Recent perspectives on Discontinuous Galerkin methods

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From joint works with

Antonietti, Arnold, Cockburn, Hughes, Marini, Masud, Süli,....

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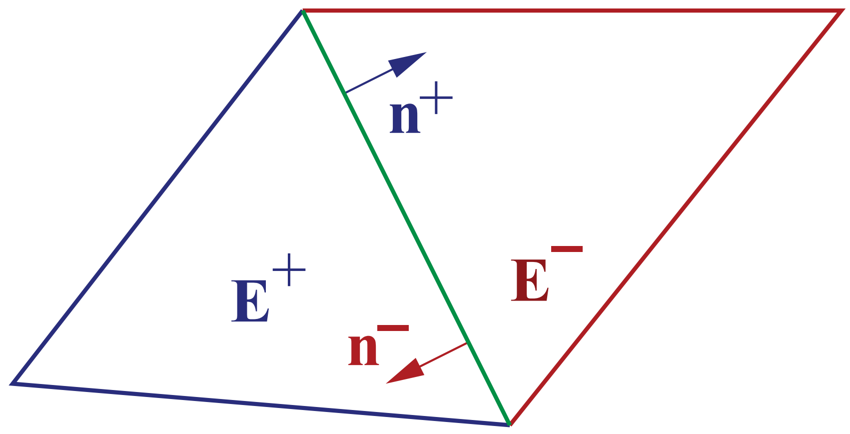
PLAN:

- Original Derivation of DG Methods
- The Weighted Residuals Formulation
- The choice of the weights
- Some numerical Results

AVERAGES AND JUMPS

$\mathscr{T}_h$: decomposition of Ω in elements K; $\mathcal{E}_h$=edges of $\mathscr{T}_h$.



Definition of average and jump on an internal edge:

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{ internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{ internal edges}$$

On the boundary edges: $[\![v]\!] = v\mathbf{n}$; $\qquad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

A MAGIC FORMULA

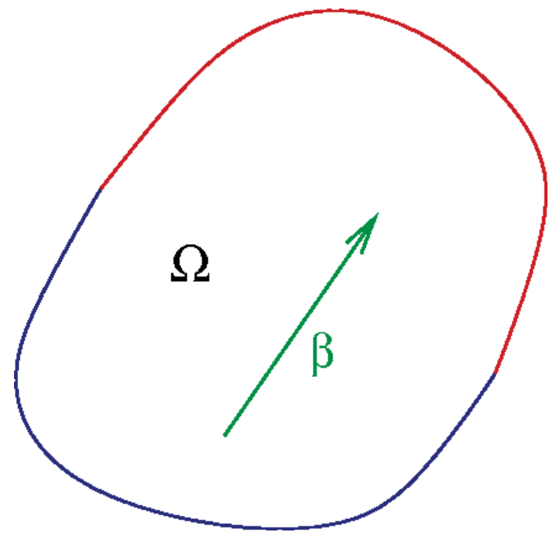
Assume that q is an edge-wise smooth scalar, and $\boldsymbol{\tau}$ an edge-wise smooth vector. We obviously accept that they have different values on the two sides of the same edge. Then, using the above definitions of jump $[\![\cdot]\!]$ and average $\{\cdot\}$, we have

$$\sum_K \int_{\partial K} q\,\boldsymbol{\tau}\cdot\mathbf{n}\,\mathrm{d}s = \sum_e \int_e [\![q]\!]\cdot\{\boldsymbol{\tau}\}\,\mathrm{d}s + \sum_{e'} \int_{e'} \{q\}[\![\boldsymbol{\tau}]\!]\,\mathrm{d}s$$

where e ranges over all edges and e' ranges over internal edges.

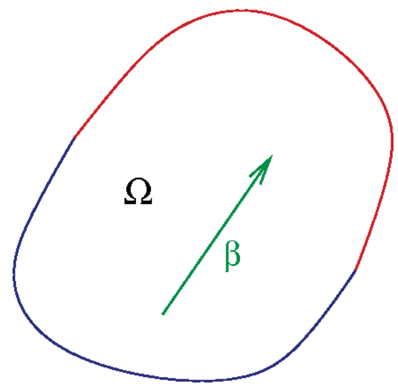
Clearly, there is **nothing** magic or deep: just *reordering terms in the sum.* But it is surely nice.

A HYPERBOLIC MODEL PROBLEM



Let Ω be a bounded polygonal domain in $\mathbf{R}^2$, and let the advective velocity field $\boldsymbol{\beta} = (\beta_1,\, \beta_2)^{\mathrm{T}}$ be a vector-valued function defined on $\overline{\Omega}$ with $\beta_i \in C^1(\overline{\Omega}),\ i = 1, 2.$

A HYPERBOLIC MODEL PROBLEM

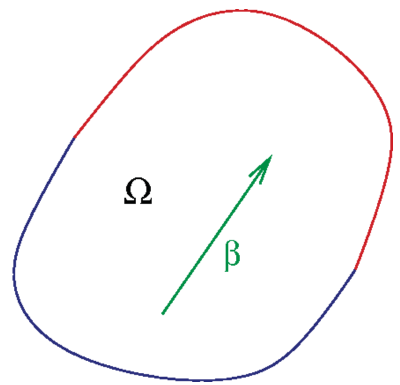


We define the *inflow* and *outflow* parts of $\Gamma = \partial\Omega$ in the usual fashion:

$$\begin{aligned}\Gamma_- &= \{x \in \Gamma : \ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) < 0\} = \text{inflow},\\ \Gamma_+ &= \{x \in \Gamma : \ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) > 0\} = \text{outflow},\end{aligned}$$

where $\mathbf{n}(x)$ denotes the unit outward normal vector to Γ at $x \in \Gamma$.

A HYPERBOLIC MODEL PROBLEM



Let moreover $\gamma \in C(\overline{\Omega})$ be the reactive term, $f \in L^2(\Omega)$ be the external source, and $g \in L^2(\Gamma_-)$ be the Dirichlet datum.

As a model problem we will consider the hyperbolic boundary value problem

$$\begin{aligned} \mathcal{L}u \equiv \operatorname{div}(\boldsymbol{\beta} u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_- \end{aligned}$$

Let us see the DG formulation (Lesaint-Raviart, Reed-Hill).

ORIGINAL DERIVATION OF THE DG METHOD

Multiply the equation by a test function v_h, and integrate over Ω:

$$\int_\Omega (\mathrm{div}(\boldsymbol{\beta} u) + \gamma u) v_h \mathrm{d}x = \int_\Omega f v_h \mathrm{d}x.$$

Then integrate by parts the first term:

$$\sum_{K \in \mathscr{T}_h} \int_K (-u\,(\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u v_h) \mathrm{d}x + \sum_{K \in \mathscr{T}_h} \int_{\partial K} (\boldsymbol{\beta} \cdot \mathbf{n})\, u\, v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

Using the magic formula, $[\![\boldsymbol{\beta} u]\!] = 0$, and $u = g$ on Γ_-:

$$\sum_{K \in \mathscr{T}_h} \int_K (-u\,(\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u v_h) \mathrm{d}x + \sum_{e \not\subset \Gamma_-} \int_e \{\boldsymbol{\beta} u\} \cdot [\![v_h]\!] \mathrm{d}s + \sum_{e \subset \Gamma_-} \int_e \boldsymbol{\beta} \cdot \mathbf{n}\, g\, v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

Then you write the equation putting u_h instead of u

$$\sum_{K\in\mathscr{T}_h}\int_K(-u_h\,(\boldsymbol{\beta}\cdot\nabla v_h)+\gamma u_h v_h)\mathrm{d}x \\ +\sum_{e\not\subset\Gamma_-}\int_e\{\boldsymbol{\beta}u_h\}\cdot[\![v_h]\!]\mathrm{d}s+\sum_{e\subset\Gamma_-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}\,g\,v_h\mathrm{d}s=\int_\Omega f v_h\mathrm{d}x.$$

Finally you substitute the *average* $\{\boldsymbol{\beta}u_h\}$ by the *upwind average* $\{\boldsymbol{\beta}u\}_{up}$, defined, on every edge, as the value of u_h coming from the upwind triangle:

$$\sum_{K\in\mathscr{T}_h}\int_K(-u_h\,(\boldsymbol{\beta}\cdot\nabla v_h)+\gamma u_h v_h)\mathrm{d}x \\ +\sum_{e\not\subset\Gamma_-}\int_e\{\boldsymbol{\beta}u_h\}_{up}\cdot[\![v_h]\!]\mathrm{d}s+\sum_{e\subset\Gamma_-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}\,g\,v_h\mathrm{d}s=\int_\Omega f v_h\mathrm{d}x.$$

AN ELLIPTIC MODEL PROBLEM (DARCY)

Let $\kappa \in \mathcal{L}^\infty(\Omega)$ be the diffusion coefficient, and consider the problem:

$$\begin{cases} A\,u \equiv -\mathrm{div}(\kappa \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

It is often convenient to introduce the flux $\boldsymbol{\sigma} = -\kappa \nabla u$ so that the problem splits in two equations

$$\begin{cases} \kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0 & \text{in } \Omega \\ \mathrm{div}\boldsymbol{\sigma} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Let us see its DG formulation (Arnold, Wheeler, Douglas-Dupont)

You multiply the equation by a test function v_h and integrate over Ω. Then integrate by parts.

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v_h \mathrm{d}x - \sum_K \int_{\partial K} \kappa\nabla u \cdot \mathbf{n}\, v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

Rearranging terms with the magic formula you have

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v_h - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa\nabla u\} - \sum_{e'} \int_{e'} [\![\kappa\nabla u]\!]\{v_h\} = \int_\Omega f v_h.$$

Now you say:"Oh, but I know that u is smooth: hence $[\![\kappa\nabla u]\!]$ is zero and I can forget about it!" You then write

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v_h \mathrm{d}x - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa\nabla u\} \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

You had

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v_h \mathrm{d}x - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa \nabla u\} \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x.$$

Then you say: "Since u is smooth, then also $[\![u]\!] = 0$!" And you add a term to restore *symmetry*

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v_h - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa \nabla u\} - \sum_e \int_e [\![u]\!] \cdot \{\kappa \nabla_h v_h\} = \int_\Omega f v_h.$$

Then you write u_h in place of u

$$\int_\Omega \kappa \nabla_h u_h \cdot \nabla_h v_h - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa \nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\kappa \nabla_h v_h\} = \int_\Omega f v.$$

Your discrete problem is now

$$\int_\Omega \kappa \nabla_h u_h \cdot \nabla_h v_h - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa \nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\kappa \nabla_h v_h\} = \int_\Omega f v_h.$$

Then you say: ”Gosh! My method is unstable! However, since $[\![u]\!] = 0$, I can add a stabilizing term”

$$\int_\Omega \kappa \nabla_h u_h \cdot \nabla_h v_h - \sum_e \int_e [\![v_h]\!] \cdot \{\kappa \nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\kappa \nabla_h v_h\} + \sum_e \frac{\kappa\gamma}{|e|} \int_e [\![u_h]\!] \cdot [\![v_h]\!] = \int_\Omega f v_h.$$

And you are happy. This is ”IP” (Interior Penalty).

DG MIXED FORMULATION

Let us see the DG mixed formulation (Bassi-Rebay, Cockburn-Shu).

You multiply the equations $\kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0$ and $\mathrm{div}\boldsymbol{\sigma} = f$ by the test functions $\boldsymbol{\tau}$ and v, respectively, and integrate by parts

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \sum_K \int_K u\,\mathrm{div}\,\boldsymbol{\tau}\mathrm{d}x + \sum_K \int_K \hat{u}\,\boldsymbol{\tau}\cdot\mathbf{n}\mathrm{d}s = 0$$

$$-\sum_K \int_K \boldsymbol{\sigma}\cdot\nabla v\mathrm{d}x + \sum_K \int_K v\,\hat{\boldsymbol{\sigma}}\cdot\mathbf{n}\mathrm{d}s = \int_\Omega f v$$

where $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are the numerical fluxes meant to approximate $u_{|\partial E}$ and $\boldsymbol{\sigma}_{|\partial E} \equiv -\kappa\nabla u_{|\partial E}$ (respectively), to be modelled later on.

Then you apply the magic formula assuming that $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are single valued

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega u\,\mathrm{div}_h\boldsymbol{\tau}\,\mathrm{d}x + \sum_{e'}\int_{e'} \hat{u}\,[\![\boldsymbol{\tau}]\!]\mathrm{d}s = 0$$

$$-\int_\Omega \boldsymbol{\sigma}\cdot\nabla_h v\,\mathrm{d}x + \sum_e \int_e \hat{\boldsymbol{\sigma}}\,[\![v]\!]\mathrm{d}s = \int_\Omega f v$$

Then you put $\boldsymbol{\sigma}_h$ in place of $\boldsymbol{\sigma}$ and u_h in place of u, and you integrate by parts back (!) the first equation

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{\tau} + \int_\Omega \nabla_h u_h\cdot\boldsymbol{\tau} + \sum_{e'}\int_{e'}\{\hat{u}-u_h\}[\![\boldsymbol{\tau}]\!]ds + \sum_e\int_e [\![u_h]\!]\{\boldsymbol{\tau}\}ds = 0$$

$$-\int_\Omega \boldsymbol{\sigma}\cdot\nabla_h v\,\mathrm{d}x + \sum_e \int_e \hat{\boldsymbol{\sigma}}\,[\![v]\!]\mathrm{d}s = \int_\Omega f v$$

If your choice of $\hat{u}$ depends only on u_h, and if the gradients of the discretized scalars are contained in the space of discretized vectors, you can use the first equation

$$\int_\Omega \kappa^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_\Omega \nabla u_h \cdot \boldsymbol{\tau} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\boldsymbol{\tau}]\!] ds - \sum_e \int_e [\![u_h]\!]\{\boldsymbol{\tau}\} ds = 0$$

to express $\boldsymbol{\sigma}_h$ directly as an explicit function of u_h

$$\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h(u_h)$$

and substitute in the second

$$-\int_\Omega \boldsymbol{\sigma}(u_h) \cdot \nabla_h v \mathrm{d}x + \sum_e \int_e \hat{\boldsymbol{\sigma}} [\![v]\!] \mathrm{d}s = \int_\Omega f v.$$

For various choices of $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ you get a whole ZOO of methods (Arnold-B-Cockburn-Marini).

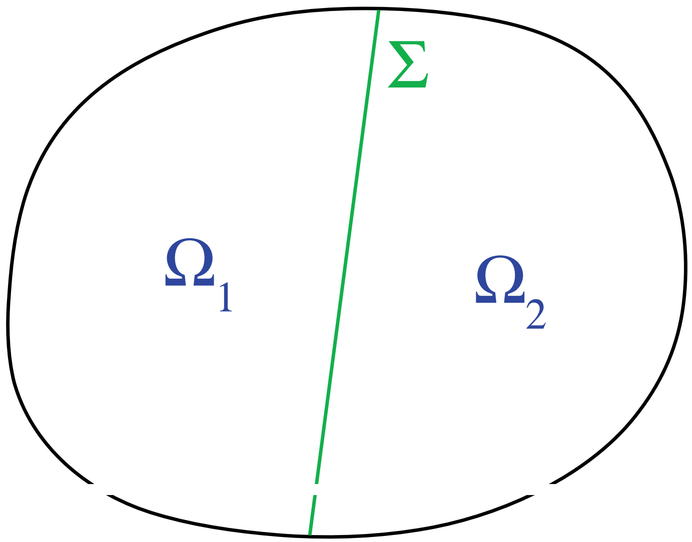
THE WEIGHTED RESIDUALS APPROACH -1

(B-Cockburn-Marini-Süli - 2006)

Let us see the basic ideas behind it. Assume that we want to solve the problem

$$\operatorname{div}(\boldsymbol{\beta} u) + \gamma u = f \text{ in } \Omega \equiv \Omega_1 \cup \Sigma \cup \Omega_2$$

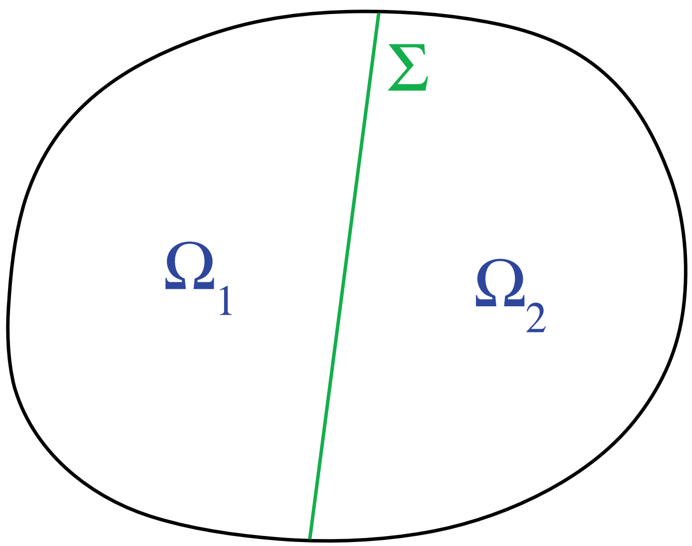
with, say, $\boldsymbol{\beta} \cdot \mathbf{n} u = 0$ on Γ_-. Then we have



$$\operatorname{div}(\boldsymbol{\beta} u) + \gamma u = f \text{ in } \Omega_i \ (i = 1, 2)$$

$$[\![\boldsymbol{\beta}\, u]\!] = 0 \text{ on } \Sigma \text{ and on } \Gamma_-$$

THE WEIGHTED RESIDUALS APPROACH - 2



$\mathrm{div}(\boldsymbol{\beta} u) + \gamma u = f$ in Ω_i $(i = 1, 2)$

$[\![\boldsymbol{\beta}\, u]\!] = 0$ on Σ and on Γ_-

Accordingly you could choose two operators, B_0 and B_1 and write

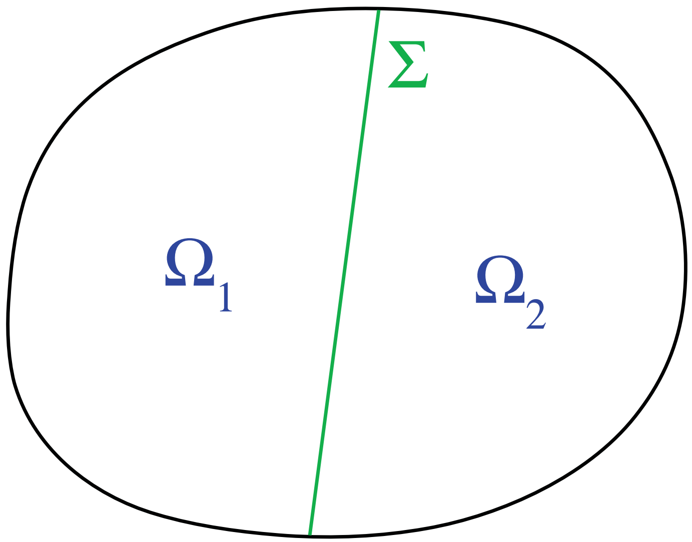
$$\sum_{i=1}^{2} \int_{\Omega_i} (\mathrm{div}(\boldsymbol{\beta} u_h) + \gamma u_h - f)\, B_0 v_h \mathrm{d}x + \int_{\Sigma \cup \Gamma_-} [\![\boldsymbol{\beta} u_h]\!] B_1 v_h \mathrm{d}s = 0 \quad \forall v_h$$

THE WEIGHTED RESIDUALS APPROACH -3

Similarly, if you want to solve the problem

$$-\operatorname{div}(\kappa\nabla u) = f \text{ in } \Omega \equiv \Omega_1 \cup \Sigma \cup \Omega_2$$

with, say, $u = 0$ on Γ, then you have instead

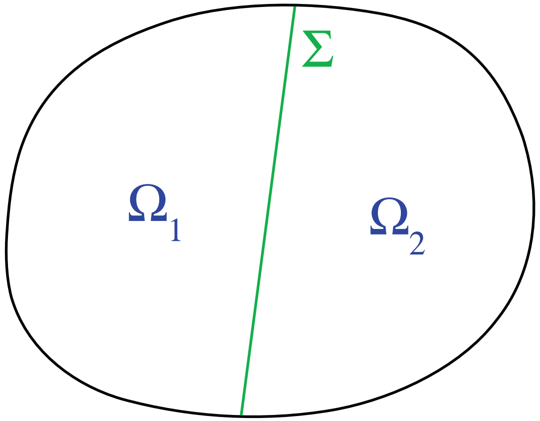


$$-\operatorname{div}(\kappa\nabla u) = f \text{ in } \Omega_i \ (i = 1, 2)$$

$$[\![u]\!] = 0 \text{ on } \Sigma \text{ and on } \Gamma$$

$$[\![\kappa\nabla u]\!] = 0 \text{ on } \Sigma$$

THE WEIGHTED RESIDUALS APPROACH - 4



$$-\operatorname{div}(\kappa\nabla u) = f \text{ in } \Omega_i \ (i = 1, 2)$$

$$[\![u]\!] = 0 \text{ on } \Sigma \text{ and on } \Gamma$$

$$[\![\kappa\nabla u]\!] = 0 \text{ on } \Sigma$$

Then you could choose **three** operators, B_0, B_1 and B_2, and write

$$\sum_{i=1}^{2} \int_{\Omega_i} (-\operatorname{div}(\kappa\nabla u_h) - f)\, B_0 v_h \mathrm{d}x + \int_{\Sigma\cup\Gamma} [\![u_h]\!] \cdot B_1 v_h \mathrm{d}s + \int_{\Sigma} [\![\kappa\nabla u_h]\!]\, B_2 v_h \mathrm{d}s = 0 \quad \forall v_h$$

CHOICE OF THE WEIGHTS -1

Starting from

$$((\mathrm{div}(\boldsymbol{\beta} u_h) + \gamma u_h - f), B_0 v_h)_h + < [\![\boldsymbol{\beta} u_h]\!], B_1 v_h >_- = 0 \quad \forall v_h$$

you can choose $B_0 v_h \equiv v_h$ and integrate by parts

$$\begin{aligned}(u_h, -\boldsymbol{\beta} \cdot \nabla v_h + \gamma v_h)_h - (f, v_h)_h \\ + < \{\boldsymbol{\beta} u_h\}, [\![v_h]\!] > + < [\![\boldsymbol{\beta} u_h]\!], \{v_h\} >_- \\ + < [\![\boldsymbol{\beta} u_h]\!], B_1 v_h >_- = 0 \quad \forall v_h\end{aligned}$$

CHOICE OF THE WEIGHTS - 2

Having

$$(u_h, -\boldsymbol{\beta}\cdot\nabla v_h + \gamma v_h)_h - (f, v_h)_h + < \{\boldsymbol{\beta} u_h\}, [\![v_h]\!] > + < [\![\boldsymbol{\beta} u_h]\!], \{v_h\} >_- + < [\![\boldsymbol{\beta} u_h]\!], B_1 v_h >_- = 0 \quad \forall v_h$$

you can choose $B_1 v_h = -\{v_h\} + \frac{|\boldsymbol{\beta}\cdot\mathbf{n}|}{2|\boldsymbol{\beta}|}\mathbf{n}\cdot[\![v_h]\!]$ and get

$$(u_h, -\boldsymbol{\beta}\cdot\nabla v_h + \gamma v_h)_h - (f, v_h)_h + < \{\boldsymbol{\beta} u_h\}_{upw}, [\![v_h]\!] >_- = 0 \quad \forall v_h$$

CHOICE OF THE WEIGHTS - 3

On the other hand from

$$(\mathrm{div}(\boldsymbol{\beta} u_h) + \gamma u_h - f, B_0 v_h)_h + < [\![\boldsymbol{\beta} u_h]\!], B_1 v_h >_- = 0 \quad \forall v_h$$

you can (brutally) choose $B_0 v_h \equiv \mathrm{div}(\boldsymbol{\beta} v_h) + \gamma u_h$ and $B_1 v_h = \mu(h)[\![\boldsymbol{\beta} v_h]\!]$ to get

$$(\mathrm{div}(\boldsymbol{\beta} u_h) + \gamma u_h - f, \mathrm{div}(\boldsymbol{\beta} v_h) + \gamma v_h)_h + \mu(h) < [\![\boldsymbol{\beta} u_h]\!], [\![\boldsymbol{\beta} v_h]\!] >_- = 0 \quad \forall v_h$$

CHOICE OF THE WEIGHTS - 4

$$(\mathrm{div}(\boldsymbol{\beta} u_h) + \gamma u_h - f, B_0 v_h)_h + < [\![\boldsymbol{\beta} u_h]\!], B_1 v_h >_- = 0 \quad \forall v_h$$

However, you can also take $B_0 v_h \equiv v_h$ and $B_1 v_h = \mu(h)[\![\boldsymbol{\beta} v_h]\!]$ for $\mu(h)$ big enough to get

$$(\mathrm{div}(\boldsymbol{\beta} u_h) + \gamma u_h - f, v_h)_h + \mu(h) < [\![\boldsymbol{\beta} u_h]\!], [\![\boldsymbol{\beta} v_h]\!] >_- = 0 \quad \forall v_h$$

that surely works for $\gamma + \nabla \cdot \boldsymbol{\beta} > 0$.

CHOICE OF THE WEIGHTS FOR DARCY-1

Coming back to the Darcy problem $-\operatorname{div}(\kappa\nabla u) = f$, with zero Dirichlet boundary conditions, our residual equations are now:

- $\operatorname{div} \kappa\nabla u + f = 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\kappa\nabla u]\!] = 0$ on each internal edge

CHOICE OF THE WEIGHTS FOR DARCY-2

Then we have to choose **three** operators B_0, B_1, and B_2, and to write

$$(A\,u_h - f,\, B_0\,(v_h))_h + \\ < [\![u_h]\!],\, B_1(v_h) >_h + < [\![\kappa \nabla u_h]\!],\, B_2(v_h) >_h^0 = 0 \quad \forall v_h$$

where again

$$(u,\, v)_h = \sum_{K \in \mathscr{T}_h} \int_K u\, v\, \mathrm{d}x \qquad < u,\, v >_h = \sum_{e \in \mathcal{E}_h} \int_e u\, v\, \mathrm{d}s$$

and $< u,\, v >_h^0$ runs only on internal edges

A POSSIBLE CHOICE FOR B_0

Choosing $B_0(v) \equiv v$ and using the magic formula, we can write:

$$
\begin{aligned}
(Au, B_0 v)_h &\equiv (-\mathrm{div}(\kappa \nabla u_h),\, v_h)_h \\
&= (\kappa \nabla u_h,\, \nabla v_h)_h - \sum_K \int_{\partial K} \kappa \nabla u_h \cdot \mathbf{n}\, v \,\mathrm{d}s \\
&= (\kappa \nabla u_h,\, \nabla v_h)_h - < \{\kappa \nabla u_h\}, [\![v_h]\!] >_h - < [\![\kappa \nabla u_h]\!], \{v_h\} >_h^0 .
\end{aligned}
$$

A POSSIBLE CHOICE FOR B_0

With the previous choice $B_0(v) \equiv v$ our method becomes then

$$
\begin{aligned}
(\kappa \nabla u_h, \nabla v_h)_h - (f, v_h) \\
&- < \{\kappa \nabla u_h\}, [\![v_h]\!] >_h - < [\![\kappa \nabla u_h]\!], \{v_h\} >_h^0 \\
&+ < [\![u_h]\!], B_1(v_h) >_h + < [\![\kappa \nabla u_h]\!], B_2(v_h) >_h^0 = 0
\end{aligned}
$$

for all v_h in the discrete space.

A POSSIBLE CHOICE FOR B_2

With the previous choice $B_0(v) \equiv v$ our method becomes then

$$
\begin{aligned}
(\kappa\nabla u_h,\, \nabla v_h)_h - (f, v_h) & \\
&- < \{\kappa\nabla u_h\}, [\![v_h]\!] >_h - < [\![\kappa\nabla u_h]\!], \{v_h\} >_h^0 \\
&+ < [\![u_h]\!],\, B_1(v_h) >_h \,+\, < [\![\kappa\nabla u_h]\!],\, B_2(v_h) >_h^0 = 0
\end{aligned}
$$

for all v_h in the discrete space. The **strong** temptation is to use $B_2(v_h) = \{v_h\}$ (in order to kill the second purple term)

.

A POSSIBLE CHOICE FOR B_1

With the previous choice $B_0(v) \equiv v$ our method becomes then

$$\begin{aligned}(\kappa\nabla u_h,\, \nabla v_h)_h - (f, v_h) \\ - < \{\kappa\nabla u_h\}, [\![v_h]\!] >_h \\ + < [\![u_h]\!], B_1(v_h) >_h \quad = 0\end{aligned}$$

for all v_h in the discrete space. The **strong** temptation is to use $B_2(v_h) = \{v_h\}$ (in order to kill the second purple term) and $B_1(v_h) = -\{\kappa\nabla u_h\}$ in order to symmetrize the first purple term.

A POSSIBLE CHOICE FOR B_1

With the previous choice $B_0(v) \equiv v$ our method becomes then

$$(\kappa\nabla u_h,\, \nabla v_h)_h - (f, v_h) - < \{\kappa\nabla u_h\}, [\![v_h]\!] >_h - < [\![u_h]\!], \{\kappa\nabla u_h\} >_h = 0$$

for all v_h in the discrete space. The **strong** temptation is to use $B_2(v_h) = \{v_h\}$ (in order to kill the second purple term) and $B_1(v_h) = -\{\kappa\nabla u_h\}$ in order to symmetrize the first purple term. But you could also use $B_1(v_h) = +\{\kappa\nabla u_h\}$ to get B-O

A POSSIBLE CHOICE FOR B_1

Surely, in all cases, you would need to add some stabilizing term to B_1. Hence, for instance, with $B_0(v) \equiv v$, $B_2(v_h) = \{v_h\}$, and $B_1(v_h) = -\{\kappa\nabla u_h\} + C_1(h)[\![v_h]\!]$ the method becomes

$$(\kappa\nabla u_h,\, \nabla v_h)_h - (f, v_h)$$
$$- < \{\kappa\nabla u_h\}, [\![v_h]\!] >_h$$
$$- < [\![u_h]\!], \{\kappa\nabla u_h\} >_h + C_1(h) < [\![u_h]\!], [\![u_h]\!] >_h \; = 0 \quad \forall v_h$$

OTHER CHOICES OF THE WEIGHTS FOR DARCY

We had to choose **three** operators B_0, B_1, and B_2 in

$$(A\,u_h - f,\, B_0\,(v_h))_h + \\ < [\![u_h]\!],\, B_1(v_h) >_h + < [\![\kappa\nabla u_h]\!],\, B_2(v_h) >_h^0 = 0 \quad \forall v_h$$

A simple minded but viable choice would be to take $B_0\,(v_h) = A\,v_h$, $B_1(v_h) = C_1(h)[\![v_h]\!]$, and $B_2(v_h) = C_2(h)[\![\kappa\nabla u_h]\!]$ to reach

$$(A\,u_h - f,\, A\,v_h)_h + \\ C_1(h) < [\![u_h]\!],\, [\![v_h]\!]) >_h + C_2(h) < [\![\kappa\nabla u_h]\!],\, [\![\kappa\nabla u_h]\!] >_h^0 = 0 \quad \forall v_h$$

OTHER CHOICES OF THE WEIGHTS FOR DARCY

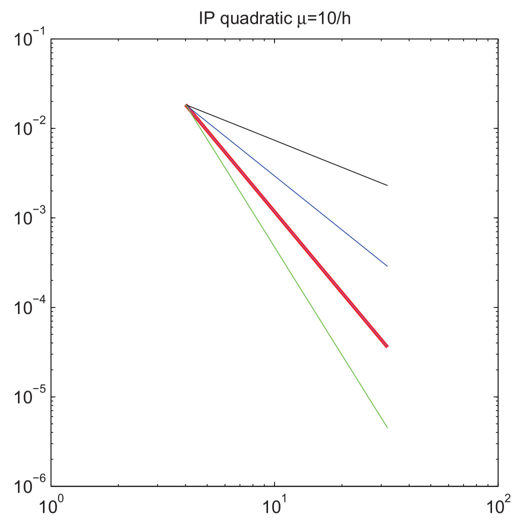
We had to choose **three** operators B_0, B_1, and B_2 in

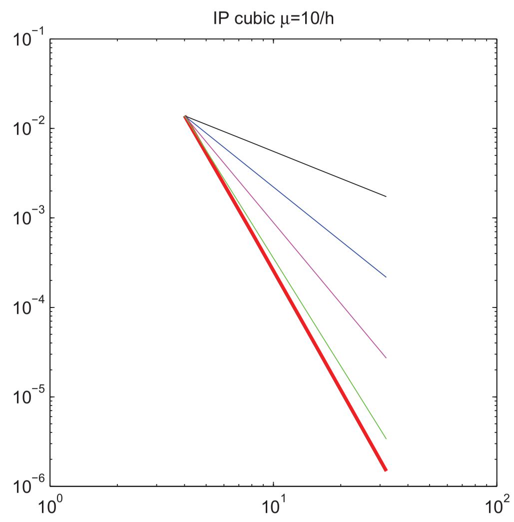
$$(A\,u_h - f,\, B_0\,(v_h))_h + \\ < [\![u_h]\!],\, B_1(v_h) >_h + < [\![\kappa\nabla u_h]\!],\, B_2(v_h) >_h^0 = 0 \quad \forall v_h$$

Now you could "play dumb" and take $B_0\,(v_h) = v_h$,
$B_1(v_h) = C_1(h)[\![v_h]\!]$, and $B_2(v_h) = C_2(h)[\![\kappa\nabla u_h]\!]$ to reach

$$(A\,u_h - f,\, v_h)_h + \\ C_1(h) < [\![u_h]\!],\, [\![v_h]\!]) >_h + C_2(h) < [\![\kappa\nabla u_h]\!],\, [\![\kappa\nabla u_h]\!] >_h^0 = 0 \quad \forall v_h$$

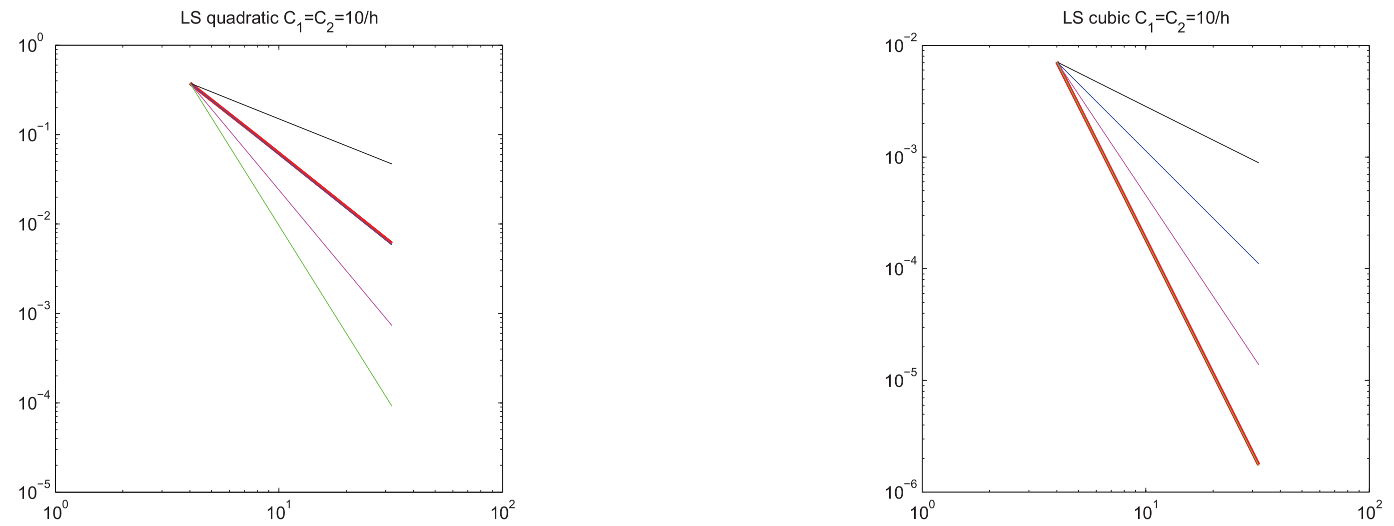
THE CLASSICAL CHOICE (IP)





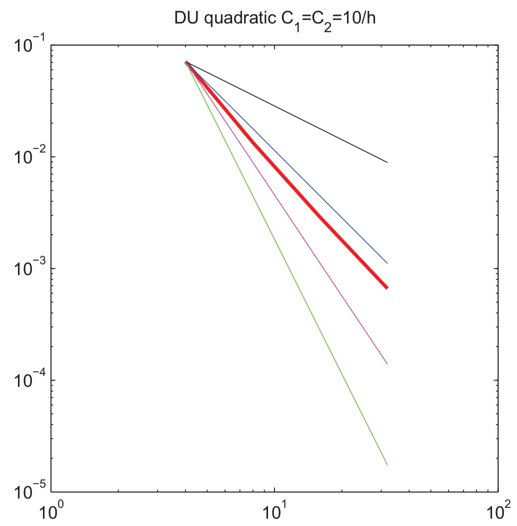
Left: Quadratic elements, $C_1 = 10/h$. Right: Cubic elements, $C_1 = 10/h$.

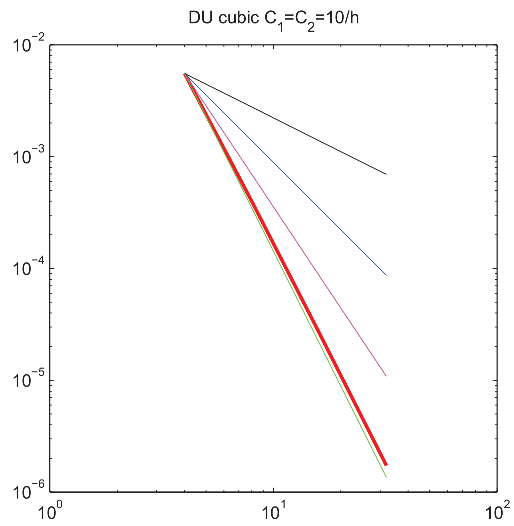
THE LEAST SQUARES CHOICE



Left: Quadratic elements, $C_1 = C_2 = 10/h$. Right: Cubic elements, $C_1 = C_2 = 10/h$.

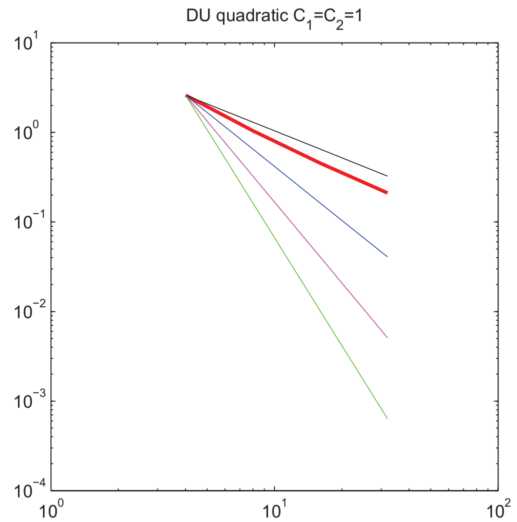
THE DUMB CHOICE

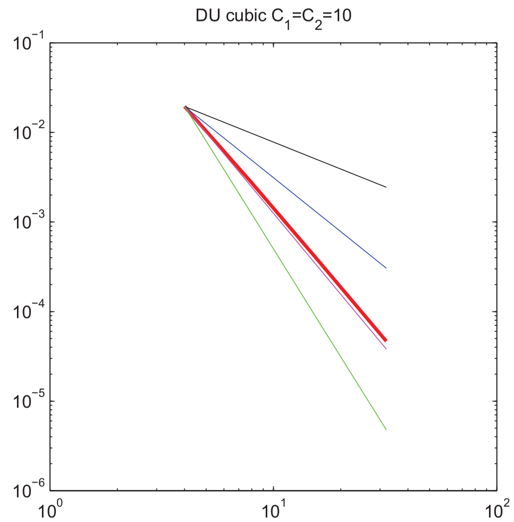




Left: Quadratic elements, $C_1 = C_2 = 10/h$. Right: Cubic elements, $C_1 = C_2 = 10/h$.

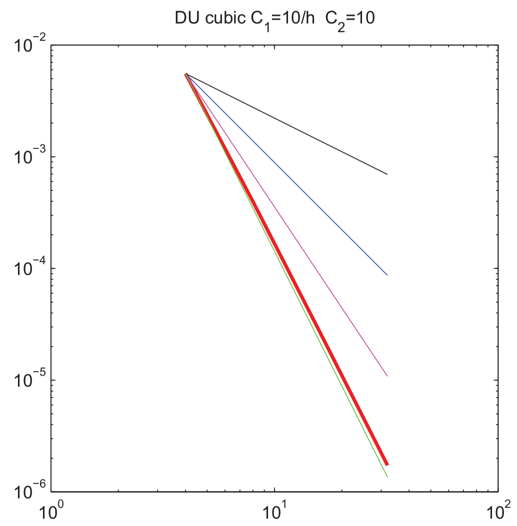
THE DUMB CHOICE: SENSITIVITY

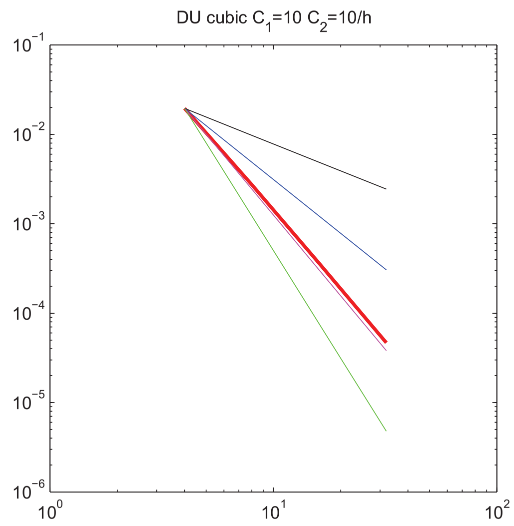




Left: Quadratic elements, $C_1 = C_2 = 1$. Right: Cubic elements, $C_1 = C_2 = 10$.

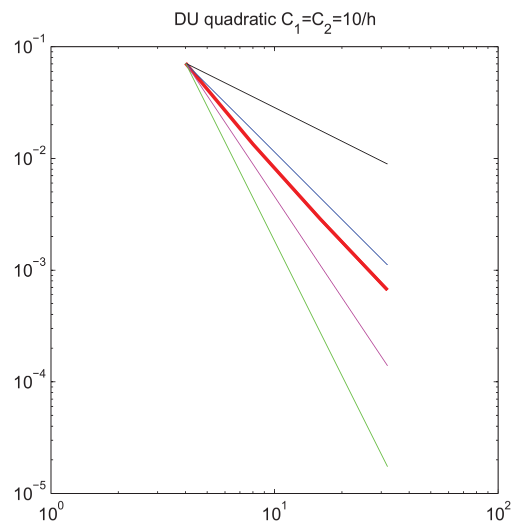
THE DUMB CHOICE: SENSITIVITY

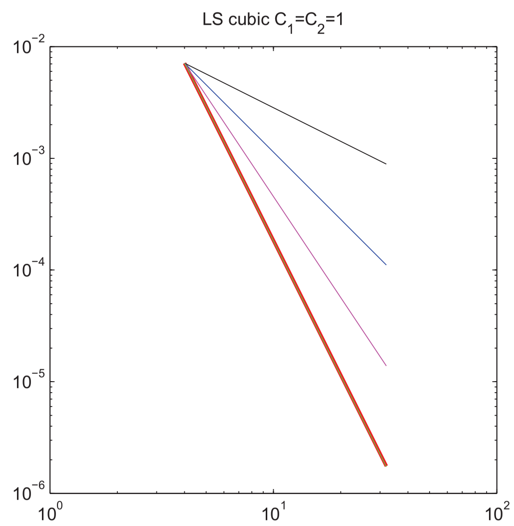




Left: Cubic elements, $C_1 = 10/h \quad C_2 = 10$. Right: Cubic elements, $C_1 = 10 \quad C_2 = 10/h$.

OTHER RESULTS





Left: Dumb, quadratic elements, $C_1 = C_2 = 10/h$. Right: Least Squares, cubic elements, $C_1 = 1 \quad C_2 = 1$.

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- You could use WRDG because it is new, and there is much to be done
- You could use WRDG because finally you know what you are doing