	194,311	8
0.295	54,097,026	605	397,716	15	395,876	16
0.149	OOM	OOM	2,177,666	79	2,079,081	76

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A nonconforming method (DG)

[Georgoulis and Lakkis, vorg]

Discontinuous Galerkin (DG) bilinear form discretizing Δ is given by

$$B(w,v) := \sum_{K\in\mathscr{T}} \int_K \nabla w \nabla v + \int_\Gamma \left(\theta\{\nabla v\}[\![w]\!] - \{\nabla w\}[\![v]\!] + \sigma [\![w]\!][\![v]\!]\right) \mathrm{d}s$$

where $[\![\phi]\!] := \phi_S^+ \boldsymbol{n}^+ + \phi^- \boldsymbol{n}^-$ is the jump of the scalar ϕ, and $\{\boldsymbol{\phi}\} := (\boldsymbol{\phi}^+ + \boldsymbol{\phi}^-)/2$ is the average of field $\boldsymbol{\phi}$, across the triangulations sides; θ, σ are method, penalty parameters, respectively.
DG space with respect to triangulation $\mathscr{T}$, with no hanging nodes restriction:

$$\mathbb{V}_h = \{v \in \mathrm{L}_2(\Omega) : \ v|_K \in \mathbb{P}^p\}$$

where p is a fixed polynomial degree.

Nonconforming Elliptic Reconstruction

[Georgoulis and Lakkis, vorg]

Definition (the DG elliptic reconstruction)

Let $U_{\mathrm{DG}}(t)$ be the (semidiscrete) DG solution at time $t \in [0, T]$, define elliptic reconstruction $w(t) \in H_0^1(\Omega)$, of $U_{\mathrm{DG}}(t)$ solves elliptic problem

$$\mathscr{A} w(t) = g(t) \, \forall \, t \in [0, T],$$

where

$$g(t) := A U_{\mathrm{DG}}(t) + f - \Pi f,$$

and $A : \mathbb{V}_h \to \mathbb{V}_h$ is the discrete DG-operator defined by for $V \in \mathbb{V}_h$ by

$$\langle AV, \Phi \rangle = B(V, \Phi) \;\; \forall \, \Phi \in \mathbb{V}_h.$$

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Nonconforming Elliptic Reconstruction

[Georgoulis and Lakkis, vorg]

Here $w \in \mathrm{H}^1_0(\Omega)$ and $u \in \mathrm{H}^1_0(\Omega) \Rightarrow \rho \in \mathrm{H}^1_0(\Omega)$. Define discontinuous part:

$$e_d := U_d := \epsilon_d,$$

and its continous part:

$$e_c := e - e_d = e - U_d = \rho + \epsilon_c.$$

Lemma (basic nonconforming energy estimate)

$$\frac{1}{2}\,\mathrm{d}_t \left\|e_c\right\|^2 + \left\|\rho\right\|_a^2 = \underbrace{B\left(\epsilon_c, \rho\right)}_{\textit{elliptic}} + \underbrace{\left\langle \partial_t U_d, e_c \right\rangle}_{\textit{nonconforming}} + \underbrace{l_{n-1}\left\langle A^{n-1}U^{n-1} - A^n U^n, e_c\right\rangle}_{\textit{time \& mesh-change}}$$

$$+ \underbrace{\left\langle \left(I^n U^{n-1} - U^{n-1}\right)/\tau_n + f^n - f, e_c \right\rangle}_{\textit{data \& mesh-change}}.$$

Closing remarks and outlook

Conclusions

- Elliptic reconstruction unifies known a posteriori analysis (with Makridakis, Nochetto).
- Leads to new optimal-order estimates (with Demlow, Makridakis).
- Rigorous justification the use of recovery techniques (with Pryer).
- Nonconforming methods (with Georgoulis).

Current developments (new directions?)

- Wave equation (with Georgoulis & Makridakis).
- Semilinear equations, e.g., Allen–Cahn (with Georgoulis & Makridakis), with applications to stochastic Monte-Carlo simulations (with Katsoulakis, Kossioris & Romito).
- Quasilinear equations, e.g, MCF of function graphs.

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Taxpayer's support

Bibliography I

Ainsworth, M. and Oden, J. T. (2000).
A posteriori error estimation in finite element analysis.
Wiley-Interscience [John Wiley & Sons], New York.

Bergam, A., Bernardi, C., and Mghazli, Z. (2005).
A posteriori analysis of the finite element discretization of some parabolic equations.
Math. Comp., 74(251):1117–1138 (electronic).

Bernardi, C. and Süli, E. (2005).
Time and space adaptivity for the second-order wave equation.
Math. Models Methods Appl. Sci., 15(2):199–225.

Binev, P., Dahmen, W., and DeVore, R. (2004).
Adaptive finite element methods with convergence rates.
Numer. Math., 97(2):219–268.

Bibliography II

Chen, Z. and Jia, F. (2004).
An adaptive finite element algorithm with reliable and efficient error control for linear parabolic problems.
Math. Comp., 73(247):1167–1193 (electronic).

Demlow, A., Lakkis, O., and Makridakis, C. (to appear, preprint 0711-3928@arXiv.org).
A posteriori error estimates in the maximum norm for parabolic problems.
SIAM J. Numer. Anal.

Eriksson, K. and Johnson, C. (1991).
Adaptive finite element methods for parabolic problems. I. A linear model problem.
SIAM J. Numer. Anal., 28(1):43–77.

US University of Sussex

Feng, X. and Wu, H.-j. (2005).
A posteriori error estimates and an adaptive finite element method for the Allen-Cahn equation and the mean curvature flow.
Journal of Scientific Computing, 24(2):121–146.

Georgoulis, E. and Lakkis, O. (to appear, preprint 0804.4262@arXiv.org).
A posteriori error control for discontinuous Galerkin methods for parabolic problems.
SIAM J. Numer. Anal.

Kessler, D., Nochetto, R. H., and Schmidt, A. (2004).
A posteriori error control for the Allen-Cahn problem: Circumventing Gronwall's inequality.
M2AN Math. Model. Numer. Anal., 38(1):129–142.

Bibliography IV

Lakkis, O. and Makridakis, C. (2006).
Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems.
Math. Comp., 75(256):1627–1658 (electronic).

Lakkis, O. and Makridakis, C. (2007).
A posteriori error control for parabolic problems via elliptic reconstruction and duality.
Technical Report 0709.0916, arXiv.

Lakkis, O. and Pryer, T. (2008).
Gradient recovery in adaptive methods for parabolic equations.
Technical Report To be submitted, University of Sussex, Brighton, UK.

University of Sussex

Bibliography V

Leykekhman, D. and Wahlbin, L. (2006).
A posteriori error estimates by recovered gradients in parabolic finite element equations.
Technical report, University of Texas, Austin.
Preprint (submitted to Math. Comp.).

Makridakis, C. and Nochetto, R. H. (2003).
Elliptic reconstruction and a posteriori error estimates for parabolic problems.
SIAM J. Numer. Anal., 41(4):1585–1594 (electronic).

Nochetto, R. H. (1995).
Pointwise a posteriori error estimates for elliptic problems on highly graded meshes.
Math. Comp., 64(209):1–22.

University of Sussex

- Nochetto, R. H. (2008).
  Adaptive finite element methods for elliptic pde.
  Lecture notes, University of Maryland.
- Nochetto, R. H., Savaré, G., and Verdi, C. (2000).
  A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations.
  Comm. Pure Appl. Math., 53(5):525–589.
- Nochetto, R. H., Schmidt, A., Siebert, K. G., and Veeser, A. (2006).
  Pointwise a posteriori error estimates for monotone semi-linear equations.
  Numer. Math., 104(4):515–538.
- Picasso, M. (1998).
  Adaptive finite elements for a linear parabolic problem.
  Comput. Methods Appl. Mech. Engrg., 167(3-4):223–237.

Bibliography VII

Schwab, C. and Stevenson, R. (2008).
Space-time adaptive wavelet methods for parabolic evolution problems.
Report 01, Seminar für Angewandte Mathematik ETH, Eidgenössische Technische Hochschule CH-8092 Zürich.

Verfürth, R. (2003).
A posteriori error estimates for finite element discretizations of the heat equation.
Calcolo, 40(3):195–212.

University of Sussex