Krylov Subspace-Based Dimension Reduction of Large-Scale Linear Dynamical Systems

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Outline

- Motivation
- From **AWE** to **PVL**
- Descriptor systems
- Preserving RCL structures
- **SPRIM**
- Using restarted Krylov subspace methods
- Concluding remarks

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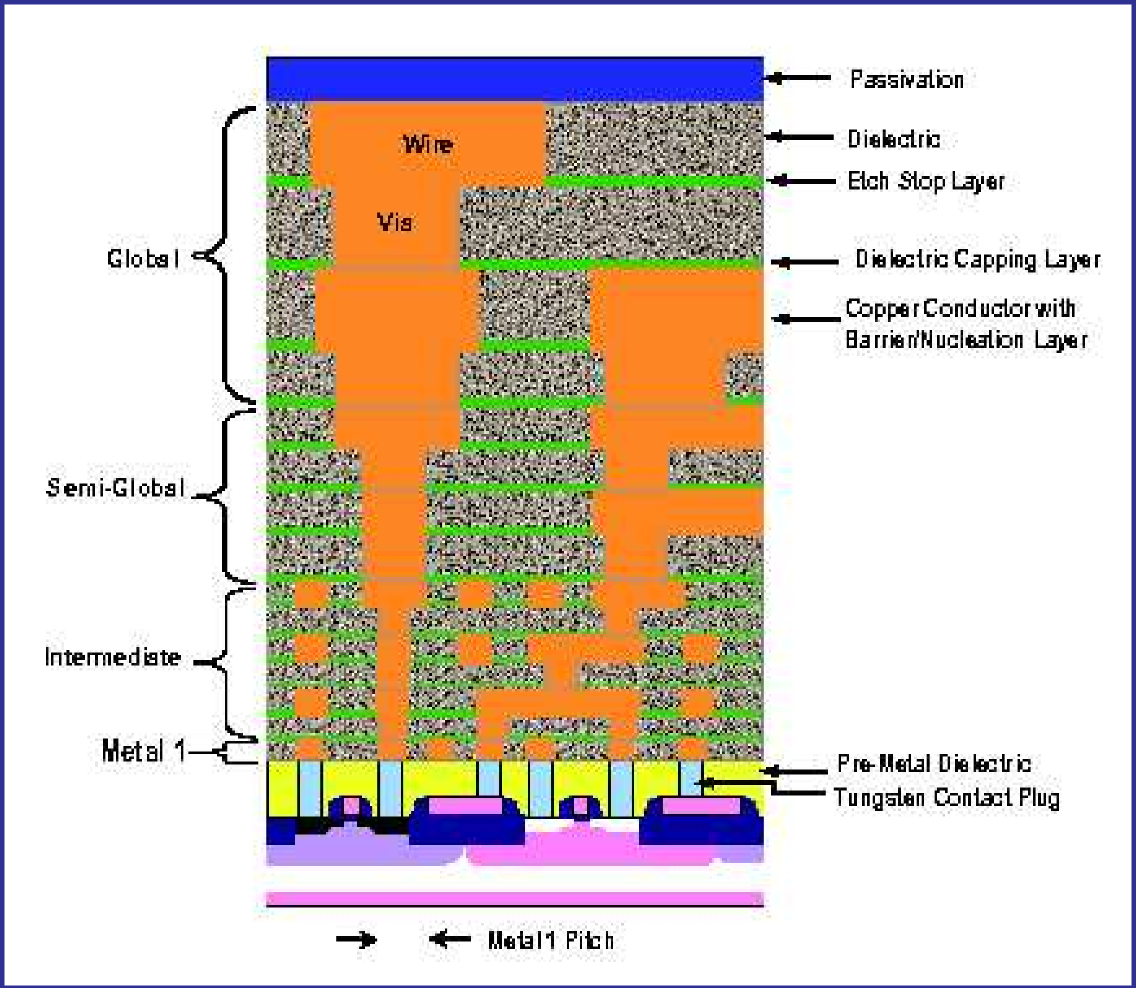
State-of-the-art VLSI circuits

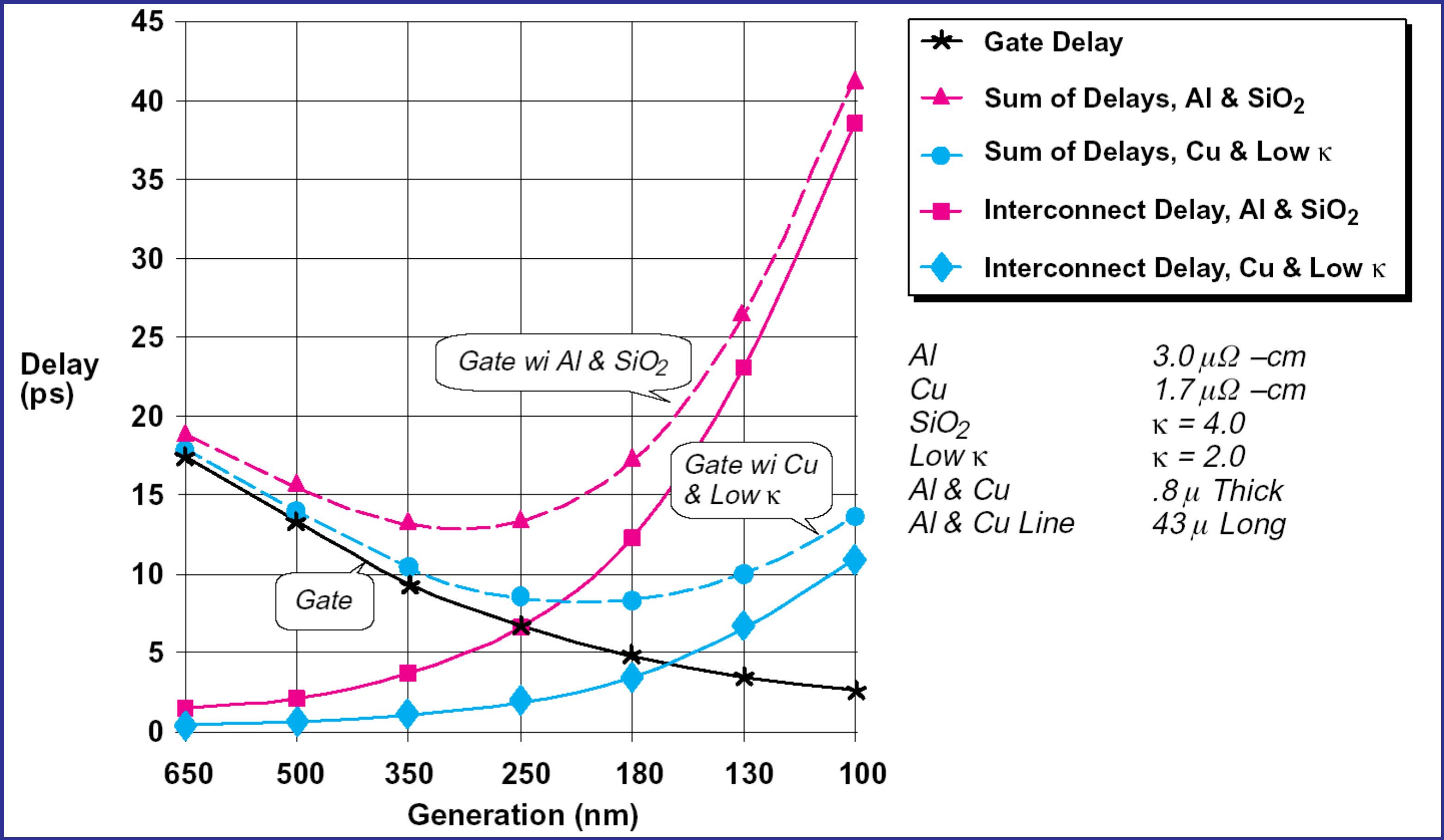
- 45 nm feature size
- $\mathcal{O}(10^9)$ transistors
- $\mathcal{O}(10)$ km of 'wires' (the *interconnect*)
- Up to 15 layers



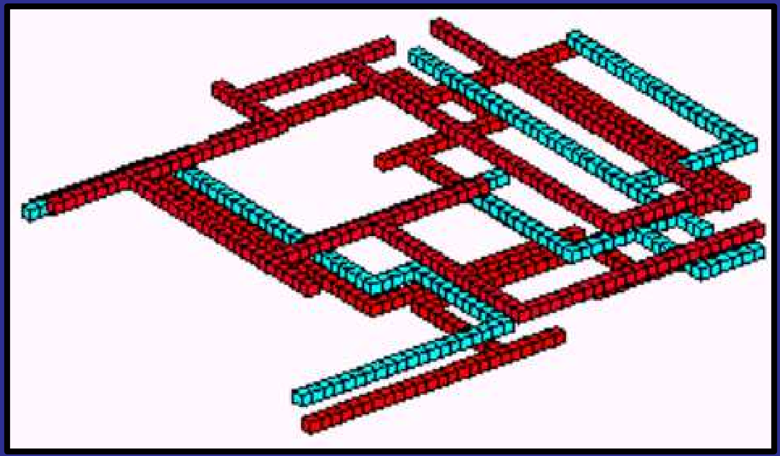
VLSI interconnect

- Wires are not ideal:
  - Resistance
  - Capacitance
  - Inductance
- Consequences:
  - *Timing behavior*
  - Noise
  - Energy consumption
  - Power distribution

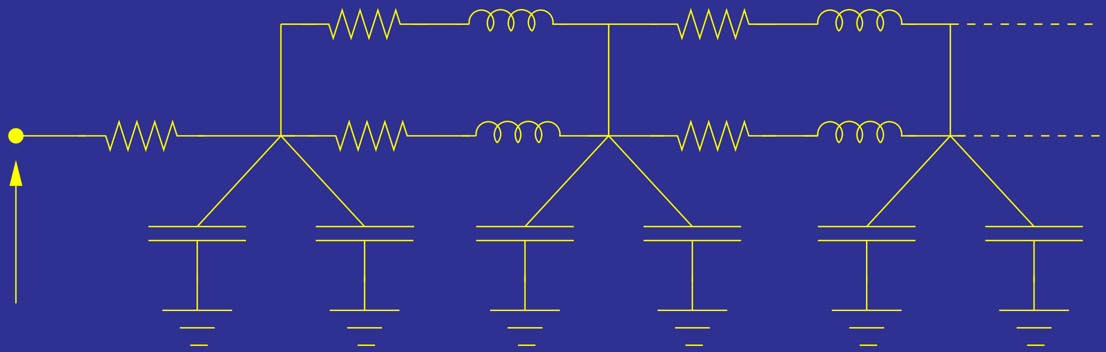
Interconnect now dominates



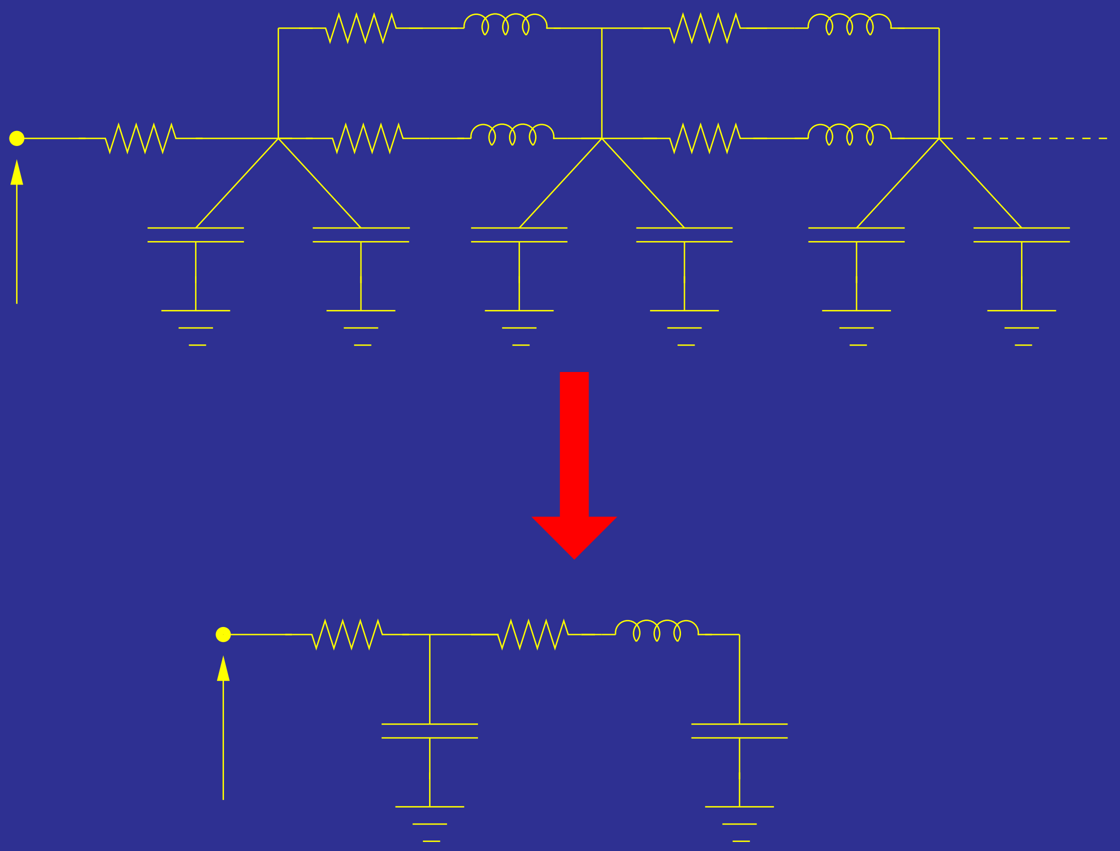
Lumped-circuit paradigm



- Replace 'pieces' of the interconnect by RCL networks:



Need for dimension reduction



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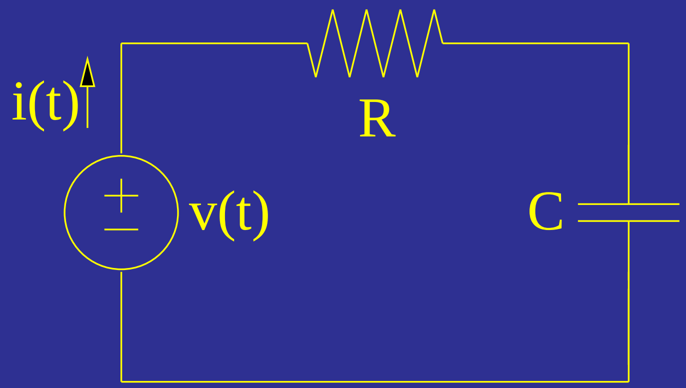
A simple RC circuit



- Impulse response:

$$i(t) = \frac{1}{R^2C} \exp\left(-\frac{t}{RC}\right), \quad t \geq 0$$

- In frequency domain:

$$\mathbf{I}(s) = \frac{1/R^2C}{s + 1/RC} \quad \left(=: \mathbf{H}(s)\right), \quad s \in \mathbb{C}$$

General case

- Impulse response:

$$i(t) = \sum_{i=1}^{N} k_i \, \mathsf{exp}(tp_i), \quad t \geq 0$$

- In frequency domain:

$$\mathbf{H}(s) = \sum_{i=1}^{N} \frac{k_i}{s - p_i}, \quad s \in \mathbb{C}$$

- Simplest RC reduced-order model

$$\mathbf{H}_1(s) := \frac{\tilde{k}}{s - \tilde{p}} \approx \mathbf{H}(s)$$

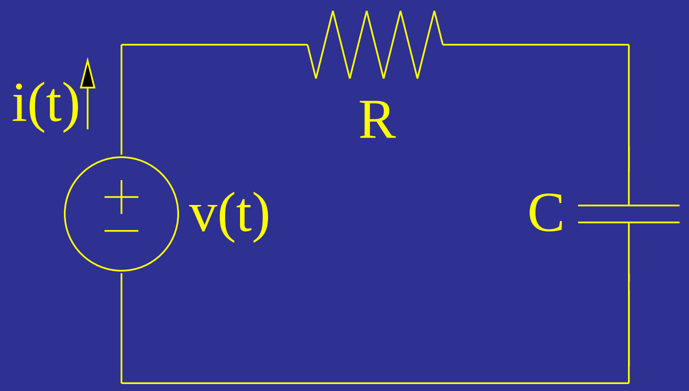
Simplest RC reduced-order model

- Moment matching: Choose $\tilde{p}$ and $\tilde{k}$ such that

$$\mathbf{H}_1(s) = \mathbf{H}(s) + \mathcal{O}\left(s^2\right)$$

- Reduced circuit: Set

$$R := -\frac{\tilde{p}}{\tilde{k}} \quad \text{and} \quad C := \frac{\tilde{k}}{\tilde{p}^2}$$



- Elmore delay:

$$\tau := RC, \quad i(t) = i(0)\ \mathrm{exp}\left(-\frac{t}{\tau}\right)$$

AWE (Pillage and Rohrer, '90)

- Transfer function of RCL network:

$$\mathbf{H}(s) = \sum_{i=1}^{N} \frac{k_i}{s - p_i}$$

- Reduced-order model via approximation

$$\mathbf{H}_n(s) = \sum_{i=1}^{n} \frac{\tilde{k}_i}{s - \tilde{p}_i}, \quad \text{where} \quad n \ll N$$

- Moment matching: Choose the $\tilde{k}_i$'s and $\tilde{p}_i$'s such that

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left(s^{2n}\right)$$

- AWE generates $\mathbf{H}_n$ via explicit moment computations

PVL (Feldmann and F., '94)

- Based on the classical Lanczos-Padé connection

- Write the transfer function in state-space form:

  $$\mathbf{H}(s) = \mathbf{l}^{\mathsf{T}} \Big(\mathbf{I} - (s - s_0)\,\mathbf{A}\Big)^{-1} \mathbf{r}, \quad \text{where} \quad \mathbf{A} \in \mathbb{R}^{N \times N}, \quad \mathbf{r}, \mathbf{l} \in \mathbb{R}^{N}$$

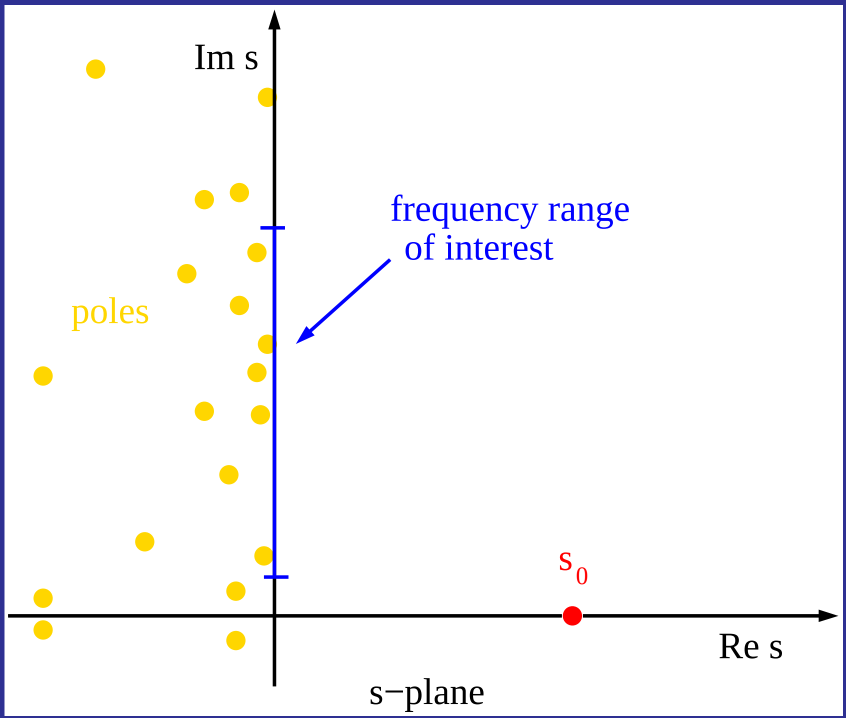
- Run n steps of the Lanczos process (applied to $\mathbf{A}$, starting vectors $\mathbf{r}$ and $\mathbf{l}$) to obtain $n \times n$ tridiagonal matrix $\mathbf{T}_n$

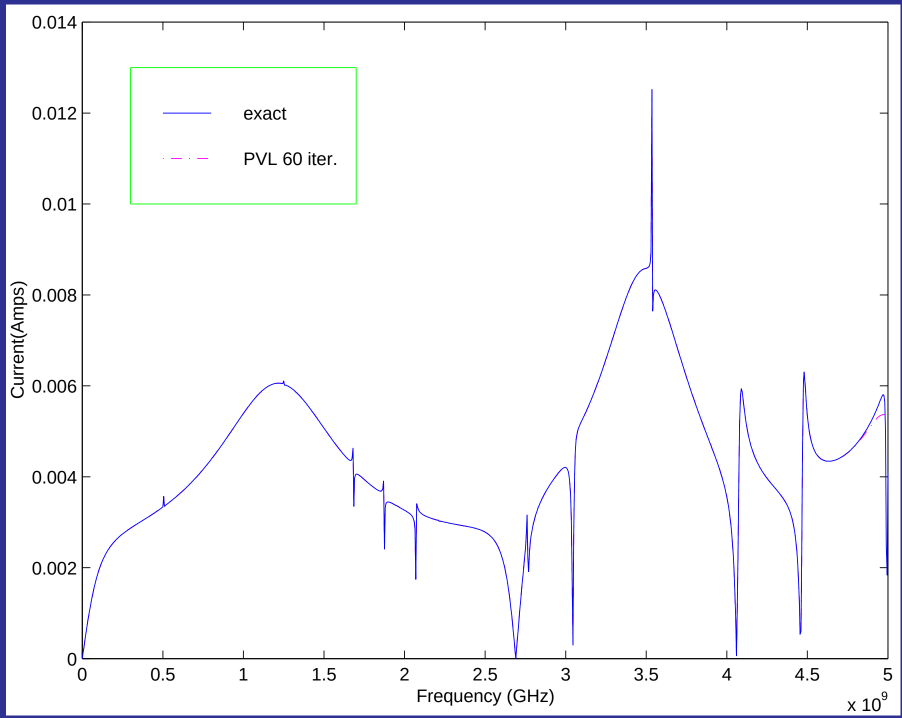
- **Theorem** (Gragg, '74):
  The n-th Padé approximant $\mathbf{H}_n$ of $\mathbf{H}$ is given by

  $$\mathbf{H}_n(s) = \Big(\mathbf{l}^{\mathsf{T}} \mathbf{r}\Big)\; \mathbf{e}_1^{\mathsf{T}} \Big(\mathbf{I} - (s - s_0)\,\mathbf{T}_n\Big)^{-1} \mathbf{e}_1$$

  where $\mathbf{e}_1$ is the first unit vector

An RCL network with mostly C's and L's





Exact and reduced-order model of size $n = 60$

The multi-input multi-output case

- Matrix-valued transfer function

$$\mathbf{H}(s) = \mathbf{L}^{\mathsf{H}}(\mathbf{I} - (s - s_0)\,\mathbf{A})^{-1}\mathbf{R}$$

  where $\mathbf{A} \in \mathbb{C}^{N \times N}$, $\mathbf{R} \in \mathbb{C}^{N \times m}$, $\mathbf{L} \in \mathbb{C}^{N \times p}$

- Band Lanczos process for any m and p
  Aliaga, Boley, F., and Hernández, '94, '96, and '00
  F., '00, '03, and '09

- **MPVL** (Matrix-Padé Via Lanczos) algorithm
  (Feldmann and F., '95)

- 'Symmetric' algorithm tailored to RC networks: **SyMPVL**
  (Feldmann and F., '97 and '98)

Full chip SIV — statistics

interconnect nets	24,607
pins	91,277
total capacitors	5,413,127
cross-coupled C's	4,955,020
grounded C's	458,107
resistors	265,941
potential violations	602
nets in cluster	2–7
total run time	2.5 hours
SyMPVL run time	15 minutes

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RCL networks as descriptor systems

- System of linear time-invariant DAEs of the form

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\mathsf{H}}\mathbf{x}(t)$$

  where $\mathbf{C},\ \mathbf{G} \in \mathbb{C}^{N\times N}$ and $\mathbf{B} \in \mathbb{C}^{N\times m}$

- $\mathbf{x}(t) \in \mathbb{C}^{N}$ is the unknown vector of state variables

- m inputs, m outputs

Reduced-order models

- System of DAEs of the same form:

$$\mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n\,\mathbf{z}(t) = \mathbf{B}_n\,\mathbf{u}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{B}_n^{\mathsf{H}}\,\mathbf{z}(t)$$

- But now:

$$\mathbf{C}_n,\ \mathbf{G}_n \in \mathbb{C}^{n\times n} \quad \text{and} \quad \mathbf{B}_n \in \mathbb{C}^{n\times m}$$

where $n \ll N$

Transfer functions

- Original descriptor system:

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{H}}\,(s\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

- Reduced-order model:

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\mathsf{H}}\,(s\,\mathbf{C}_n + \mathbf{G}_n)^{-1}\,\mathbf{B}_n$$

- 'Good' reduced-order model

$\Longleftrightarrow$ 'Good' approximation $\mathbf{H}_n \approx \mathbf{H}$

- Original dimension $N \approx 10^{4-6}$

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{H}} \left(s\,\mathbf{C} + \mathbf{G} \right)^{-1} \mathbf{B}$$

- Reduced dimension $n \ll N$ $(n \approx 10^{0-2})$

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\mathsf{H}} \left(s\,\mathbf{C}_n + \mathbf{G}_n \right)^{-1} \mathbf{B}_n$$

Padé approximation

- Choose expansion point $s_0 \in \mathbb{C}$ such that the matrix $s_0\,\mathbf{C} + \mathbf{G}$ is nonsingular

- $\mathbf{C}_n,\ \mathbf{G}_n \in \mathbb{C}^{n \times n},\ \mathbf{B}_n \in \mathbb{C}^{n \times m}$ are such that

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{q(n)}\right)$$

  and $q(n)$ is maximal

- $q(n) \geq 2 \left\lfloor \frac{n}{m} \right\rfloor$ with equality in the 'generic' case

Padé-type approximation

- Padé approximants have undesirable properties in general

- Remedy: relax approximation property

- $\mathbf{C}_n,\ \mathbf{G}_n \in \mathbb{C}^{n\times n},\ \mathbf{B}_n \in \mathbb{C}^{n\times m}$ are such that

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}(n)}\right)$$

  where $\tilde{q}(n)$ is no longer maximal

- Typical: $\tilde{q}(n) \geq \left\lfloor \frac{n}{m} \right\rfloor$ with equality in the 'generic' case

Reduction to one matrix

- Transfer function:

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{H}} (s\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B} = \mathbf{B}^{\mathsf{H}} \Big(s_0\,\mathbf{C} + \mathbf{G} + (s - s_0)\,\mathbf{C} \Big)^{-1} \mathbf{B}$$

- Set

$$\mathbf{A} := -\,(s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C} \quad \text{and} \quad \mathbf{R} := (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

- Rewriting $\mathbf{H}$ gives

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{H}} (\mathbf{I} - (s - s_0)\,\mathbf{A})^{-1}\,\mathbf{R}$$

Krylov subspaces and Padé

- Expanding about s_0 gives

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{H}} \left(\mathbf{I} - (s - s_0)\,\mathbf{A}\right)^{-1} \mathbf{R}$$

$$= \sum_{i=0}^{\infty} \mathbf{B}^{\mathsf{H}} \left(\mathbf{A}^{i}\,\mathbf{R}\right) (s - s_0)^{i}$$

$$= \sum_{i=0}^{\infty} \left(\left(\mathbf{A}^{\mathsf{H}}\right)^{i} \mathbf{B}\right)^{\mathsf{H}} \mathbf{R}\, (s - s_0)^{i}$$

- Right and left block Krylov sequences:

$$\left[\, \mathbf{R} \quad \mathbf{A}\,\mathbf{R} \quad \cdots \quad \mathbf{A}^{i}\mathbf{R} \quad \cdots \,\right] \quad \text{and} \quad \left[\, \mathbf{B} \quad \mathbf{A}^{\mathsf{H}}\,\mathbf{B} \quad \cdots \quad \left(\mathbf{A}^{\mathsf{H}}\right)^{i} \mathbf{B} \quad \cdots \,\right]$$

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Problem of structure preservation

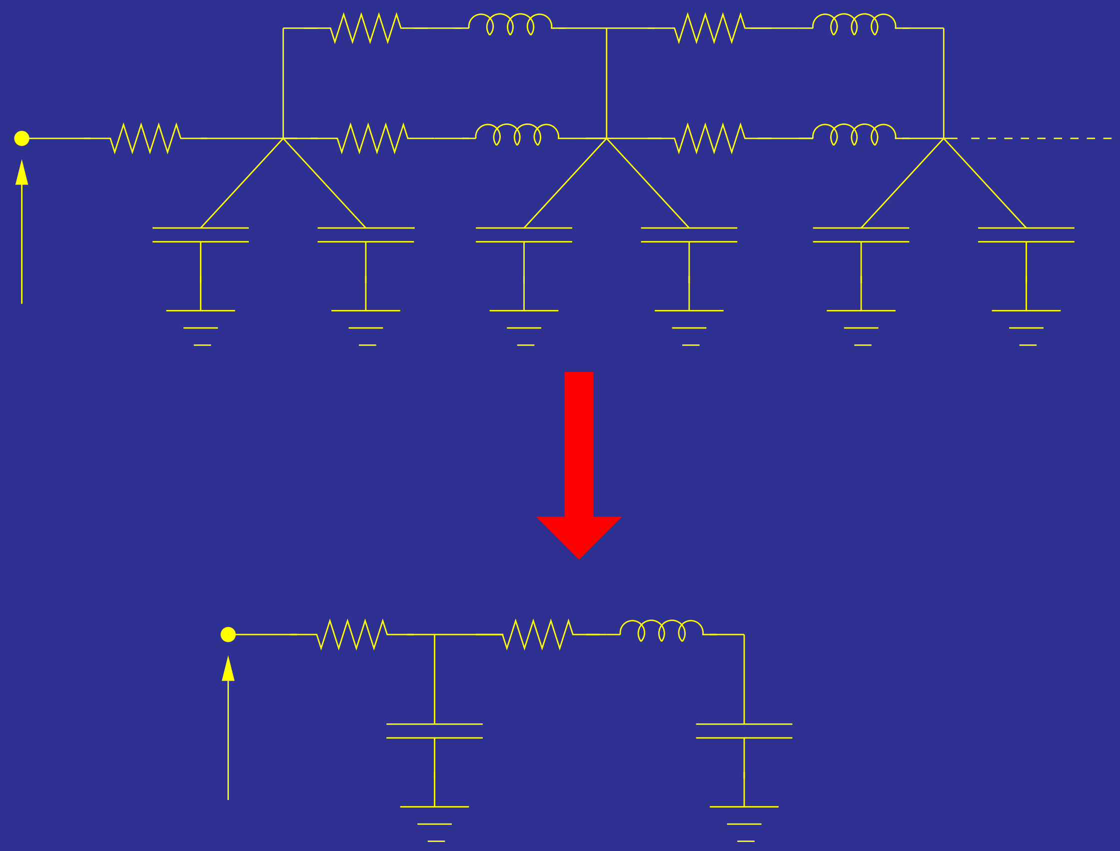
- Any RCL network is stable, passive, . . .

- Reduced-order model should be stable, passive, . . .

- More difficult problem:
  Reduced-order model of an RCL network should be synthesizable as an RCL network

- Padé reduced-order models are not even stable in general!

Preservation of RCL structure



General RCL network equations

- System of linear time-invariant DAEs of the form

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\mathsf{H}}\mathbf{x}(t)$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \\ -\mathbf{G}_2^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{G}_3^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$$

- Moreover:

$$\mathbf{C} \succeq \mathbf{0} \quad \text{and} \quad \mathbf{G} + \mathbf{G}^{\mathsf{H}} \succeq \mathbf{0}$$

(This implies passivity!)

Dimension reduction via projection

- **PRIMA**
  Passive Reduced Interconnect Macromodeling Algorithm
  (Odabasioglu, '96; Odabasioglu, Celik, and Pileggi, '97)

- **SPRIM**
  Structure-Preserving Reduced Interconnect Macromodeling
  (F., '04 and '09)

- PRIMA and SPRIM satisfy a Padé-type property:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^j\right)$$

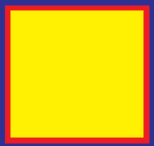
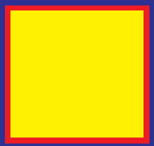
  for some $j = j(n)$

PRIMA does not preserve RCL structure

- Structure of the data matrices:

$$C = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2 & G_3 \\ -G_2^{\mathsf{H}} & 0 & 0 \\ -G_3^{\mathsf{H}} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_2 \end{bmatrix}$$

- Structure of PRIMA reduced-order matrices:

$C_n =$  , $G_n =$  , $B_n =$ 

SPRIM does preserve RCL structure

- Structure of SPRIM reduced-order matrices:

$$\mathbf{C}_n = \begin{bmatrix} \tilde{\mathbf{C}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{C}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \; \mathbf{G}_n = \begin{bmatrix} \tilde{\mathbf{G}}_1 & \tilde{\mathbf{G}}_2 & \tilde{\mathbf{G}}_3 \\ -\tilde{\mathbf{G}}_2^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \\ -\tilde{\mathbf{G}}_3^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \; \mathbf{B}_n = \begin{bmatrix} \tilde{\mathbf{B}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{B}}_2 \end{bmatrix}$$

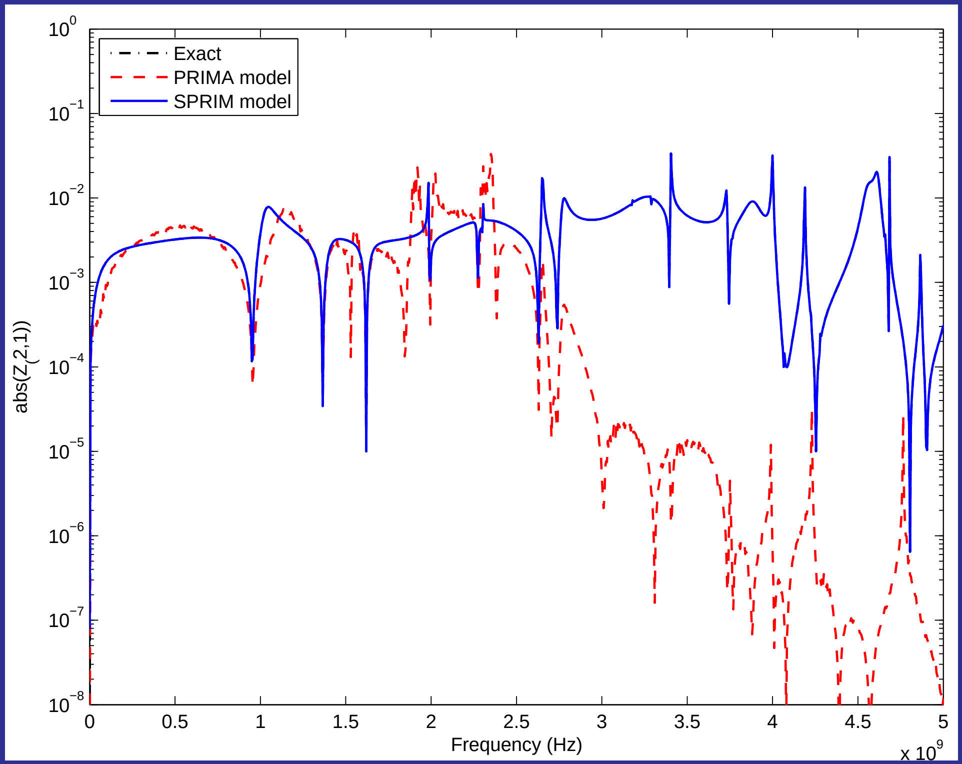
- Padé-type property:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^j\right)$$

  with j the same integer as for PRIMA

- For SPRIM, we even have $j \Rightarrow 2j$. Why?

An RCL network with mostly C's and L's



Exact and models corresponding to
block Krylov subspace of dimension $\widehat{n} = 120$

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Projection-based reduction

- Let $\mathbf{V}_n \in \mathbb{C}^{N \times n}$ be any matrix with full column rank n

- Use $\mathbf{V}_n$ to explicitly project the data matrices of

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\mathsf{H}}\mathbf{x}(t)$$

  onto the subspace spanned by the columns of $\mathbf{V}_n$

Projection-based reduction, continued

- Resulting reduced-order model

$$\mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n\,\mathbf{z}(t) = \mathbf{B}_n\,\mathbf{u}(t)$$
$$\widetilde{\mathbf{y}}(t) = \mathbf{B}_n^{\mathsf{H}}\mathbf{z}(t)$$

where

$$\mathbf{C}_n = \mathbf{V}_n^{\mathsf{H}}\,\mathbf{C}\,\mathbf{V}_n, \quad \mathbf{G}_n = \mathbf{V}_n^{\mathsf{H}}\,\mathbf{G}\,\mathbf{V}_n, \quad \mathbf{B}_n = \mathbf{V}_n^{\mathsf{H}}\,\mathbf{B}$$

- Passivity is preserved:

$$\mathbf{C} \succeq \mathbf{0}, \ \mathbf{G} + \mathbf{G}^{\mathsf{H}} \succeq \mathbf{0} \quad \Rightarrow \quad \mathbf{C}_n \succeq \mathbf{0}, \ \mathbf{G}_n + \mathbf{G}_n^{\mathsf{H}} \succeq \mathbf{0}$$

Projection + Krylov

- Choose an expansion point $s_0 \in \mathbb{C}$ and re-write the original transfer function:

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{H}} (s\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

$$= \mathbf{B}^{\mathsf{H}} (\mathbf{I} - (s - s_0)\,\mathbf{A})^{-1}\,\mathbf{R}$$

where

$$\mathbf{A} := -\left(s_0\,\mathbf{C} + \mathbf{G}\right)^{-1}\mathbf{C} \quad \text{and} \quad \mathbf{R} := \left(s_0\,\mathbf{C} + \mathbf{G}\right)^{-1}\mathbf{B}$$

- **Block Krylov sequence**:

$$\mathbf{R}, \mathbf{A}\mathbf{R}, \mathbf{A}^2\mathbf{R}, \ldots, \mathbf{A}^i\mathbf{R}, \ldots$$

Projection + Krylov, continued

- $\widehat{n}$-th **block Krylov subspace**:

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) := \mathrm{colspan}_{\widehat{n}} \left[\mathbf{R} \quad \mathbf{A}\mathbf{R} \quad \mathbf{A}^2\mathbf{R} \quad \cdots \right]$$

- Choose the projection matrix $\mathbf{V}_n$ such that

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) \subseteq \mathrm{Range}\, \mathbf{V}_n$$

- Projection + Krylov subspace = **Padé-type approximant**:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^j \right), \quad \text{where} \quad j \geq \lfloor \widehat{n}/m \rfloor$$

SPRIM

- Let $\widehat{\mathbf{V}}_{\widehat{n}}$ be any matrix such that

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) = \text{Range}\, \widehat{\mathbf{V}}_{\widehat{n}}$$

- Recall:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \\ -\mathbf{G}_2^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{G}_3^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$$

SPRIM, continued

- Partition $\hat{\mathbf{V}}_{\widehat{n}}$ accordingly:

$$\hat{\mathbf{V}}_{\widehat{n}} = \begin{bmatrix} \mathbf{V}_{\widehat{n}}^{(1)} \\ \mathbf{V}_{\widehat{n}}^{(2)} \\ \mathbf{V}_{\widehat{n}}^{(3)} \end{bmatrix}$$

- For $l = 1, 2, 3$:
  If $\text{Rank}\, \mathbf{V}_{\widehat{n}}^{(i)} < \widehat{n}$, replace $\mathbf{V}_{\widehat{n}}^{(i)}$ by matrix of full column rank

SPRIM, continued

- Set

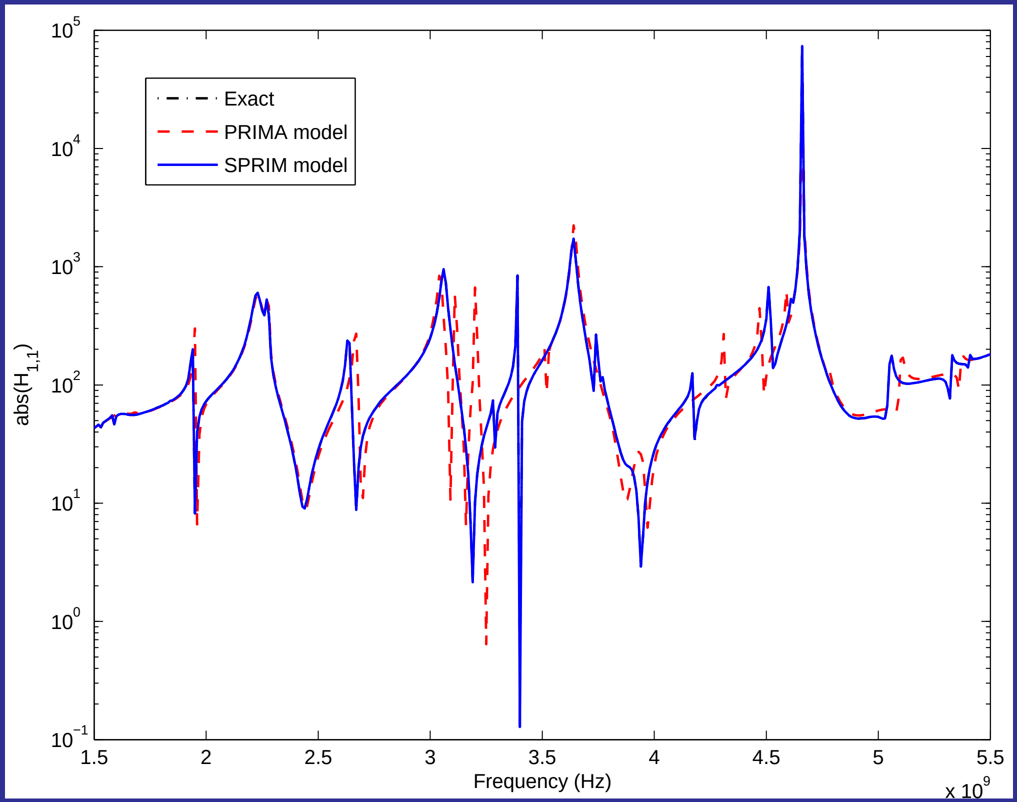
$$\mathbf{V}_n = \begin{bmatrix} \mathbf{V}_n^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_n^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_n^{(3)} \end{bmatrix}$$

- Block structure is preserved:

$$\mathbf{C}_n = \begin{bmatrix} \tilde{\mathbf{C}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{C}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \tilde{\mathbf{G}}_1 & \tilde{\mathbf{G}}_2 & \tilde{\mathbf{G}}_3 \\ -\tilde{\mathbf{G}}_2^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \\ -\tilde{\mathbf{G}}_3^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \tilde{\mathbf{B}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{B}}_2 \end{bmatrix}$$

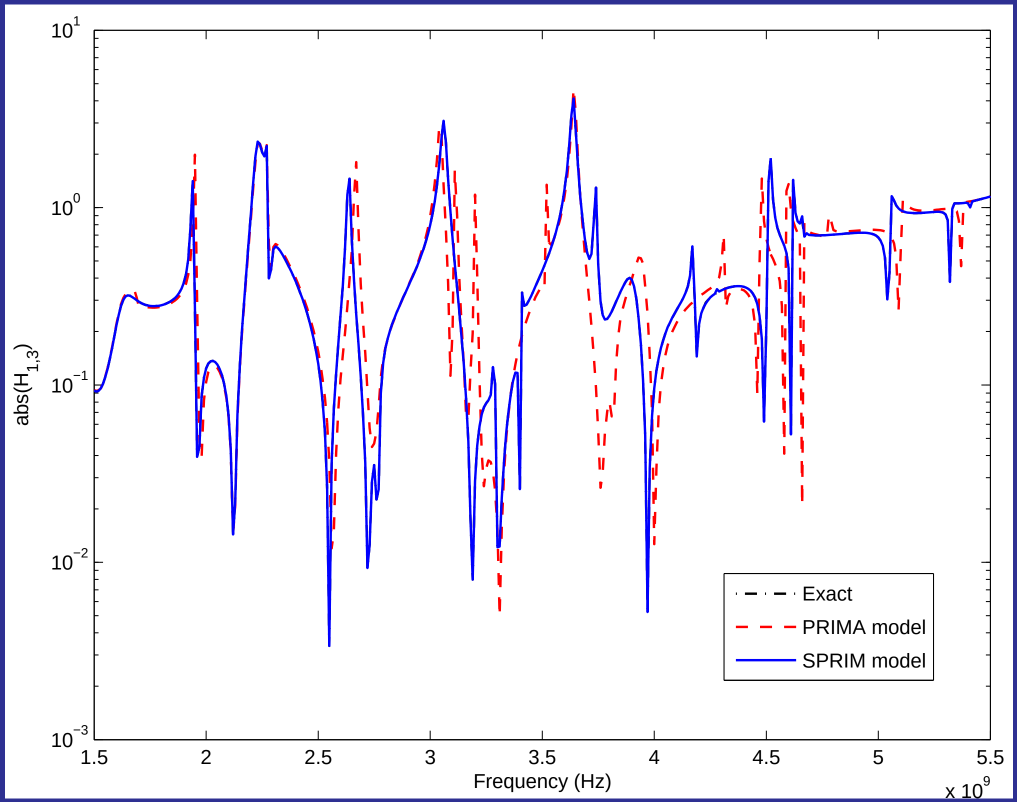
- $\mathcal{K}_{\hat{n}}(\mathbf{A}, \mathbf{R}) = \text{Range}\, \mathbf{V}_{\hat{n}} \subseteq \text{Range}\, \mathbf{V}_n \quad \Rightarrow$ Padé-type property!

An RCL network with mostly C's and L's



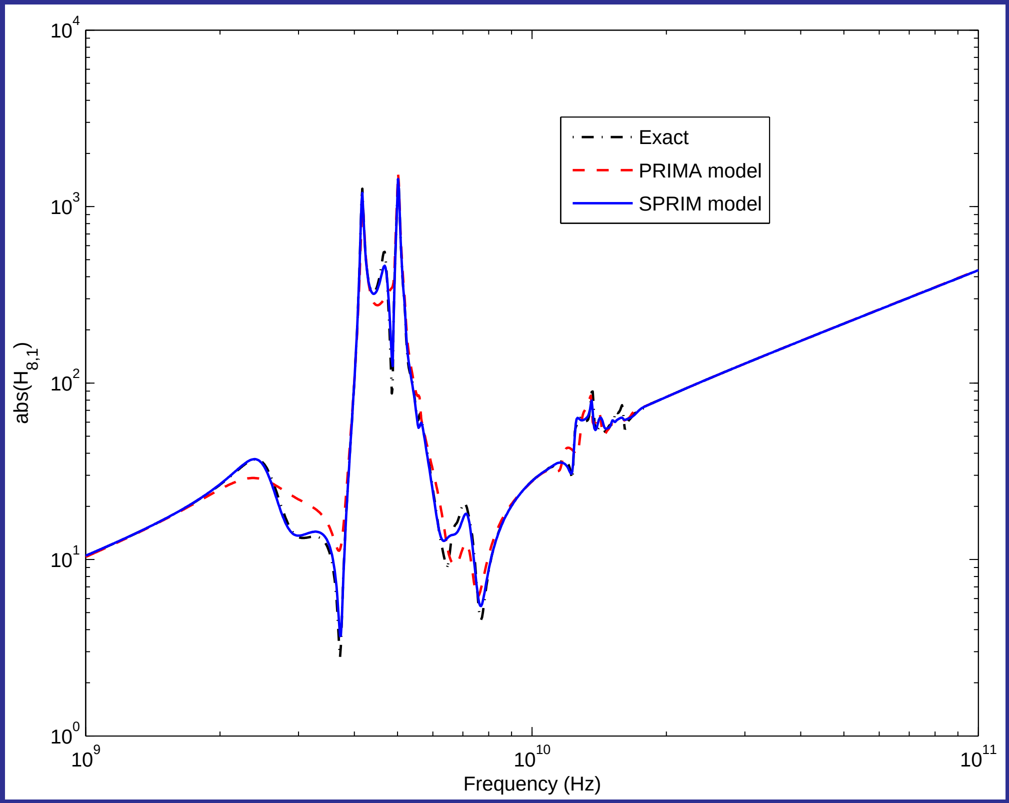
Exact and models corresponding to $\widehat{n} = 90$

An RCL network with mostly C's and L's



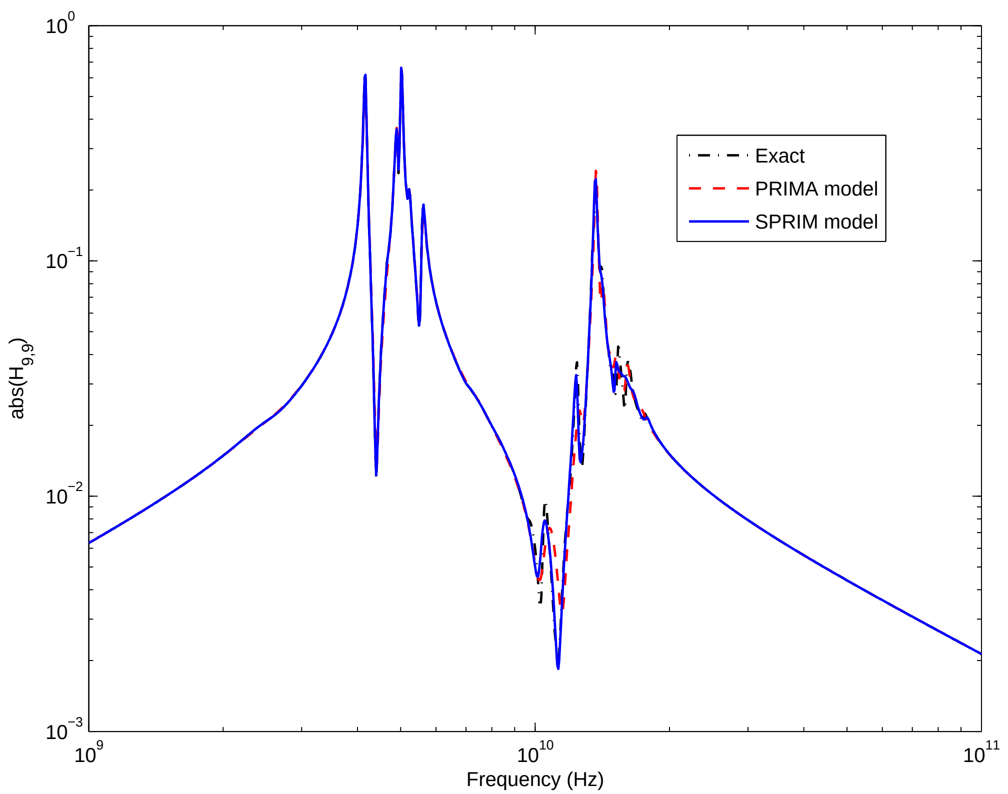
Exact and models corresponding to $\hat{n} = 90$

A package example



Exact and models corresponding to $\widehat{n} = 128$

A package example



Exact and models corresponding to $\widehat{n} = 128$

Padé-type property

- So far, we only know that both PRIMA and SPRIM produce Padé-type reduced-order models with

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^q\right), \quad \text{where} \quad q \geq \lfloor \widehat{n}/m \rfloor$$

- Can we say more in the case of SPRIM?

- Easy in the case of no third subblock $\mathbf{V}_n^{(3)}$ (F. '05)

- General case: $\mathbf{J}$-Hermitian linear dynamical systems (F. '08)

J-Hermitian systems

- Recall:

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\mathsf{H}}\mathbf{x}(t)$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \\ -\mathbf{G}_2^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{G}_3^{\mathsf{H}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$$

- $\mathbf{C}$ and $\mathbf{G}$ are $\mathbf{J}$-Hermitian:

$$\mathbf{J}\,\mathbf{C} = \mathbf{C}^{\mathsf{H}}\mathbf{J} \quad \text{and} \quad \mathbf{J}\,\mathbf{G} = \mathbf{G}^{\mathsf{H}}\mathbf{J}, \quad \text{where} \quad \mathbf{J} := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix}$$

$\mathbf{J}$-Hermitian systems, continued

- The input-output matrix $\mathbf{B}$ satisfies

$$\mathrm{Range}(\mathbf{J}\,\mathbf{B}) = \mathrm{Range}(\mathbf{B})$$

$\mathbf{J}_n$-Hermitian property of SPRIM models

- The SPRIM models

$$\mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n \mathbf{z}(t) = \mathbf{B}_n \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}_n^{\mathsf{H}} \mathbf{z}(t)$$

  preserve the structure of $\mathbf{C}_n$, $\mathbf{G}_n$, $\mathbf{B}_n$

- Therefore, $\mathbf{C}_n$ and $\mathbf{G}_n$ are $\mathbf{J}_n$-Hermitian with

$$\mathbf{J}_n := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathsf{Range}(\mathbf{J}_n \mathbf{B}_n) = \mathsf{Range}(\mathbf{B}_n)$$

- Moreover, the projection matrix $\mathbf{V}_n$ satisfies

$$\mathbf{J}\,\mathbf{V}_n = \mathbf{V}_n \mathbf{J}_n$$

Padé-type property

- **Theorem** (F., '08)

  For $\mathbf{J}$-Hermitian systems and real expansion points s_0, the n-th SPRIM model is $\mathbf{J}_n$-Hermitian and satisfies

  $$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq 2\left\lfloor \widehat{n}/m \right\rfloor$$

- Twice as accurate as PRIMA!

Outline

- Motivation
- From **AWE** to **PVL**
- Descriptor systems
- Preserving RCL structures
- **SPRIM**
- *Using restarted Krylov subspace methods*
- Concluding remarks

Using restarts (with Efrem Rensi)

- To obtain a Padé-type property, we need to generate a matrix $\widehat{\mathbf{V}}_{\widehat{n}}$ such that

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) = \text{Range}\, \widehat{\mathbf{V}}_{\widehat{n}}$$

- Use suitable variant of the Arnoldi process

- But: prohibitive for large $\widehat{n}$

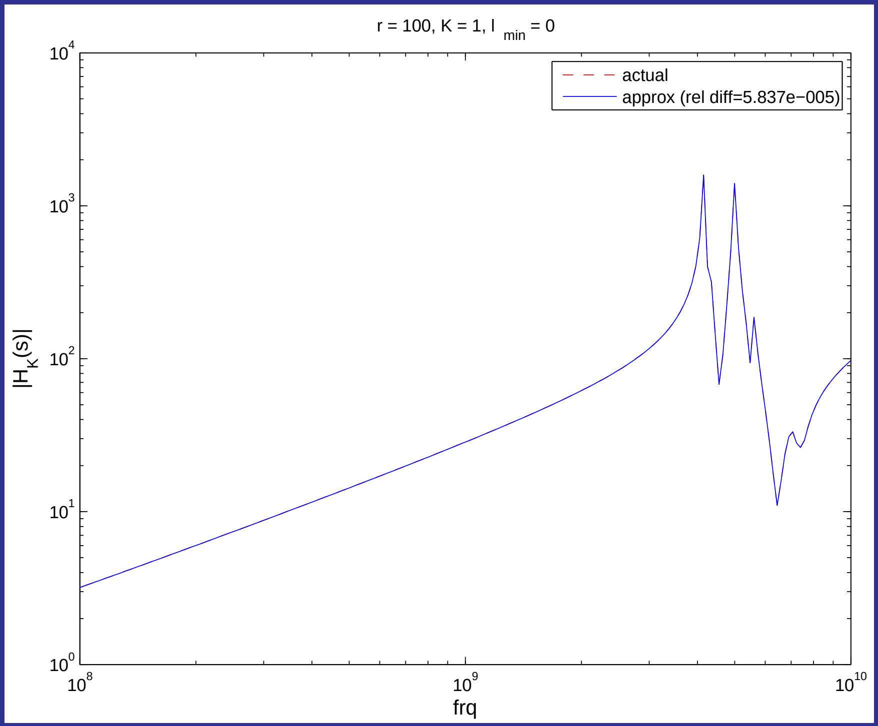
- Remedy: (thick) restarts

Using restarts, continued

- Motivated by recent work by Eiermann et al.
- Restart after each cycle of r Arnoldi steps
- Extract 'good' eigenvector information Y from the last batch of r Arnoldi vectors
- Use the columns of Y as the first vectors in the next cycle
- At each restart allow for changing expansion point:

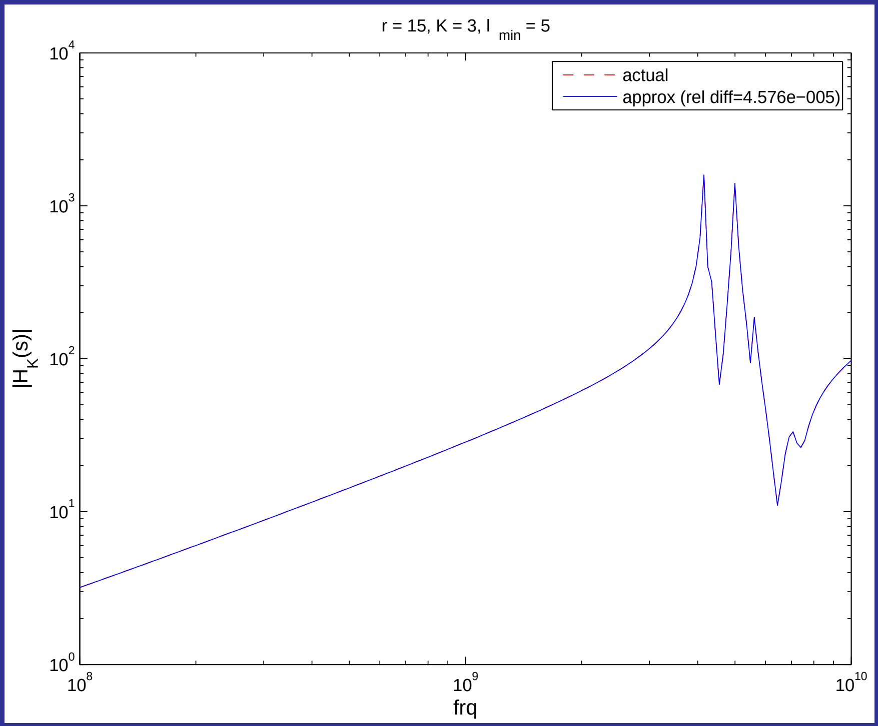
$$\mathbf{A}(s_0) = -\,(s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C} \quad \Rightarrow \quad \mathbf{A}(\tilde{s}_0) = -\,(\tilde{s}_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C}$$

Single vs. multiple expansion points



Single point — no restarts

$n = 100$



3 points — thick restarts

$n = 45$

Outline

- Motivation
- From **AWE** to **PVL**
- Descriptor systems
- Preserving RCL structures
- **SPRIM**
- Using restarted Krylov subspace methods
- *Concluding remarks*

Concluding remarks

- Practical use of Krylov subspace-based dimension reduction was motivated by need to handle large-scale RCL networks

- Lead to the development of new Krylov subspace methods

- How to avoid the need to store $N \times n$ dense matrix in projection methods?

- Krylov subspace methods with thick restarts?

- Use with multiple expansion points?