The interplay between computation and analysis in the study of 3D incompresible flows

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Motivation

The question of whether the 3D incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data is one of the seven Clay Millennium Problems.

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0$. Define vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, then $\boldsymbol{\omega}$ is governed by

$$\boldsymbol{\omega}_t + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \nabla \mathbf{u} \cdot \boldsymbol{\omega} + \nu \Delta \boldsymbol{\omega}. \tag{2}$$

Note $\nabla \mathbf{u}$ is formally of the same order as $\boldsymbol{\omega}$. Thus the vortex stretching term $\nabla \mathbf{u} \cdot \boldsymbol{\omega} \approx \boldsymbol{\omega}^2$.

So far, most regularity analysis uses energy estimates and treats the nonlinear terms as a small perturbation to the diffusion term. The global regularity can be obtained only for small data.

A brief review

- Global existence for small data (Leray, Ladyzhenskaya, Kato, etc). If $\|\mathbf{u}_0\|_{L^p}$ $(p \geq 3)$ or $\|\mathbf{u}_0\|_{L^2}\|\nabla \mathbf{u}_0\|_{L^2}$ is small, then the 3D Navier-Stokes equations have a globally smooth solution.

- Non-blowup criteria due to J. Serrin 63, G. Prodi 59. A weak solution $\mathbf{u}$ of the 3D Navier-Stokes equations is smooth on $[0, T] \times \mathbb{R}^3$ provided that

$$\|\mathbf{u}\|_{L_t^q L_x^p([0,T]\times\mathbb{R}^3)} < \infty$$

  for some p, q satisfying $\frac{3}{p} + \frac{2}{q} \leq 1$ with $3 \geq p \leq \infty$ and $2 \leq q < \infty$.

- **Partial regularity theory** (Caffarelli-Kohn-Nirenberg 82, F. Lin 98) For any suitable weak solution of the 3D Navier-Stokes equations on an open set in space-time, the one-dimensional Hausdorff measure of the associated singular set is zero.

Convection has been ignored in regularity analysis

- Due to the incompressibility condition, the convection term does not contribute to the energy norm of velocity or the L^p-norm of ω:

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \nabla \mathbf{u} \cdot \omega + \nu \Delta \omega,$$

- As a result, the convection term has been basically ignored in the regularity analysis for the Navier-Stokes equations.
- Most of the efforts have focused on formulating some kind of scale-invariant smallness assumption on the solution so that diffusion can control the formal quadratic nonlinear vortex stretching term.
- We will demonstrate that convection actually has a stabilizing effect.

Connection to 3D Euler Equations

- The convection term and the vortex stretching term can be reformulated as a commutator: $\boldsymbol{\omega}_t + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$.
- When we consider the two terms together, we preserve the Lagrangian structure of the solution:

$$\boldsymbol{\omega}(X(\alpha, t), t) = X_\alpha(\alpha, t)\boldsymbol{\omega}_0(\alpha), \quad \det(X_\alpha(\alpha, t)) \equiv 1$$

where $X(\alpha, t)$ is the flow map: $X_t = \mathbf{u}(X, t)$, $X(\alpha, 0) = \alpha$.

- Convection tends to severely deform and flatten the support of maximum vorticity. Such deformation tends to weaken the nonlinearity of vortex stretching dynamicallty.
- If we ignore the convection term, the vortex stretching term may indeed achieve the $O(|\omega|^2)$ scaling dynamically and develop an isotropic singularity in finite time.

Local non-blowup conditions by Deng-Hou-Yu

Motivated by the earlier work of Constantin-Majda-Fefferman [96], we proved the following localized non-blowup criterion for the 3D Euler equations using a Lagrangian approach:

Theorem 1 (Deng-Hou-Yu, 2005 and 2006, CPDE)

- Denote by $L(t)$ the arclength of a vortex line segment L_t around the maximum vorticity, $\boldsymbol{\xi} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$, and κ is curvature of L_t. If
  1. $\max_{L_t}(|\mathbf{u}\cdot\boldsymbol{\xi}| + |\mathbf{u}\cdot\mathbf{n}|) \leq C_U(T-t)^{-A}$ with $A < 1$;
  2. $C_L(T-t)^B \leq L(t) \leq C_0/\max_{L_t}(|\kappa|,\ |\nabla\cdot\boldsymbol{\xi}|)$ with $B \leq 1-A$;

  then the solution of the 3D Euler equations remains regular up to T.

This theorem provides a sharper non-blowup criterion to eliminate some of potentially candidates for 3D Euler singularities.

Computation of Hou and Li, J. Nonlinear Science, 2006

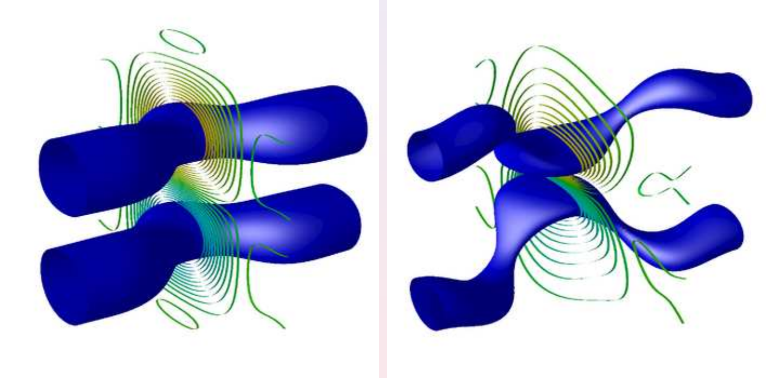


Figure: Two slightly perturbed antiparallel vortex tubes at t=0 and t=6

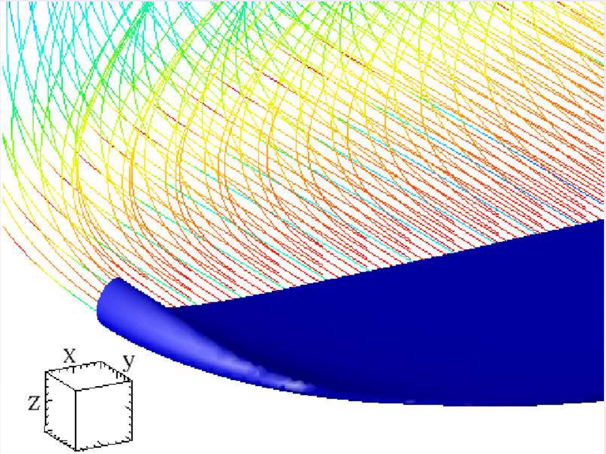


Figure: The local 3D vortex structures and vortex lines around the maximum vorticity at $t = 17$.

Maximum velocity in time

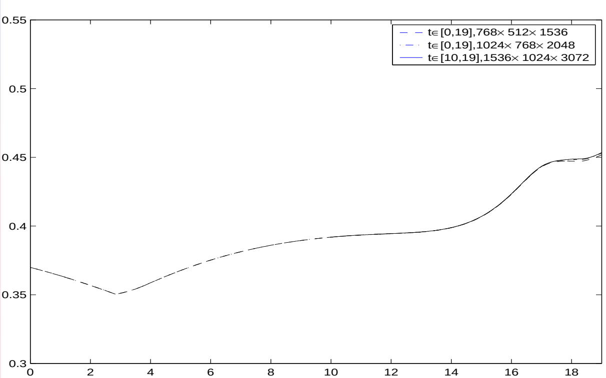


Figure: Maximum velocity $\|\mathbf{u}\|_\infty$ in time using different resolutions. With maximum velocity being bounded, the non-blowup criterion of Deng-Hou-Yu applies with $A = 0$ and $B = 1/2$, implying no blowup at least up to $T = 19$.

Dynamic depletion of vortex stretching

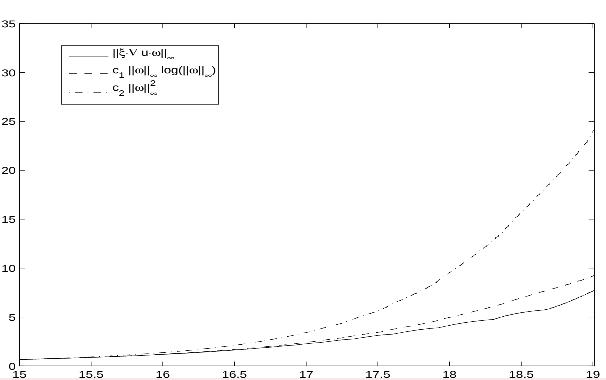


Figure: Study of the vortex stretching term in time, resolution $1536 \times 1024 \times 3072$. The fact $|\boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}| \leq c_1 |\boldsymbol{\omega}| log |\boldsymbol{\omega}|$ plus $\frac{D}{Dt}|\omega| = \boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}$ implies $|\boldsymbol{\omega}|$ bounded by doubly exponential.

Log log plot of maximum vorticity in time

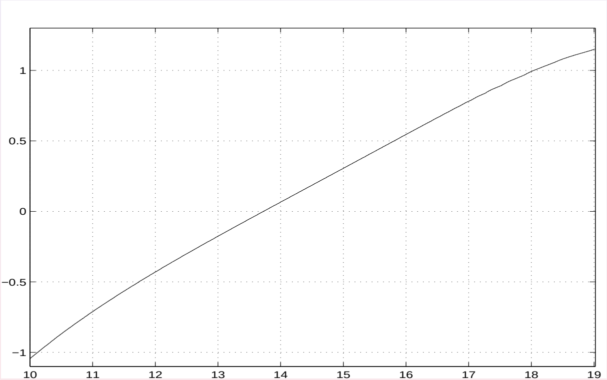


Figure: The plot of $\log\log\|\omega\|_\infty$ vs time, resolution $1536 \times 1024 \times 3072$.

Vorticity vector alignment

Recall that

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = S\cdot\boldsymbol{\omega}, \quad S = \frac{1}{2}(\nabla u + \nabla^T u).$$

Let $\lambda_1 < \lambda_2 < \lambda_3$ be the three eigenvalues of S, $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

time	$\lvert\boldsymbol{\omega}\rvert$	λ_1	θ_1	λ_2	θ_2	λ_3	θ_3
16.012	5.628	-1.508	89.992	0.206	0.007	1.302	89.998
16.515	7.016	-1.864	89.995	0.232	0.010	1.631	89.990
17.013	8.910	-2.322	89.998	0.254	0.006	2.066	89.993
17.515	11.430	-2.630	89.969	0.224	0.085	2.415	89.920
18.011	14.890	-3.625	89.969	0.257	0.036	3.378	89.979
18.516	19.130	-4.501	89.966	0.246	0.036	4.274	89.984
19.014	23.590	-5.477	89.966	0.247	0.034	5.258	89.994

Table: The alignment of the vorticity vector and the eigenvectors of S around the point of maximum vorticity with resolution $1536 \times 1024 \times 3072$. Here, θ_i is the angle between the i-th eigenvector of S and the vorticity vector.

The Stabilizing Effect of Convection

Consider the 3D axi-symmetric incompressible Navier-Stokes equations

$$u_t^\theta + u^r u_r^\theta + u^z u_z^\theta = \nu \left(\nabla^2 - \frac{1}{r^2}\right) u^\theta - \frac{1}{r} u^r u^\theta, \tag{3}$$

$$\omega_t^\theta + u^r \omega_r^\theta + u^z \omega_z^\theta = \nu \left(\nabla^2 - \frac{1}{r^2}\right) \omega^\theta + \frac{1}{r} \left((u^\theta)^2\right)_z + \frac{1}{r} u^r \omega^\theta, \tag{4}$$

$$-\left(\nabla^2 - \frac{1}{r^2}\right) \psi^\theta = \omega^\theta, \tag{5}$$

where u^θ, ω^θ and ψ^θ are the angular components of the velocity, vorticity and stream function respectively, and

$$u^r = -(\psi^\theta)_z \quad u^z = \frac{1}{r}(r\psi^\theta)_r.$$

Note that equations (3)-(5) completely determine the evolution of the 3D axisymmetric Navier-Stokes equations.

Reformulation of axisymmetric Navier-Stokes equations

In [CPAM 08], Hou and Li introduced the following new variables:

$$u_1 = u^\theta / r, \quad \omega_1 = \omega^\theta / r, \quad \psi_1 = \psi^\theta / r, \tag{6}$$

and derived the following equivalent system that governs the dynamics of u_1, ω_1 and ψ_1 as follows:

$$\begin{cases} \partial_t u_1 + u^r \partial_r u_1 + u^z \partial_z u_1 = \nu\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right) u_1 + 2u_1 \psi_{1z}, \\ \partial_t \omega_1 + u^r \partial_r \omega_1 + u^z \partial_z \omega_1 = \nu\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\omega_1 + \left(u_1^2\right)_z, \\ -\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\psi_1 = \omega_1, \end{cases} \tag{7}$$

where $u^r = -r\psi_{1z}, \quad u^z = 2\psi_1 + r\psi_{1r}$.

Liu and Wang [SINUM07] showed that if $\mathbf{u}$ is a smooth velocity field, then u^θ, ω^θ and ψ^θ must satisfy: $u^\theta\big|_{r=0} = \omega^\theta\big|_{r=0} = \psi^\theta\big|_{r=0} = 0$. Thus u_1, ψ_1 and ω_1 are well defiend.

Stabilizing effect of convection through an exact 1D model for the 3D Navier-Stokes equations

In [Hou-Li, CPAM, **61** (2008), no. 5, 661–697], we derived an excact 1D model along the z-axis for the Navier-Stokes equations:

$$(u_1)_t + 2\psi_1 (u_1)_z = \nu(u_1)_{zz} + 2(\psi_1)_z u_1, \quad (8)$$

$$(\omega_1)_t + 2\psi_1 (\omega_1)_z = \nu(\omega_1)_{zz} + \left(u_1^2\right)_z, \quad (9)$$

$$-(\psi_1)_{zz} = \omega_1. \quad (10)$$

Let $\tilde{u} = u_1$, $\tilde{v} = -(\psi_1)_z$, and $\tilde{\psi} = \psi_1$. The above system becomes

$$(\tilde{u})_t + 2\tilde{\psi}(\tilde{u})_z = \nu(\tilde{u})_{zz} - 2\tilde{v}\tilde{u}, \quad (11)$$

$$(\tilde{v})_t + 2\tilde{\psi}(\tilde{v})_z = \nu(\tilde{v})_{zz} + (\tilde{u})^2 - (\tilde{v})^2 + c(t), \quad (12)$$

where $\tilde{v} = -(\tilde{\psi})_z$, $\tilde{v}_z = \tilde{\omega}$, and $c(t)$ is an integration constant to enforce the mean of $\tilde{v}$ equal to zero.

The 1D model is exact!

A surprising result is that the above 1D model is exact.

Theorem 3. Let u_1, ψ_1 and ω_1 be the solution of the 1D model (8)-(10) and define

$$u^\theta(r,z,t) = ru_1(z,t), \quad \omega^\theta(r,z,t) = r\omega_1(z,t), \quad \psi^\theta(r,z,t) = r\psi_1(z,t).$$

Then $(u^\theta(r,z,t),\ \omega^\theta(r,z,t),\ \psi^\theta(r,z,t))$ is an exact solution of the 3D Navier-Stokes equations.

Theorem 3 tells us that the 1D model (8)-(10) preserves some essential nonlinear structure of the 3D axisymmetric Navier-Stokes equations.

Energy method does not work for the 1D model!

- A standard energy estimate for the 1D model would give we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tilde{u}^2 dz = -3\int_0^1 (\tilde{u})^2\tilde{v}\, dz - \nu\int_0^1 \tilde{u}_z^2 dz,$$
$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tilde{v}^2 dz = \int_0^1 \tilde{u}^2\tilde{v}\, dz - 3\int_0^1 (\tilde{v})^3 dz - \nu\int_0^1 \tilde{v}_z^2 dz.$$

- One can obtain essentially the same result for the corresponding reaction-diffusion model by dropping convection and $c(t)$:

$$(\tilde{u})_t = \nu(\tilde{u})_{zz} - 2\tilde{v}\tilde{u}, \quad (\tilde{v})_t = \nu(\tilde{v})_{zz} + (\tilde{u})^2 - (\tilde{v})^2, \tag{13}$$

which admits finite time blowup solutions.

- It is not clear how to control the nonlinear vortex stretching like terms by the diffusion terms, unless we assume

$$\int_0^T \|\tilde{v}\|_{L^\infty} dt < \infty, t \le T.$$

Global Well-Posedness of the full 1D Model

Theorem 4. *Assume that $\tilde{u}(z,0)$ and $\tilde{v}(z,0)$ are in $C^m[0,1]$ with $m \geq 1$ and periodic with period 1. Then the solution $(\tilde{u},\tilde{v})$ of the 1D model will be in $C^m[0,1]$ for all times and for $\nu \geq 0$.*

Proof. The key is to obtain *a priori* **pointwise** estimate for the Lyapunov function $\tilde{u}_z^2 + \tilde{v}_z^2$. Differentiating the $\tilde{u}$ and $\tilde{v}$-equations w.r.t z, we get

$$(\tilde{u}_z)_t + 2\tilde{\psi}(\tilde{u}_z)_z - 2\tilde{v}\tilde{u}_z = -2\tilde{v}\tilde{u}_z - 2\tilde{u}\tilde{v}_z + \nu(\tilde{u}_z)_{zz},$$
$$(\tilde{v}_z)_t + 2\tilde{\psi}(\tilde{v}_z)_z - 2\tilde{v}\tilde{v}_z = 2\tilde{u}\tilde{u}_z - 2\tilde{v}\tilde{v}_z + \nu(\tilde{v}_z)_{zz}.$$

Note that the **convection term contributes to stability** by cancelling one of the nonlinear terms on the right hand side. This gives

$$(\tilde{u}_z)_t + 2\tilde{\psi}(\tilde{u}_z)_z = -2\tilde{u}\tilde{v}_z + \nu(\tilde{u}_z)_{zz}, \quad (14)$$
$$(\tilde{v}_z)_t + 2\tilde{\psi}(\tilde{v}_z)_z = 2\tilde{u}\tilde{u}_z + \nu(\tilde{v}_z)_{zz}. \quad (15)$$

Multiplying (14) by $2\tilde{u}_z$ and (15) by $2\tilde{v}_z$, we have

$$(\tilde{u}_z^2)_t + 2\tilde{\psi}(\tilde{u}_z^2)_z = -4\tilde{u}\tilde{u}_z\tilde{v}_z + 2\nu\tilde{u}_z(\tilde{u}_z)_{zz}, \qquad (16)$$

$$(\tilde{v}_z^2)_t + 2\tilde{\psi}(\tilde{v}_z^2)_z = 4\tilde{u}\tilde{u}_z\tilde{v}_z + 2\nu\tilde{v}_z(\tilde{v}_z)_{zz}. \qquad (17)$$

Now, we add (16) to (17). **Surprisingly, the nonlinear vortex stretching-like terms cancel each other.** We get

$$\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_t + 2\tilde{\psi}\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_z = 2\nu\left(\tilde{u}_z(\tilde{u}_z)_{zz} + \tilde{v}_z(\tilde{v}_z)_{zz}\right).$$

Moreover we can rewrite the diffusion term in the following form:

$$\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_t + 2\tilde{\psi}\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_z = \nu\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_{zz} - 2\nu\left[(\tilde{u}_{zz})^2 + (\tilde{v}_{zz})^2\right].$$

Thus, $(\tilde{u}_z^2 + \tilde{v}_z^2)$ satisfies a **maximum principle** for all $\nu \geq 0$:

$$\|\tilde{u}_z^2 + \tilde{v}_z^2\|_{L^\infty} \leq \|(\tilde{u}_0)_z^2 + (\tilde{v}_0)_z^2\|_{L^\infty}.$$

Construction of a family of globally smooth solutions

Theorem 5. *Let* $\phi(r)$ *be a smooth cut-off function and* u_1, ω_1 *and* ψ_1 *be the solution of the 1D model. Define*

$$\begin{aligned} u^\theta(r,z,t) &= r u_1(z,t)\phi(r) + \tilde{u}(r,z,t), \\ \omega^\theta(r,z,t) &= r\omega_1(z,t)\phi(r) + \tilde{\omega}(r,z,t), \\ \psi^\theta(r,z,t) &= r\psi_1(z,t)\phi(r) + \tilde{\psi}(r,z,t). \end{aligned}$$

Then there exists a family of globally smooth functioons $\tilde{u}$, $\tilde{\omega}$ *and* $\tilde{\psi}$ *such that* u^θ, ω^θ *and* ψ^θ *are globally smooth solutions of the 3D Navier-Stokes equations with finite energy.*

A New 3D Model for NSE, [Hou-Lei, CPAM, 09]

Recall the reformulated 3D Navier-Stokes equations:

$$\begin{cases} \partial_t u_1 + u^r \partial_r u_1 + u^z \partial_z u_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big) u_1 + 2u_1\psi_{1z}, \\ \partial_t \omega_1 + u^r \partial_r \omega_1 + u^z \partial_z \omega_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\omega_1 + \big(u_1^2\big)_z, \\ -\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\psi_1 = \omega_1, \end{cases} \tag{18}$$

where $u^r = -r\psi_{1z}$, $u^z = 2\psi_1 + r\psi_{1r}$. Our 3D model is derived by simply dropping the convective term from (18):

$$\begin{cases} \partial_t u_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big) u_1 + 2u_1\psi_{1z}, \\ \partial_t \omega_1 = \nu\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\omega_1 + (u_1^2)_z, \\ -\big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\big)\psi_1 = \omega_1. \end{cases} \tag{19}$$

Note that (24) is already a closed system, and $u_1 = u^\theta/r$ characterizes the axial vorticity near $r = 0$.

Properties of the 3D Model [Hou-Lei, CPAM, 09]

This 3D model shares many important properties with the axisymmetric Navier-Stokes equations.

First of all, one can define an incompressible velocity field in the model equations (24).

$$\mathbf{u}(t,x) = u^r(t,r,z)e_r + u^\theta(t,r,z)e_\theta + u^z(t,r,z)e_z, \tag{20}$$

$$u^\theta = ru_1, \; u^r = -r\psi_{1z}, \quad u^z = 2\psi_1 + r\psi_{1r}, \tag{21}$$

where $x = (x_1, \; x_2, \; z)$, $r = \sqrt{x_1^2 + x_2^2}$. It is easy to check that

$$\nabla \cdot \mathbf{u} = \partial_r u^r + \partial_z u^z + \frac{u^r}{r} = 0, \tag{22}$$

which is the same as the Navier-Stokes equations.

Our model enjoys the following properties ([Hou-Lei, CPAM-09]):

Theorem 6. Energy identity. The strong solution of (24) satisfies

$$\frac{1}{2}\frac{d}{dt}\int \left(|u_1|^2 + 2|D\psi_1|^2\right) r^3 drdz + \int \left(|Du_1|^2 + 2|D^2\psi_1|^2\right) r^3 drdz = 0,$$

which has been proved to be equivalent to that of the Navier-Stokes equations. Here D is the first order derivative operator defined in $\mathbb{R}^5$.

Theorem 7. A non-blowup criterion of Beale-Kato-Majda type. A smooth solution (u_1, ω_1, ψ_1) of the model (24) for $0 \le t < T$ blows up at time $t = T$ if and only if

$$\int_0^T \|\nabla \times \mathbf{u}\|_{\mathrm{BMO}(\mathbb{R}^3)} dt = \infty,$$

where $\mathbf{u}$ is defined in (20)-(21).

Theorem 8. A non-blowup criterion of Serrin-Prodi type. A weak solution (u_1, ω_1, ψ_1) of the model (24) is smooth on $[0, T] \times \mathbb{R}^3$ provided that

$$\|u^\theta\|_{L_t^q L_x^p([0,T]\times\mathbb{R}^3)} < \infty \tag{23}$$

for some p, q satisfying $\frac{3}{p} + \frac{2}{q} \leq 1$ with $3 < p \leq \infty$ and $2 \leq q < \infty$.

Theorem 9. An analog of Caffarelli-Kohn-Nirenberg partial regularity result [Hou-Lei, CMP-09]. For any suitable weak solution of the 3D model equations (24) on an open set in space-time, the one-dimensional Hausdorff measure of the associated singular set is zero.

Potential singularity formation of the 3D model

It is interesting to study the invicsid model.

$$\begin{cases} \partial_t u_1 = 2u_1\psi_{1z}, \\ \partial_t \omega_1 = (u_1^2)_z, \\ -\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\psi_1 = \omega_1. \end{cases} \tag{24}$$

If we let $v = log(u_1^2)$, then we can further reduce the 3D model to the following nonlocal nonlinear wave equation:

$$v_{tt} = 4\left((-\Delta)^{-1}e^v\right)_{zz}, \tag{25}$$

where $-\Delta = -\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)$, and $\int e^v r^3 drdz \le C_0$. Note that $(-\Delta)^{-1}$ is a positive operator. This is a nonlinear nonlocal hyperbolic equation along the z-direction.

Numerical evidence for a potential finite time singularity

Innitial condition we consider in our numerical computations is given by

$$u_1(z,r,0) = (1+\sin(4\pi z))(r^2-1)^{20}(r^2-1.2)^{30}, \qquad (26)$$

$$\psi_1(z,r,0) = 0, \qquad (27)$$

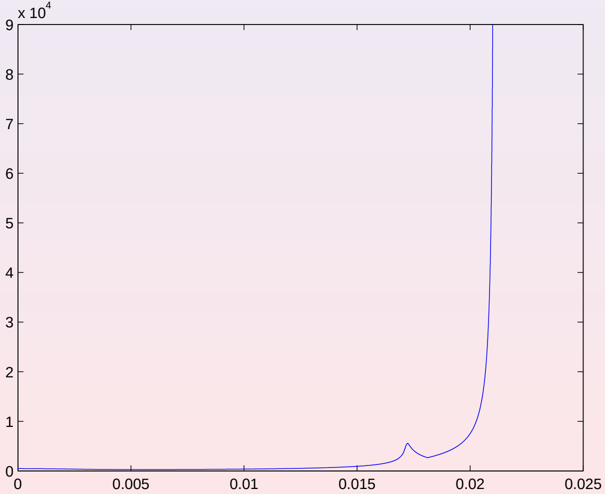
$$\omega_1(z,r,0) = 0. \qquad (28)$$

A second order finite difference discretization is used in space, and the classical fourth order Runge-Kutta method is used to discretize in time. We use the following coordinate transformation along the r-direction to achieve the adaptivity:

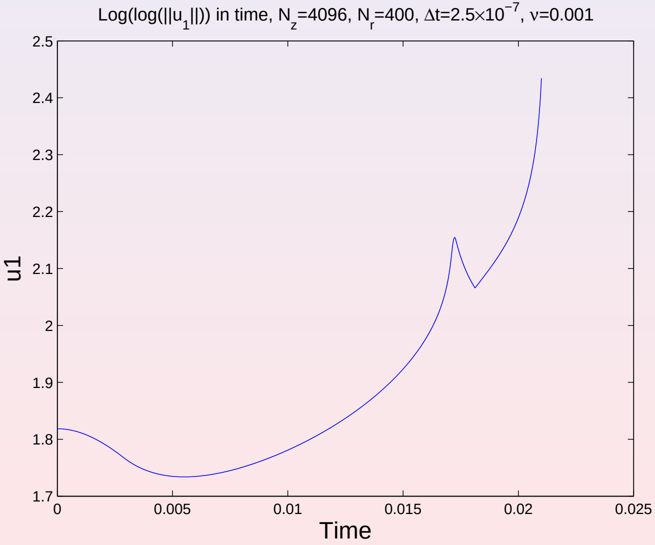
$$r = f(\alpha) \equiv \alpha - 0.9\sin(\pi\alpha)/\pi. \qquad (29)$$

We use an effective resolution up to 4096^3 for the 3D problem.

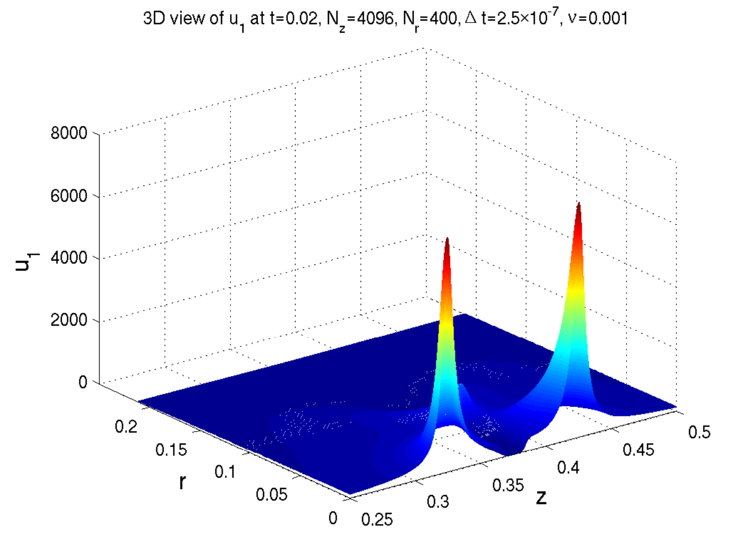
$\|u_1\|_\infty$ as a function of time over the interval $[0,\ 0.021]$, the viscous model with $\nu = 0.001$.



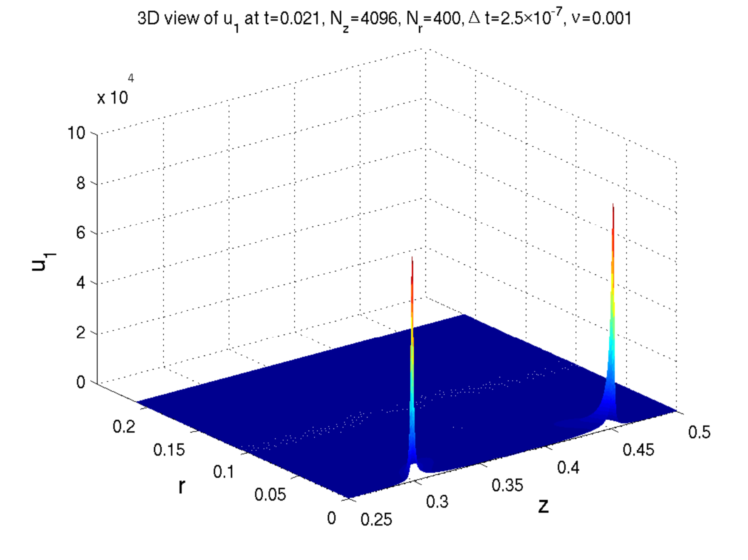
$\log(\log(\|u_1\|_\infty))$ as a function of time over the interval $[0,\ 0.021]$, the viscous model with $\nu = 0.001$.



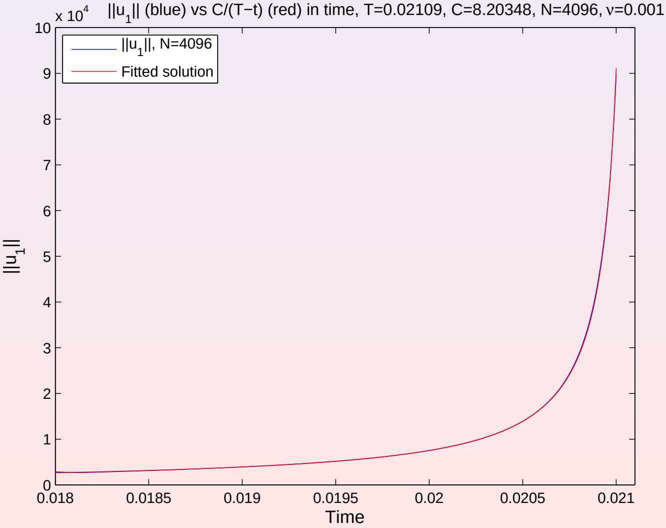
A 3D view of u_1 at $t = 0.02$.



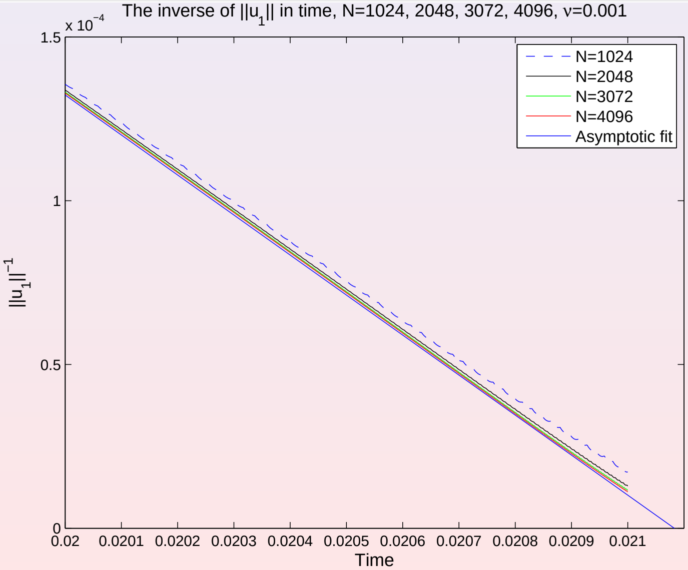
A 3D view of u_1 at $t = 0.021$.



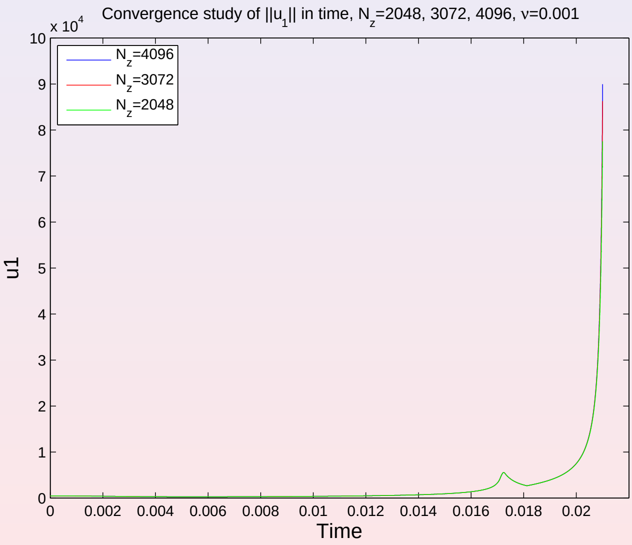
Asymptotic blowup rate: $\|u_1\|_\infty \approx \frac{C}{(T-t)}$, with $T = 0.02109$ and $C = 8.20348$.



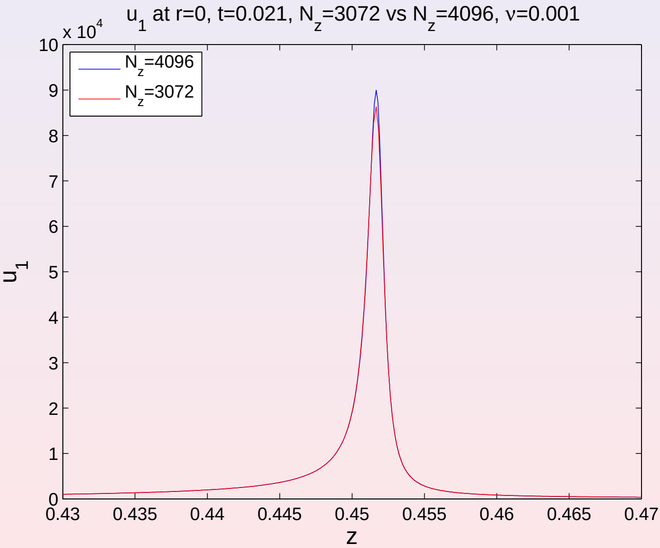
Asymptotic blowup fit: $\|u_1\|_\infty^{-1} \approx \frac{(T-t)}{C}$, with limiting values $T = 0.021083$ and $C = 8.1901$.



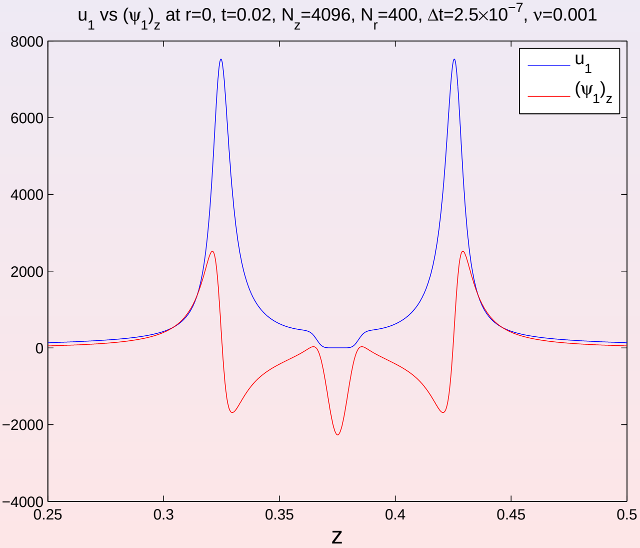
Convergence study of $\|u_1\|_\infty$ in time.



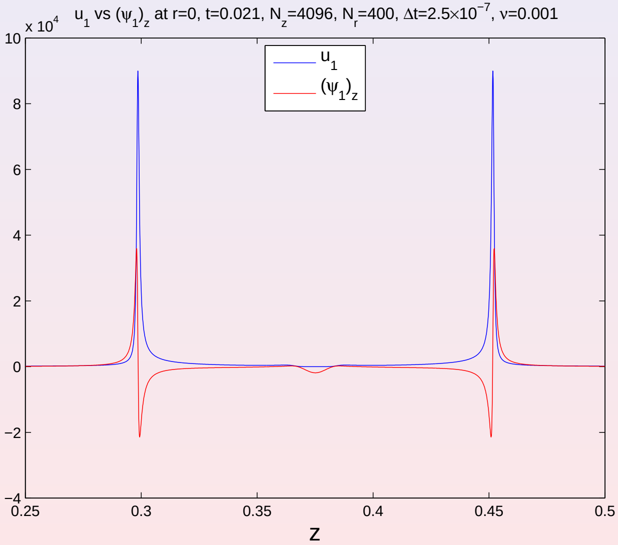
Convergence study of u_1 at $r = 0$ and $t = 0.021$



Local alignment of u_1 and ψ_{1z} at $t = 0.02$. Recall $(u_1)_t = 2u_1\psi_{1z} + \nu\Delta u_1$, $(\omega_1)_t = (u_1^2)_z + \nu\Delta\omega_1$.

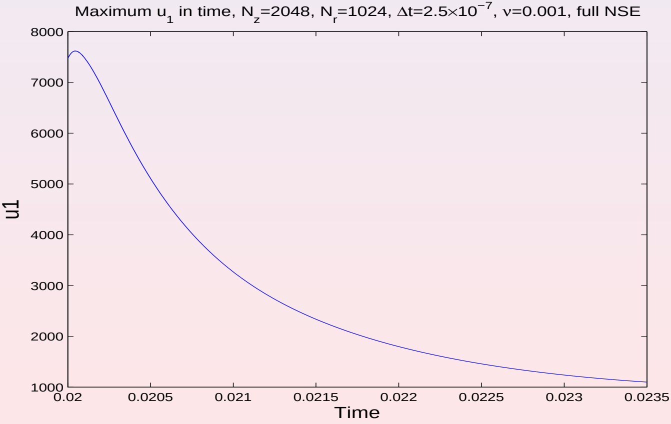


Local alignment of u_1 and ψ_{1z} at $t = 0.021$.

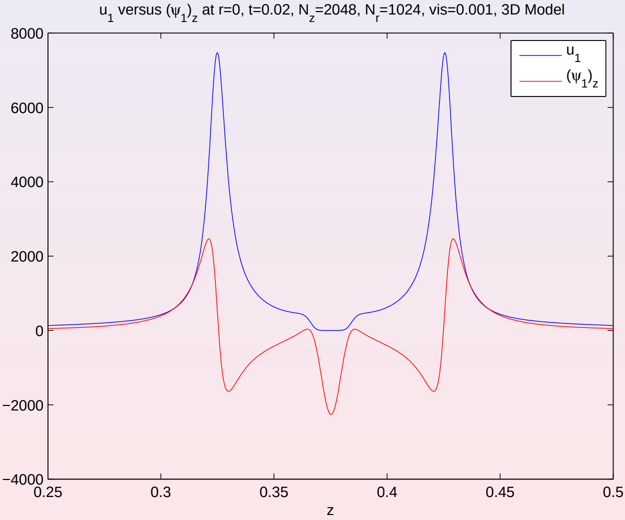


Stabilizing effect of convection.

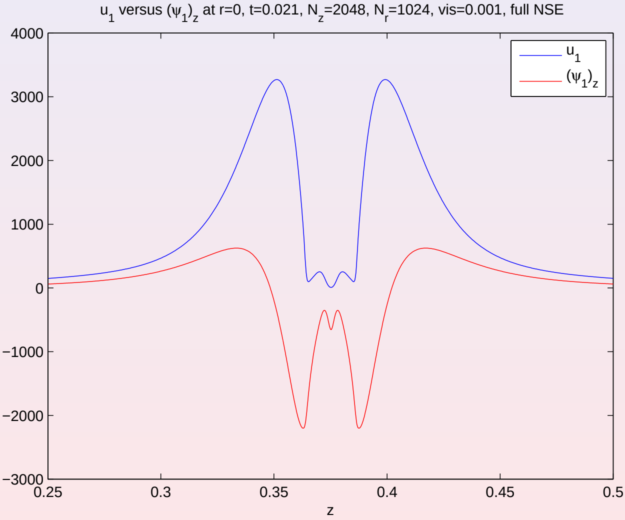
To study the stabilizing effect of convection, we add the convection term back to the 3D model and solve the Navier-Stokes equations using the solution of the 3D model at $t = 0.02$ as the initial condition.



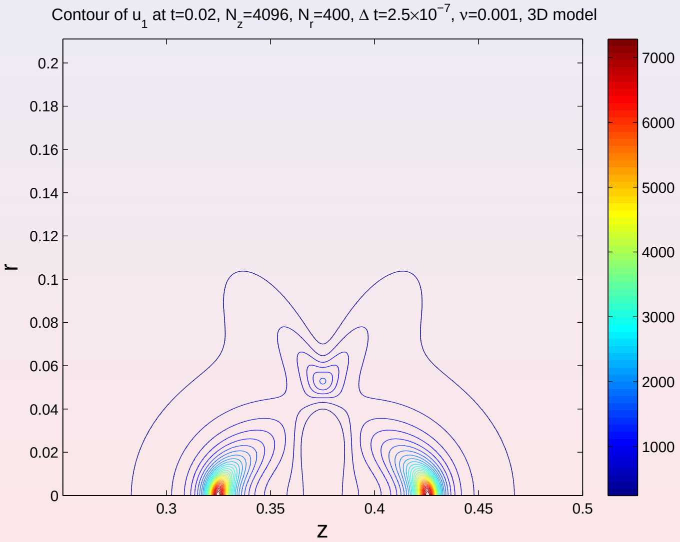
Depletion of vortex stretching due to convection



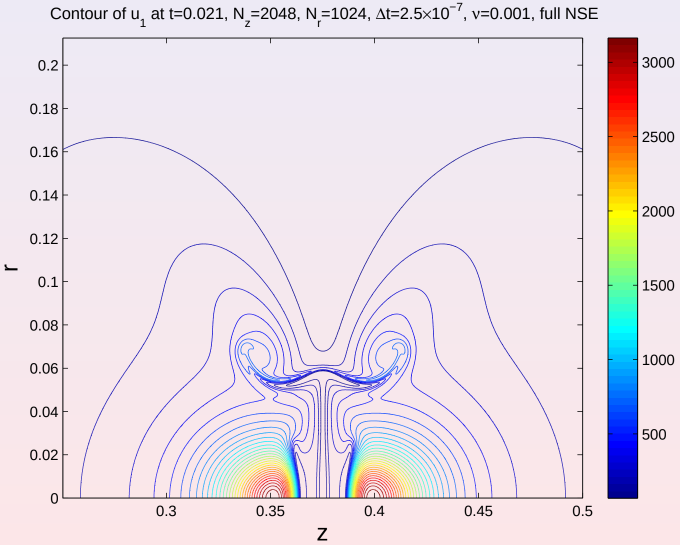
Depletion of vortex stretching due to convection



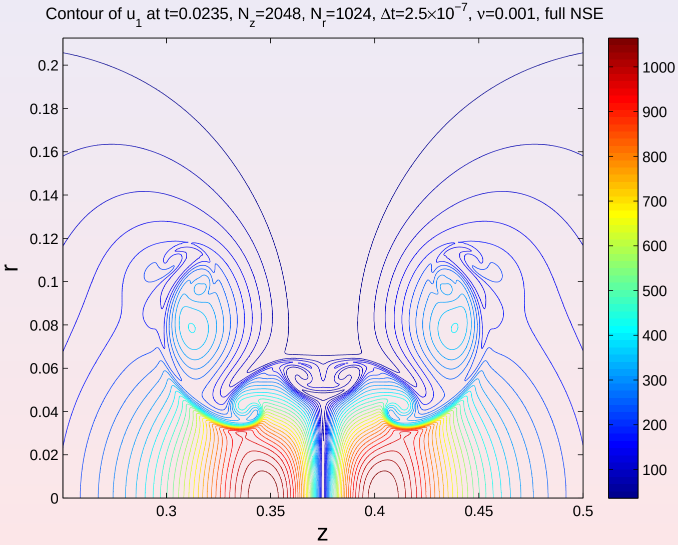
Contours of initial data for u_1.



Contours of u_1 at $t = 0.021$, solution of full NSE.



Contours of u_1 at $t = 0.0235$, solution of full NSE.



Recent theortical progress for the 3D model

Theorem 10. [Hou-Shi-Wang, 09]. Consider the 3D inviscid model

$$\begin{cases} u_t = 2u\psi_z, \quad \omega_t = (u^2)_z, \\ -(\partial_x^2 + \partial_z^2)\psi = \omega, \quad 0 \le x \le 1,\ 0 \le z \le 1, \end{cases} \tag{30}$$

with boundary condition $\psi = 0$ at $x = 0,\ 1,\ z = 1$, and $(\alpha \frac{\partial \psi}{\partial z} + \psi)|_{z=0} = 0$ for some $0 < \alpha < 1$. If the initial conditions, u_0 and ψ_0, are smooth, satisfying $u_0 = 0$ at $z = 0,\ 1$, and

$$\int_0^1 \int_0^1 \log(u_0)\phi(x,z)dxdz \ge 0, \quad \int (\psi_0)_z \phi(x,z)dxdz > 0, \tag{31}$$

where $\phi(x,z) = \sin(x)\cosh(\alpha(1-z))$, then the 3D inviscid model must develop a finite time singularity. Moreover, if the ω-equation is viscous and ω satisfies the same boundary condition as ψ, then the 3D model with partial viscosity must develop a finite time singularity.

Theorem 11. [Hou-Shi-Wang, 09]. Consider the 3D inviscid model

$$\begin{cases} u_t = 2u\psi_z, \quad \omega_t = (u^2)_z, \\ -(\partial_x^2 + \partial_z^2)\psi = \omega, \quad 0 \le x \le 1,\ 0 \le z \le 1, \end{cases} \tag{32}$$

with boundary condition $\psi = 0$ at $x = 0,\ 1,\ z = 1$, and $\psi_z|_{z=0} = 0$. Assume that the initial condition, u_0 and ψ_0, are smooth, satisfying $u_0 = 0$ at $z = 0,\ 1$, and

$$\int_0^1 \int_0^1 \log(u_0)\phi dxdz \ge 0, \quad \int_0^1 \int_0^1 (\psi_0)_z \phi dxdz > 0, \tag{33}$$

where $\phi(x, z) = \sin(x)\cosh(\alpha(1 - z))$ for some $0 < \alpha < 1$. Then the 3D inviscid model must develop a finite time singularity provided that

$$\int_0^1 \sin(x)\left(\psi|_{z=0}(t) - \psi_0|_{z=0}\right) dx \le C_0 \int_0^1 \int_0^1 (\psi_0)_z \phi dxdz, \tag{34}$$

as long as the solution remains regular, where $C_0 < (1 - \alpha^2)/\cosh(\alpha)$.

Concluding Remarks

- Our study shows that convection could play an important role in the dynamic depletion of the nonlinear vortex stretching.
- By neglecting the convection term, we contruct a new 3D model which shares almost all properties of the Navier-Stokes equations, but could develop finite time singularities.
- This seems to suggest that one should take advantage of the stabilizing effect of convection in an essential way in our global regularity analysis of the 3D NSE.
- Convection tends to severely deform and flatten the support of maximum vorticity, which could weaken and eventually deplete the nonlinear vortex stretching.
- The current methods based on energy estimates seem too crude to capture the stabilizing effect of convection. A more localized analytic method may be required to study the global regualrity or blowup of 3D NSE.

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