Challenges in Modelling and Computing Planetary Scale Atmospheric Flows at High Resolution

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Thanks to ...

Ulrich Achatz	(Goethe-Universität, Frankfurt)
Didier Bresch	(Université de Savoie, Chambéry)
Omar Knio	(Johns Hopkins University, Baltimore)
Oswald Knoth	(IFT, Leipzig)
Piotr Smolarkiewicz	(NCAR, Boulder, CO, USA)

Deutsche
Forschungsgemeinschaft

MetStröm DFG

Motivation

The Challenge: global cloud-resolving models

Motivation

Compressible flow equations

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi = 0$$

$$\boldsymbol{(\rho w)_t + \nabla \cdot (\rho v w)} + P \pi_z = -\rho g$$

$$\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$P = p^{\frac{1}{\gamma}} = \rho \theta \,, \qquad \pi = p / \Gamma P \,, \qquad \Gamma = c_p / R \,, \qquad \boldsymbol{v} = \boldsymbol{u} + w \boldsymbol{k} \,, \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)$$

Motivation

Pseudo-incompressible model*

"sound-proof"

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + \overline{P} \nabla_{\|} \pi = 0$$

$$(\rho w)_t + \nabla \cdot (\rho \boldsymbol{v} w) + \overline{P} \pi_z = -\rho g$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

$\boldsymbol{\Delta x < 15}$ **km**

$$P \equiv \overline{P}(z)\,, \qquad \rho\theta = \overline{P}(z)\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

* **Durran (1988)**

Motivation

Hydrostatic primitve equations

"vertically sound-proof"

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
\times \qquad P \pi_z &= -\rho g \\
P_t + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

$\boldsymbol{\Delta x > 15}$ **km**

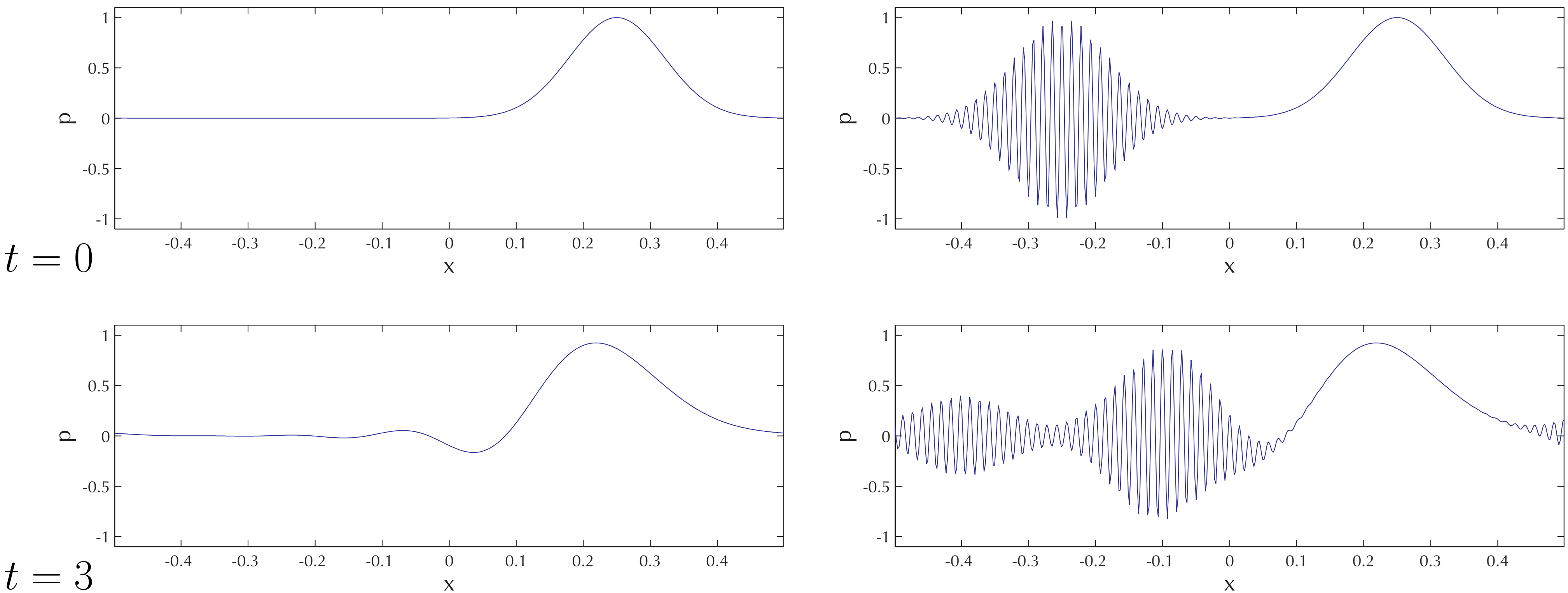
$$
P = p^{\frac{1}{\gamma}} = \rho\theta \,, \qquad \pi = p / \Gamma P \,, \qquad \Gamma = c_p / R \,, \qquad \boldsymbol{v} = \boldsymbol{u} + w \boldsymbol{k} \,, \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)
$$

Motivation

Why not simply solve the full compressible-flow equations?

Simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 *grid pts.,* $\mathrm{CFL} = 10$*)*



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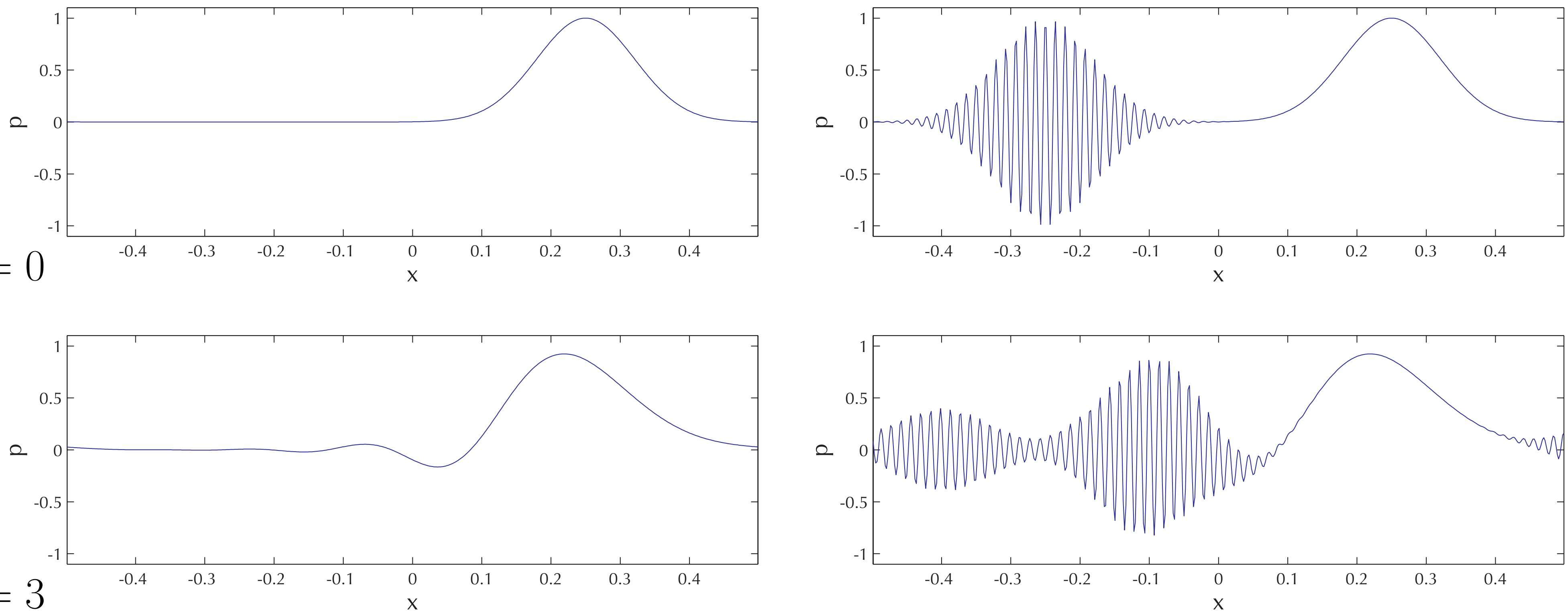


$t = 0$

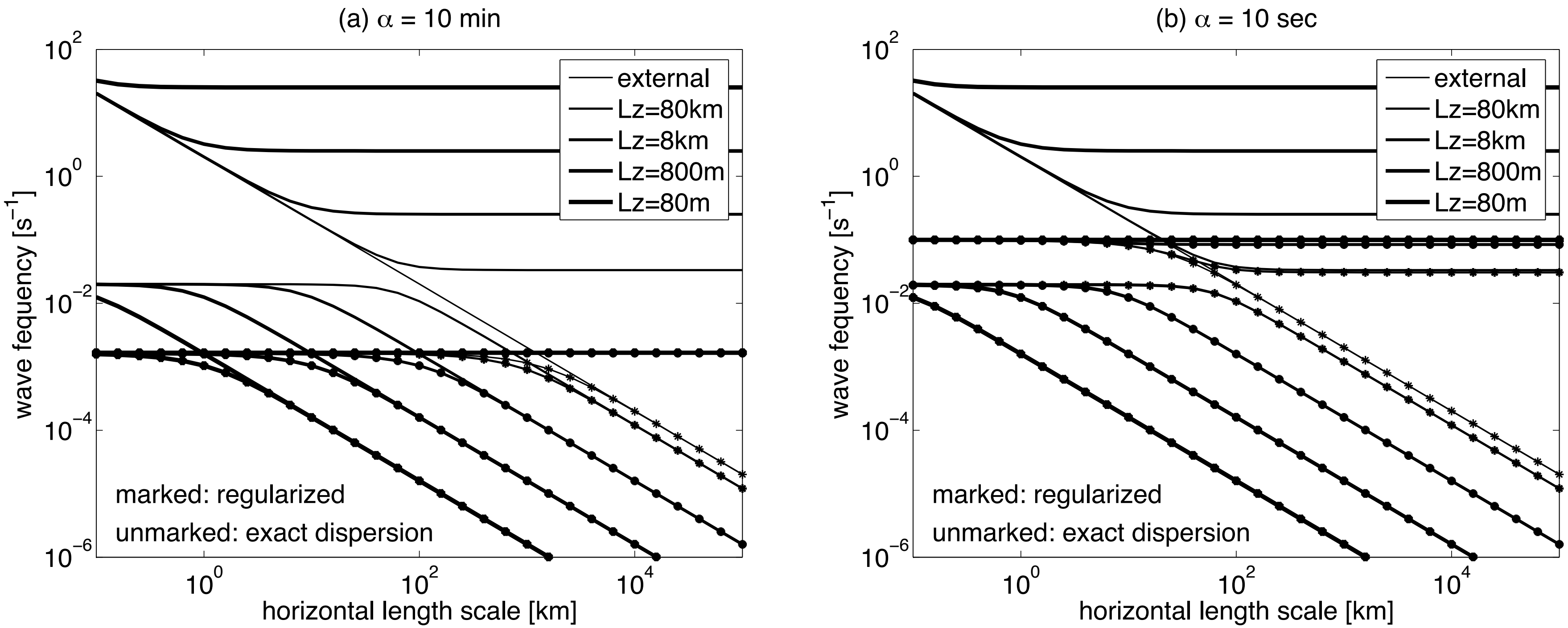
$t = 3$

Implicit discretization **regularizes** by **slowing down** the waves*

* see, e.g., Reich et al. (2007)

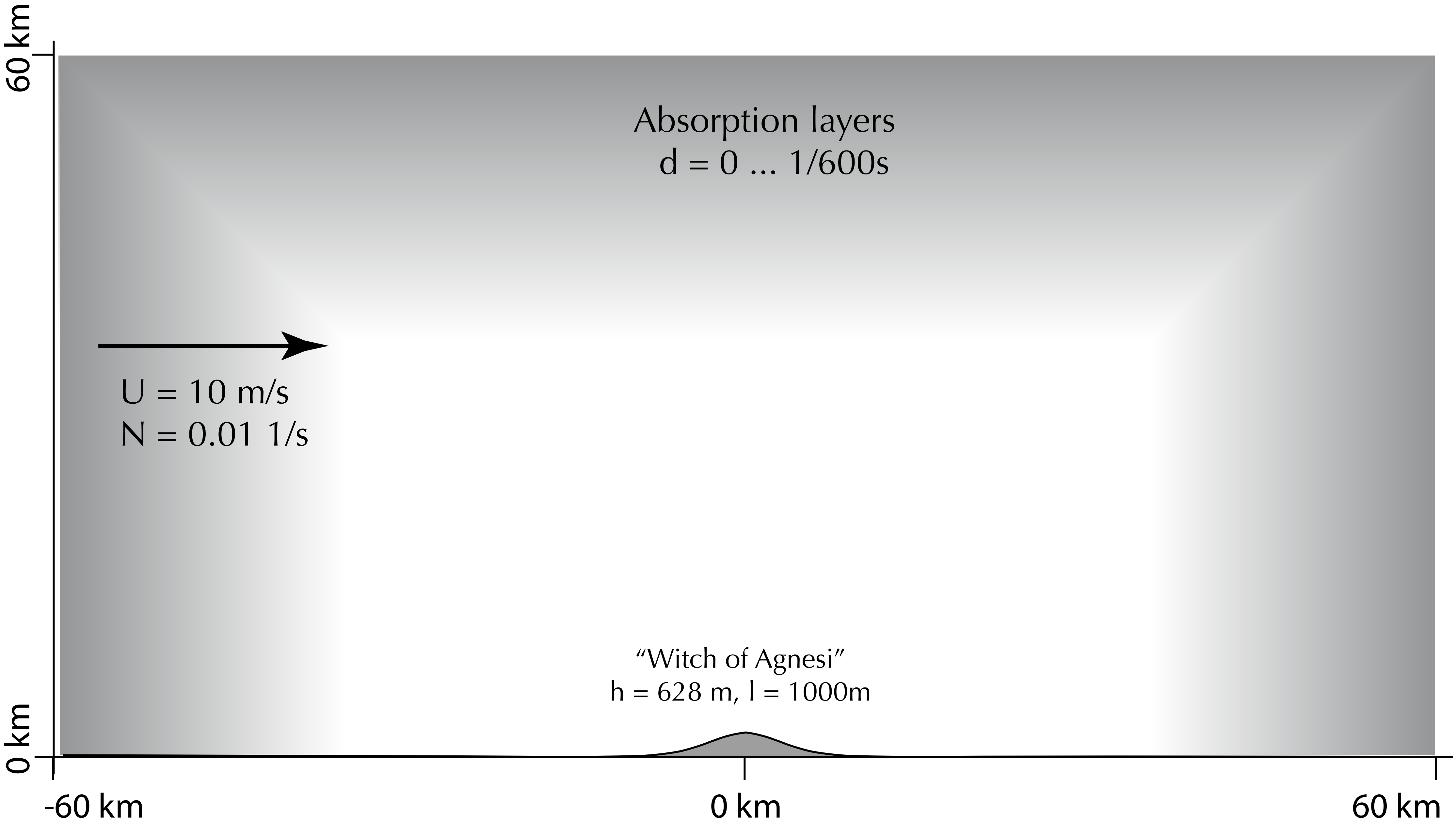
Motivation

Dispersion relation for linear waves in a compressible atmosphere

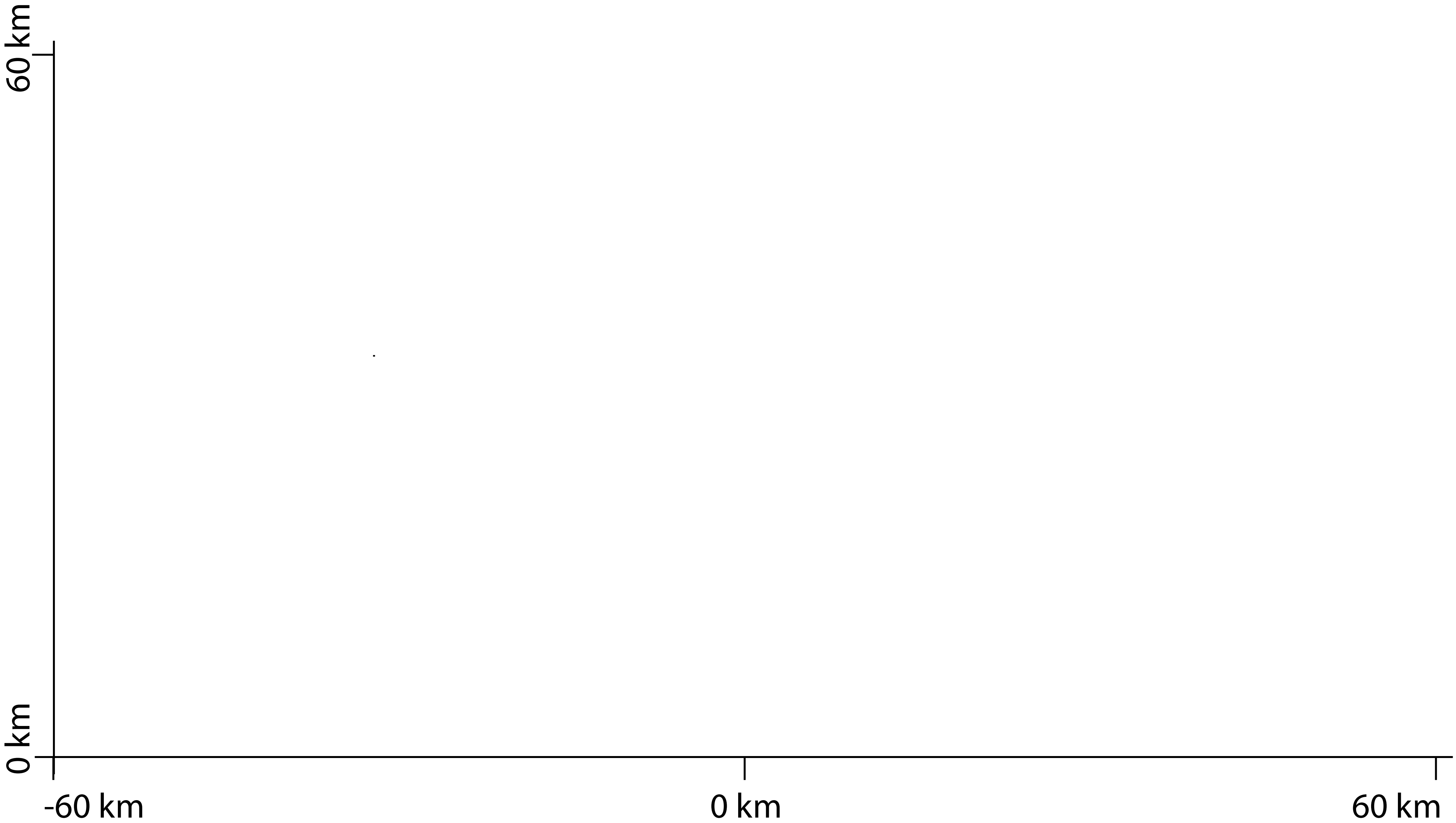


courtesy: Sebastian Reich, Potsdam University

Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Motivation

Asymptotics

Two-Scale Models

Numerics

Conclusions

Asymptotics: planetary scale 

Hydrostatic Primitive Equations: ... aspect-ratio asymptotics ✓

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
P \pi_z &= -\rho g \\
P_t + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

$$
P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k}\,, \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)
$$

Asymptotics: convective scale 

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$

Asymptotics: convective scale

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For single-scale models with advection & internal waves:*

$$\frac{h_{\text{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \qquad \text{or} \qquad \Delta\overline{\theta} \sim 0.3\ \text{K}$$

* Ogura & Phillips (1962)

Asymptotics: convective scale

Ogura & Phillips' (1962) Anelastic Model:

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi = \overline{\rho} \theta' g \boldsymbol{k}$$

$$(\overline{\rho} \theta')_t + \nabla \cdot (\overline{\rho} \theta' \boldsymbol{v}) = 0$$

For single-scale models with advection & internal waves:*

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* Ogura & Phillips (1962)

Asymptotics: convective scale 

More realistic regimes with three time scales

Mach number and stratification

$$\frac{u_{\text{ref}}}{c_{\text{ref}}} = \varepsilon \ll 1 \,, \qquad \frac{h_{\text{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \,, \quad (0 < \mu < 2)$$

Rescaled dependent variables

$$\pi = \overline{\pi}^{\varepsilon}(z) + \varepsilon \tilde{\pi} \,, \qquad \theta = \underbrace{1 + \varepsilon^{\mu} \overline{\theta}(z)}_{\overline{\theta}^{\varepsilon}} + \varepsilon^{\nu+\mu} \tilde{\theta} \,, \qquad (\mu = 2\,(1 - \nu))$$

Asymptotics: convective scale 

$$
\begin{aligned}
\tilde{\theta}_\tau \;+\; \frac{1}{\varepsilon^{\nu}}\,\tilde{w}\,\frac{d\overline{\theta}}{dz} & & &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau \;+\; \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde{\theta}}{\overline{\theta}^{\varepsilon}}\,\boldsymbol{k} &\;+\; \frac{1}{\varepsilon}\,\overline{\theta}^{\varepsilon}\nabla\tilde{\pi} & &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi}\;\;\cdot \\
\tilde{\pi}_\tau & \;+\; \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}^{\varepsilon}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}^{\varepsilon}}{dz}\right) & &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

Asymptotics: convective scale 

$$
\begin{aligned}
\tilde{\theta}_\tau \;+\; \frac{1}{\varepsilon^{\nu}}\,\tilde{w}\,\frac{d\overline{\theta}}{dz} \qquad\qquad\qquad\qquad\qquad\qquad &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau \;+\; \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde{\theta}}{\overline{\theta}^{\varepsilon}}\,\boldsymbol{k} \;+\; \frac{1}{\varepsilon}\,\overline{\theta}^{\varepsilon}\nabla\tilde{\pi} \qquad\qquad\qquad &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi} \;\; . \\
\tilde{\pi}_\tau \qquad\qquad\;\; + \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}^{\varepsilon}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}^{\varepsilon}}{dz}\right) &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

For the linear system:

✓ Conservation of weighted quadratic energy

✓ Control of time derivatives by initial data $(\tau = O(1))$

... consider internal wave scalings for $\tau = O(\varepsilon^{\nu})$:

$$
\vartheta = \frac{\tau}{\varepsilon^{\nu}}\,, \qquad \pi^{*} = \varepsilon^{\nu-1}\tilde{\pi}\,,
$$

Klainerman, Majda (1982) ... Dutrifoy, Majda, Schochet (2006,2008)

Asymptotics: convective scale

Linearized compressible / pseudo-incompressible systems

$$\tilde{\theta}_\vartheta + \tilde{w}\,\frac{d\overline{\theta}}{dz} = 0$$

$$\tilde{\boldsymbol{v}}_\vartheta + \frac{\tilde{\theta}}{\overline{\theta}^{\varepsilon}}\,\boldsymbol{k} + \overline{\theta}^{\varepsilon}\nabla\pi^* = 0$$

$$\varepsilon^{\mu}\,\pi^*_\vartheta + \left(\gamma\Gamma\overline{\pi}^{\varepsilon}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}^{\varepsilon}}{dz}\right) = 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^* \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^* & 0 & 0 & 0 \\ 0 & \boldsymbol{U}^* & 0 & 0 \\ 0 & 0 & W^* & 0 \\ 0 & 0 & 0 & \Pi^* \end{pmatrix}(z)\ \exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Asymptotics: convective scale

Relation between compressible and pseudo-incompressible vertical modes

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}}}\frac{1}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^*$$

$\varepsilon^{\mu} = 0$: pseudo-incompressible case

regular Sturm-Liouville problem for internal wave modes

$\varepsilon^{\mu} > 0$: compressible case

nonlinear Sturm-Liouville problem ...

$\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}} = O(1)$: perturbations of pseudo-incompressible modes & EVals

Asymptotics: convective scale

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^{\varepsilon\,2}}}\frac{1}{\overline{\theta}^{\varepsilon}\,\overline{P}^{\varepsilon}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}^{\varepsilon}\,\overline{P}^{\varepsilon}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}^{\varepsilon}\,\overline{P}^{\varepsilon}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^{\varepsilon\,2}} = O(1)\right)$

- pseudo-inc. modes/EVals = compressible modes/EVals + $O(\varepsilon^{\mu})$ †
- phase errors remain small for $\vartheta = t_{\text{adv}}/\varepsilon^{\nu} < O(\varepsilon^{-\mu})$
- validity for $t_{\text{adv}} = O(1) \quad \Rightarrow \quad \nu - \mu = 1 - \frac{3}{2}\mu > 0$

The pseudo-incompressible model remains relevant for stratifications

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{\text{sc}}} \lesssim 50\ \text{K}$$

not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

† rigorous proof pending $\longrightarrow$ work with D. Bresch

Asymptotics: convective scale

Anelastic

$$\times \quad \nabla\cdot(\overline{\rho}\boldsymbol{v}) = 0$$

$$(\overline{\rho}\boldsymbol{v})_t + \nabla\cdot(\overline{\rho}\boldsymbol{v}\circ\boldsymbol{v}) + \overline{\rho}\nabla\pi = \frac{\theta'}{\overline{\theta}}\,\overline{\rho} g\boldsymbol{k}$$

$$P_t + \nabla\cdot(P\boldsymbol{v}) = 0$$

$$\overline{\rho}(z)\theta = P\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\nabla\cdot(\overline{\rho}\boldsymbol{v}) = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla\pi = \frac{\theta'}{\overline{\theta}}\,g\boldsymbol{k}$$

$$\theta_t + \boldsymbol{v}\cdot\nabla\theta = 0$$

$$\theta' = \theta(z) - \overline{\theta}(z)$$

Pseudo-incompressible

baroclinic torque / modified divergence

$$\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0$$

$$(\rho\boldsymbol{v})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{v}) + \overline{P}\nabla\pi = \frac{\theta'}{\overline{\theta}}\,\rho g\boldsymbol{k}$$

$$\times \quad \nabla\cdot(\overline{P}\boldsymbol{v}) = 0$$

$$\rho(z)\theta = \overline{P}\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$(1/\theta)_t + \boldsymbol{v}\cdot\nabla(1/\theta) = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \underline{\theta\nabla\pi} = \frac{\theta'}{\overline{\theta}}\,g\boldsymbol{k}$$

$$\underline{\nabla\cdot(\overline{P}\boldsymbol{v})} = 0$$

relevant for deep atmospheres / large scales*

*see, e.g., Smolarkiewicz & Dörnbrack (2007)

Asymptotics: convective scale

Anelastic

Boussinesq approximation

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi = \frac{\theta'}{\overline{\theta}} \overline{\rho} g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$\overline{\rho}(z)\theta = P \,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \pi = \theta \, g \boldsymbol{k}$$

$$\theta_t + \boldsymbol{v} \cdot \nabla \theta = 0$$

Pseudo-incompressible

zero-Mach, variable density flow

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi = \frac{\theta'}{\overline{\theta}} \rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

$$\rho(z)\theta = \overline{P} \,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\rho_t + \boldsymbol{v} \cdot \nabla \rho = 0$$

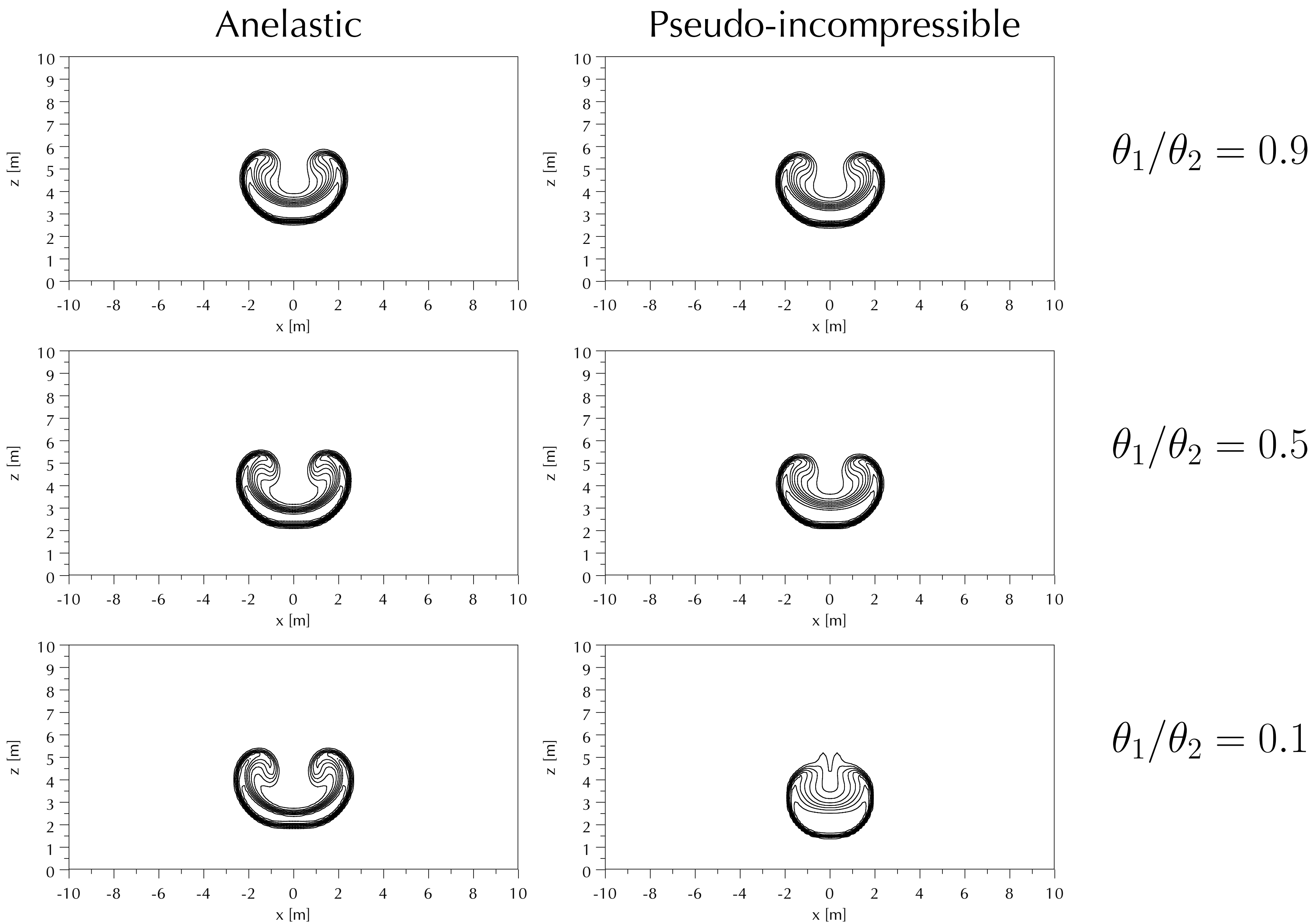
$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho} \nabla \pi = (\rho - \overline{\rho}) \, g \boldsymbol{k}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

Small scale limits

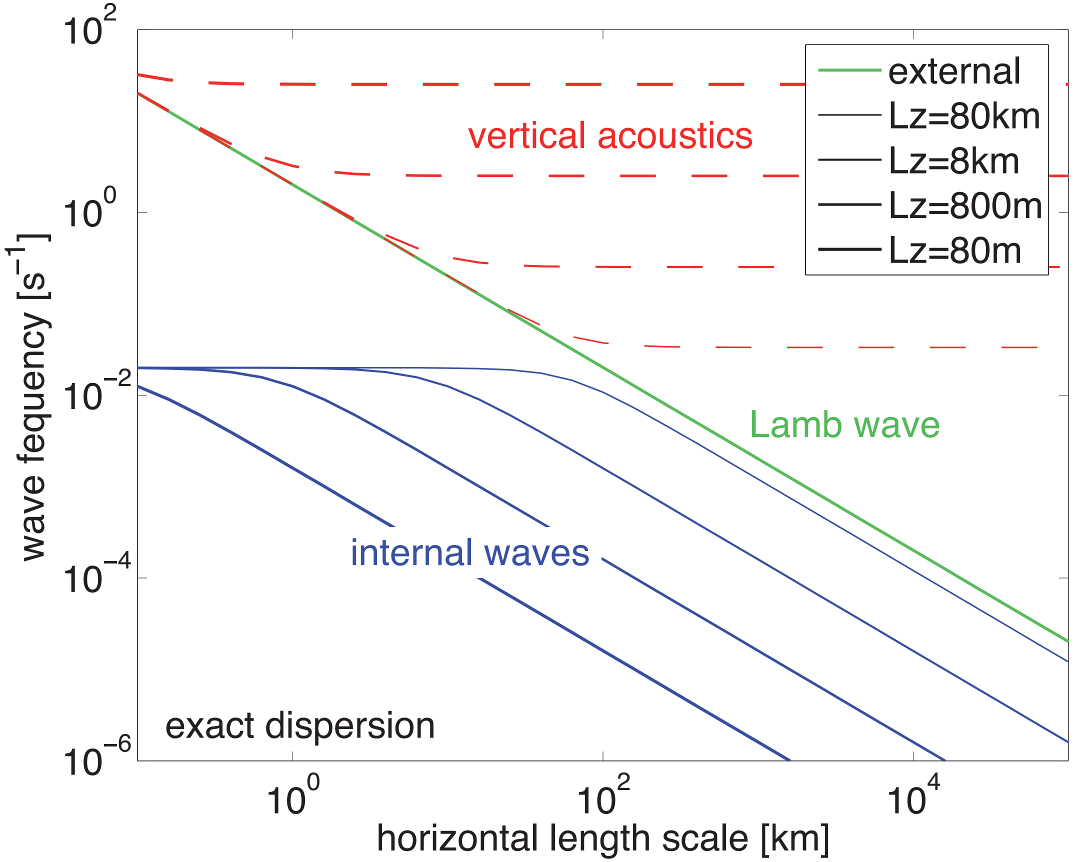
Asymptotics: convective scale

Cold air blobs at small scales



Two-Scale Models

Dispersion relation for linear waves in a compressible atmosphere



Goal:

Keep the external and internal waves, eliminate vertical acoustics

courtesy: Sebastian Reich, Potsdam University

Two-Scale Models

Durran, JFM, (08); Arakawa & Konor, MWR, (09)*

$$
\begin{aligned}
\boldsymbol{\rho}^*{}_t + \nabla \cdot (\boldsymbol{\rho}^* \boldsymbol{v}) &= 0 \\
\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{v} + w \boldsymbol{v}_z + \theta \nabla_{\|} \left(\boldsymbol{\pi_h} + \boldsymbol{\pi}' \right) &= 0 \\
w_t + \boldsymbol{u} \cdot \nabla w + w w_z + \theta \left(\boldsymbol{\pi_h} + \boldsymbol{\pi}' \right)_z &= -g \\
\theta_t + \boldsymbol{u} \cdot \nabla \theta + w \theta_z &= 0
\end{aligned}
$$

$$
\boldsymbol{\rho}^* = \rho(\theta, \boldsymbol{\pi_h}) \,, \qquad \boldsymbol{\pi_h} = \boldsymbol{\pi_S}(t, \boldsymbol{x}) - \int_{z_S}^{z} \frac{g}{\theta} \, dz \,,
$$

compressible barotropic dynamics for $\boldsymbol{\pi_S}$

* *(the gist only!)*

Motivation

Asymptotics

Two-Scale Models

Numerics

Conclusions

Numerics

Why not simply solve the full compressible equations?

Competing approaches: **model codes**

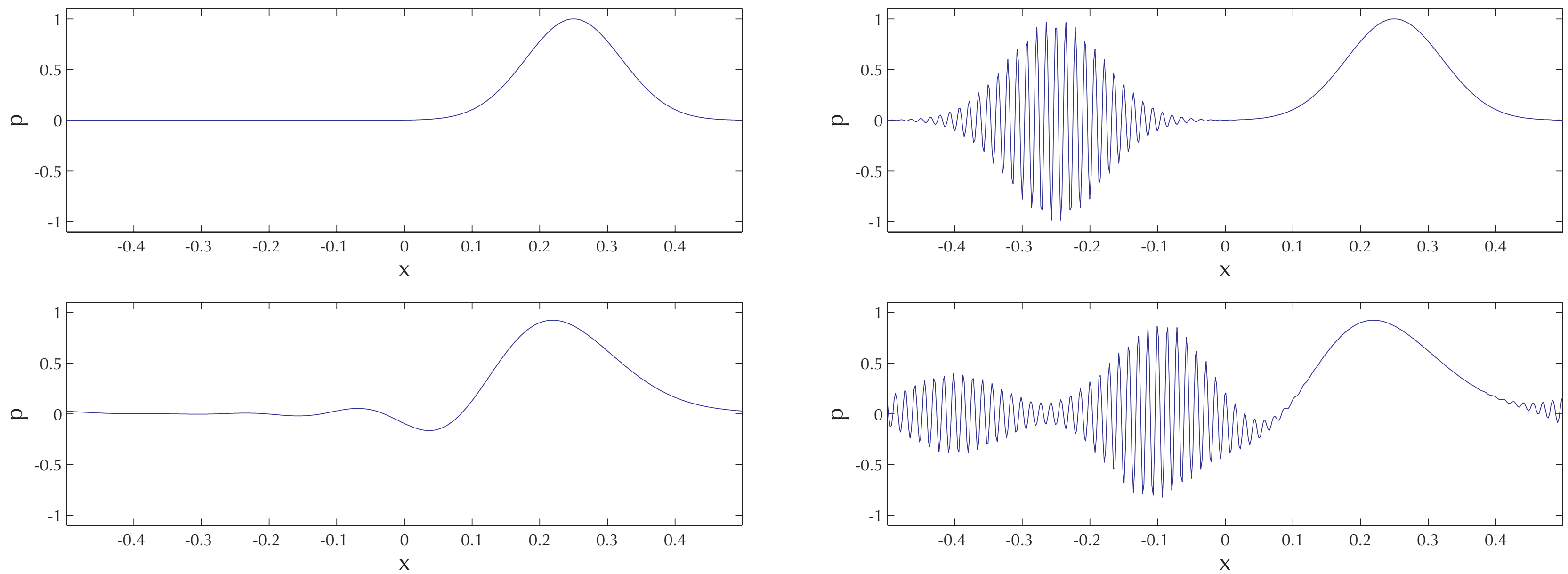
- Split-explicit / multi-rate methods, e.g.,
  - Runge-Kutta (slow) + forward-backward (fast), e.g.,
    Wicker & Skamarock, MWR, (98), ... ; *MM5, LM, WRF ...*
  - Multirate infinitesimal schemes, peer methods
    Wensch et al., BIT, (09); *ASAM, ...*
- Semi-implicit / linearly implicit schemes
  - explicit advection, damped 2nd or 1st-order schemes for fast modes, e.g.,
    Robert, Japan Met. J., (69), ... ; *UKMO, ...*
  - linearly implicit Rosenbrock-type methods, e.g.,
    Reisner et al., MWR, (05), ...; *ASAM, LANL Hurricane model, ...*
- Fully implicit integration

Numerics

Why not simply solve the full compressible equations?

Simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***



$t = 0$

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Numerics

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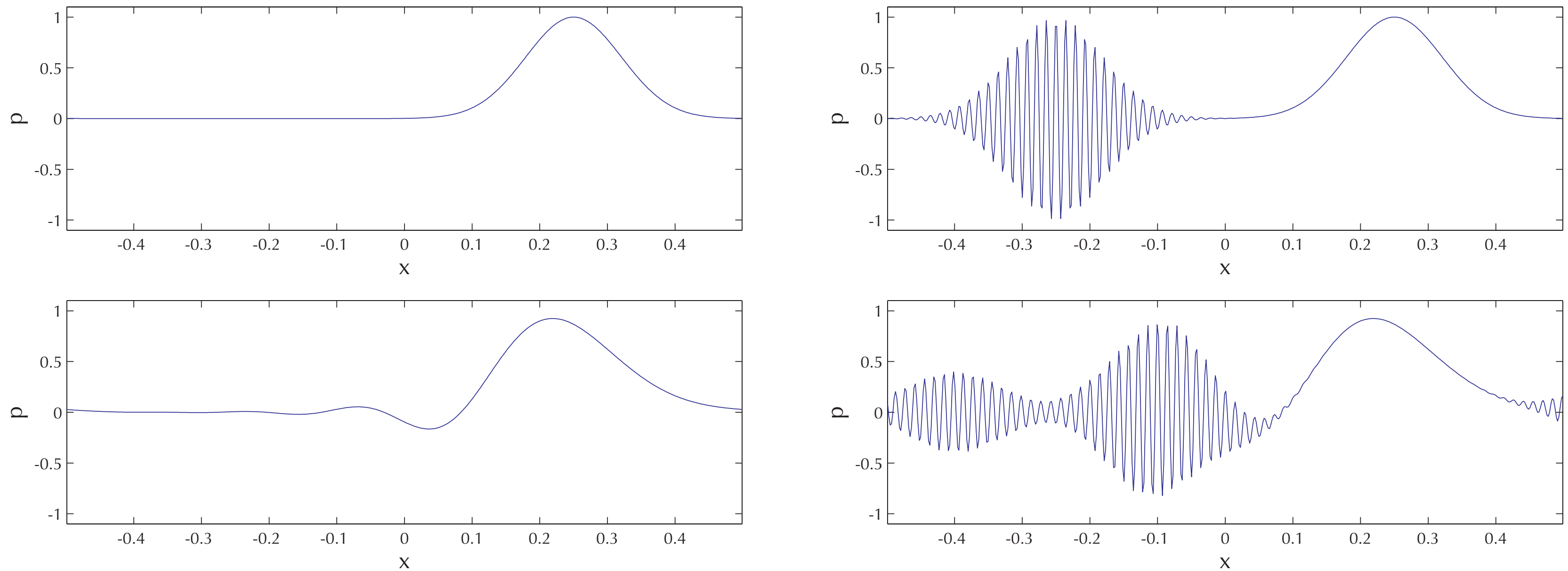

$t = 0$

$t = 3$

Ideas:

- Slave short waves $(c\Delta t/\ell > 1)$ to long waves $(c\Delta t/\ell \leq 1)$
- with pseudo-incompressible limit behavior

Numerics

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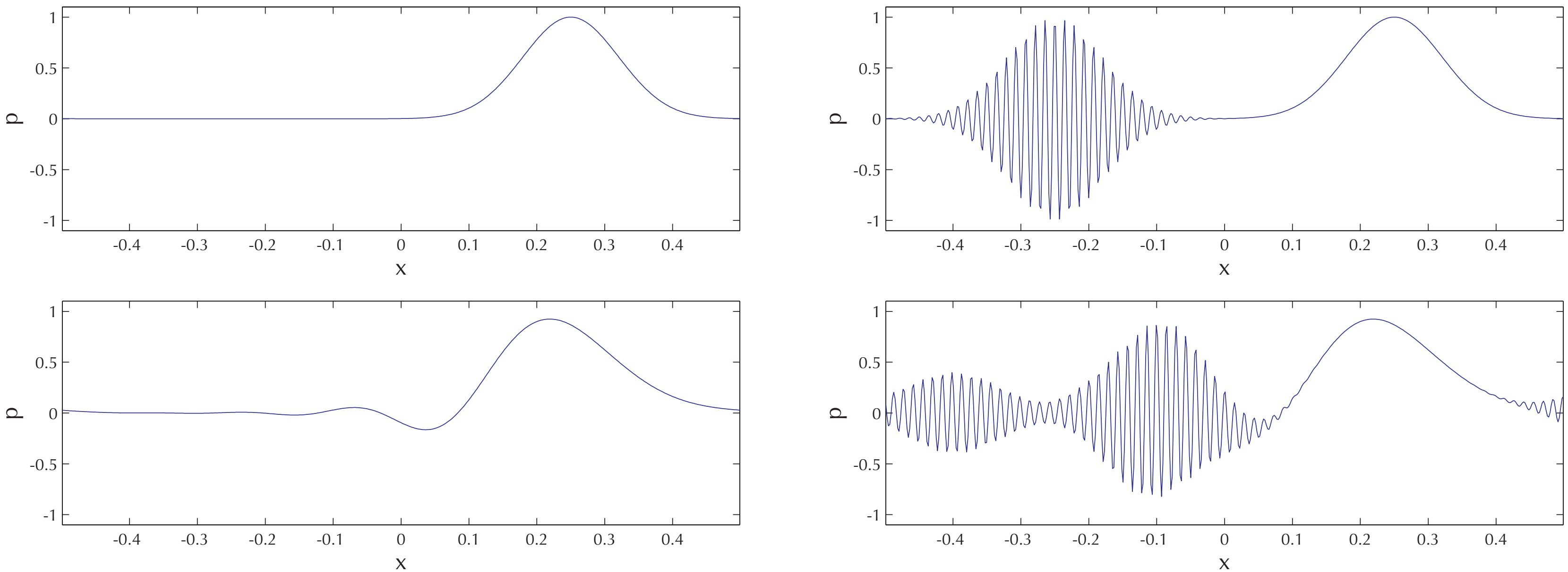

$t = 0$

$t = 3$

Ideas:

- Slave short waves $(c\Delta t/\ell > 1)$ to long waves $(c\Delta t/\ell \le 1)$
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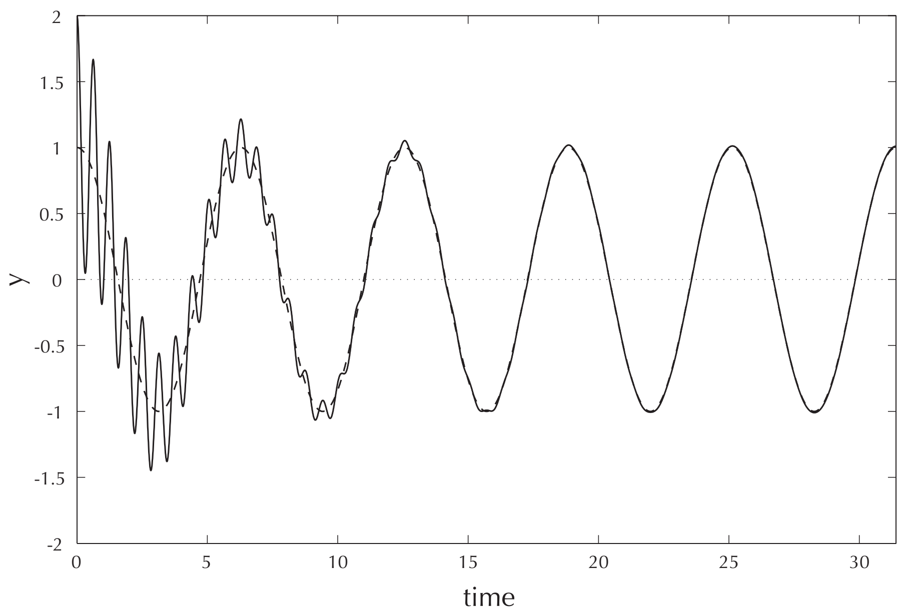
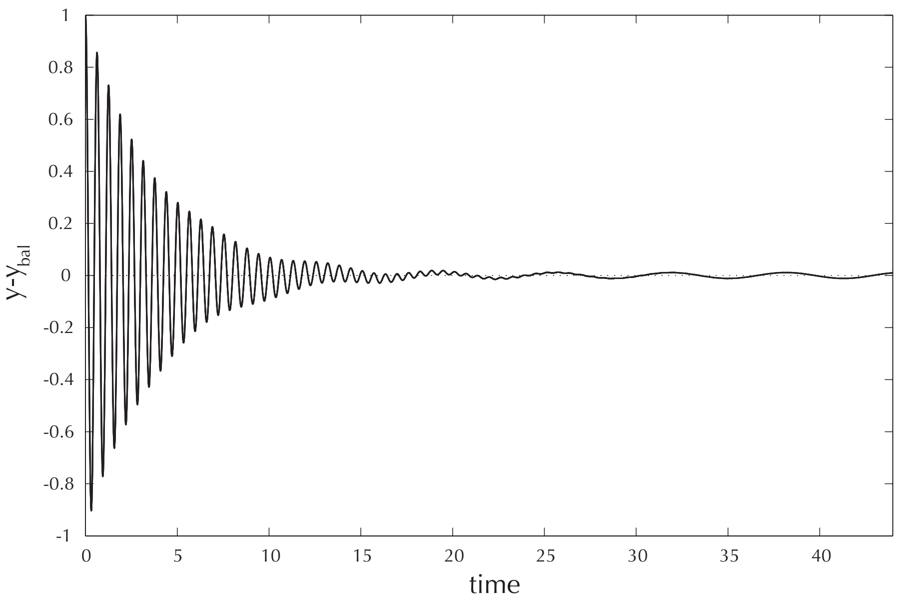
"super-implicit" scheme
non-standard multi grid
projection method

Numerics

$$\varepsilon\ddot{y} + \varepsilon\kappa\dot{y} + y = \cos(t)\,, \qquad \begin{cases} y(0) &= 1 + a \\ \dot{y}(0) &= 0 \end{cases}, \qquad (\varepsilon = 0.01)$$



$y(t)$



$y(t) - \cos(t)$

Numerics

$$\varepsilon\ddot{y} + \varepsilon\kappa\dot{y} + y = \cos(t)$$

Slow-time asymptotics for $\varepsilon \ll 1$:

$$y(t) = y^{(0)}(t) + \varepsilon y^{(1)}(t) + \ldots\,, \qquad \begin{aligned} y^{(0)}(t) &= \cos(t) \\ y^{(1)}(t) &= -(\ddot{y}^{(0)} + \kappa\dot{y}^{(0)})(t) \end{aligned}$$

Associated "super-implicit" discretization *(extreme BDF)*:

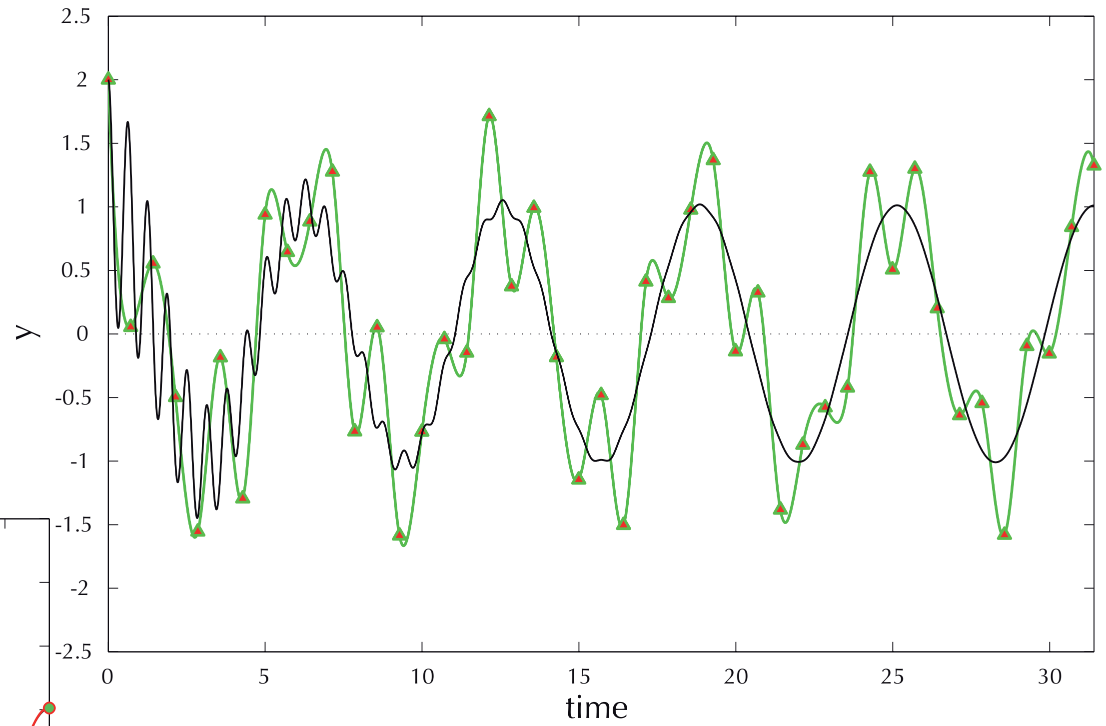
$$\begin{aligned} y^{n+1} &= \cos(t^{n+1}) - \varepsilon\left[(\delta_t + \kappa)\ \dot{y}\right]^{*,n+1} \\ \dot{y}^{n+1} &= \frac{1}{\Delta t}\left(y^{n+1} - y^n + \frac{1}{2}\left(y^{n+1} - 2y^n + y^{n-1}\right)\right) \end{aligned}$$

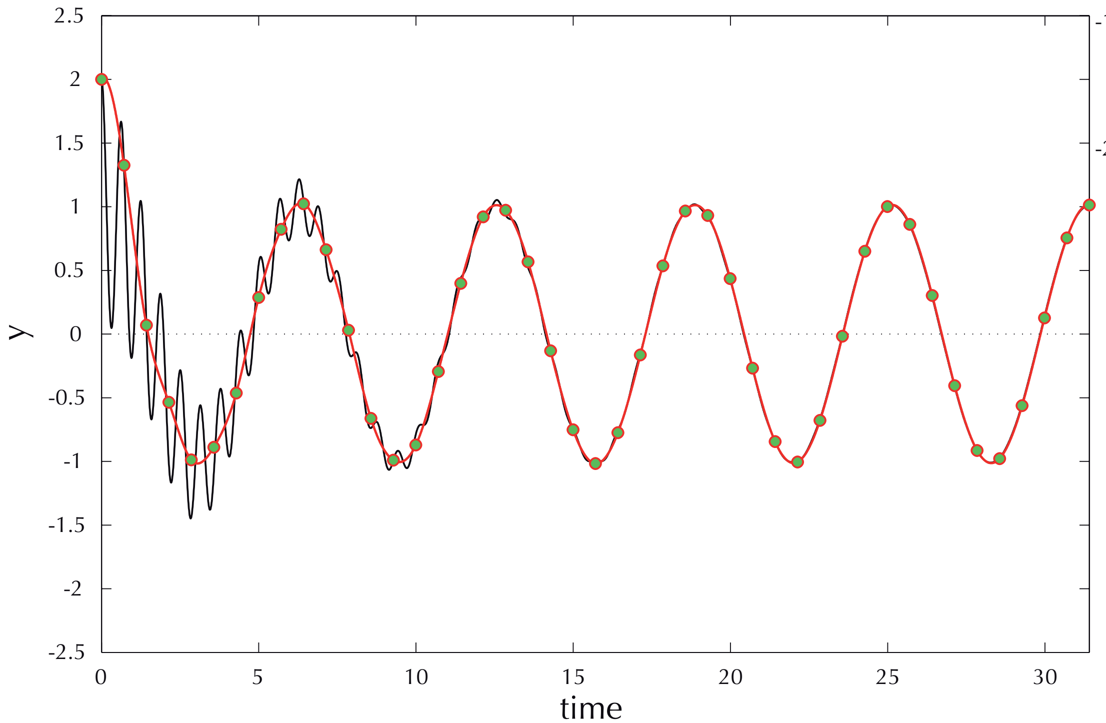
where

$$\begin{aligned} u^{*,n+1} &= 2u^n - u^{n-1} \\ (\delta_t u)^{*,n+1} &= \frac{1}{\Delta t}\left(u^n - u^{n-1} + \frac{3}{2}(u^n - 2u^{n-1} + u^{n-2})\right) \end{aligned}$$

Numerics

Implicit midpoint rule $\Delta t = 7\sqrt{\varepsilon}$

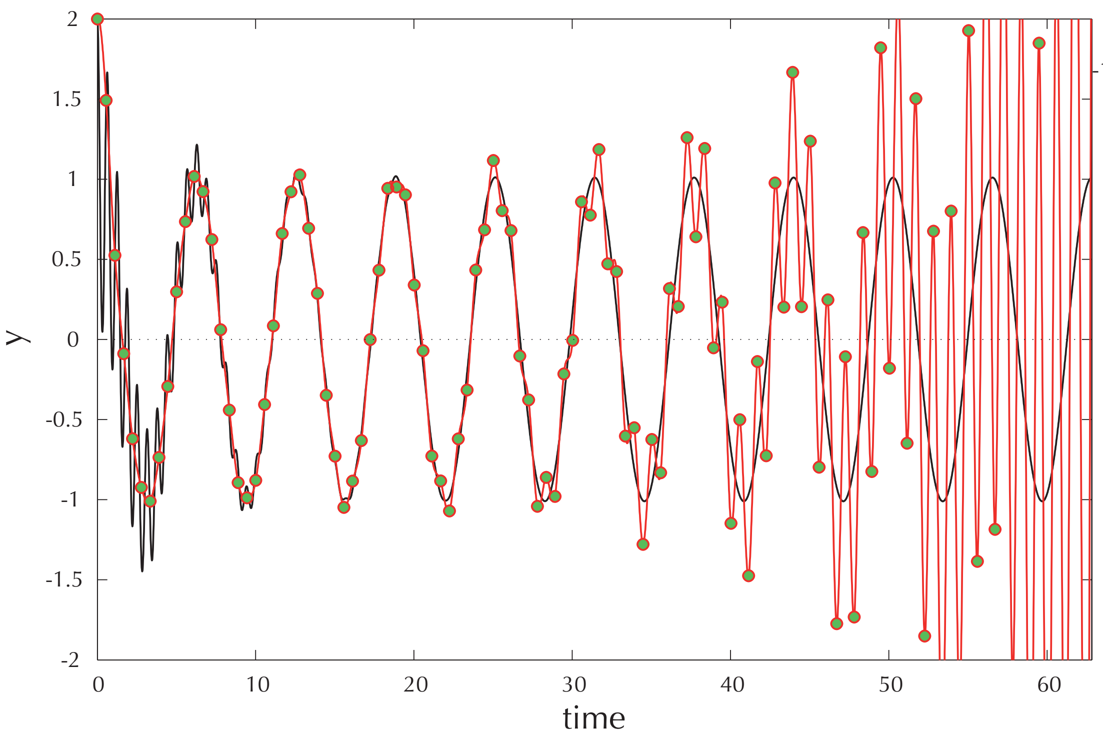


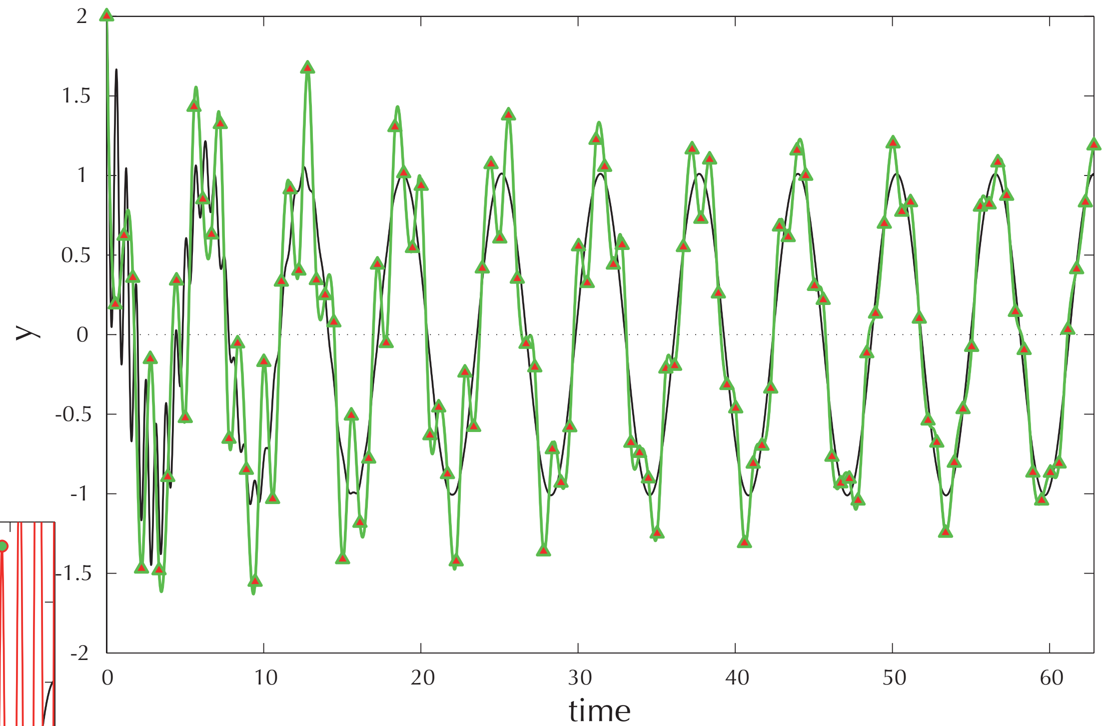


$\Delta t = 7\sqrt{\varepsilon}$ Super-implicit scheme

Numerics

Implicit midpoint rule $\Delta t = 5.55\sqrt{\varepsilon}$



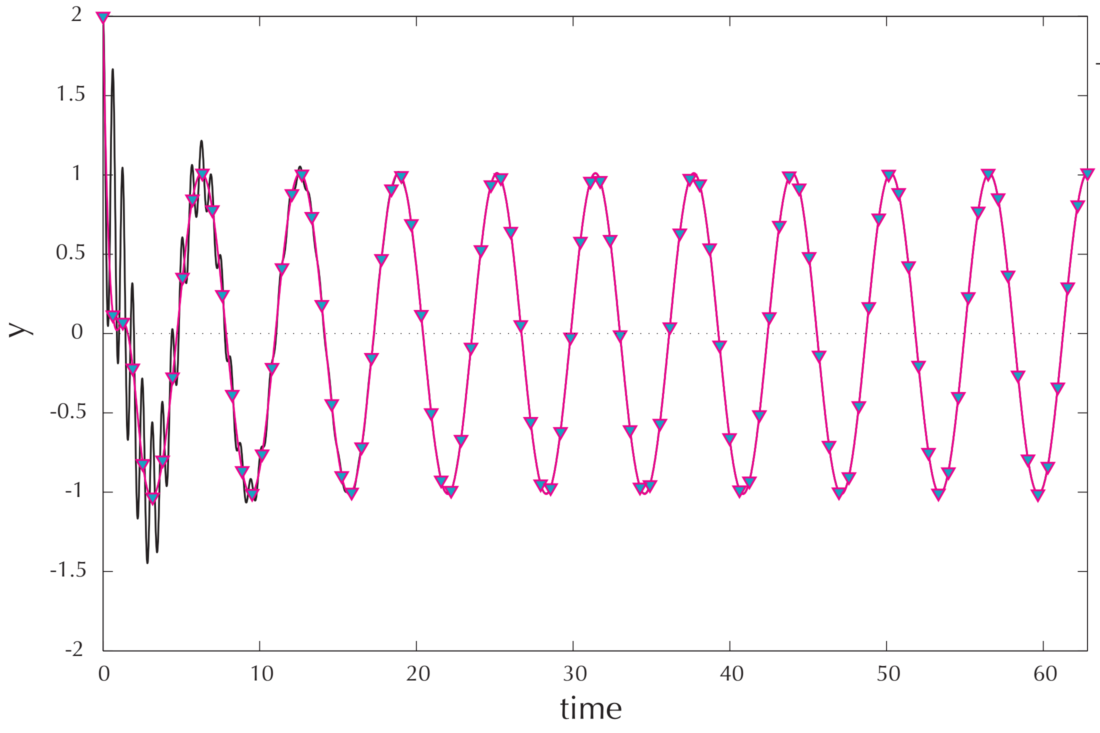


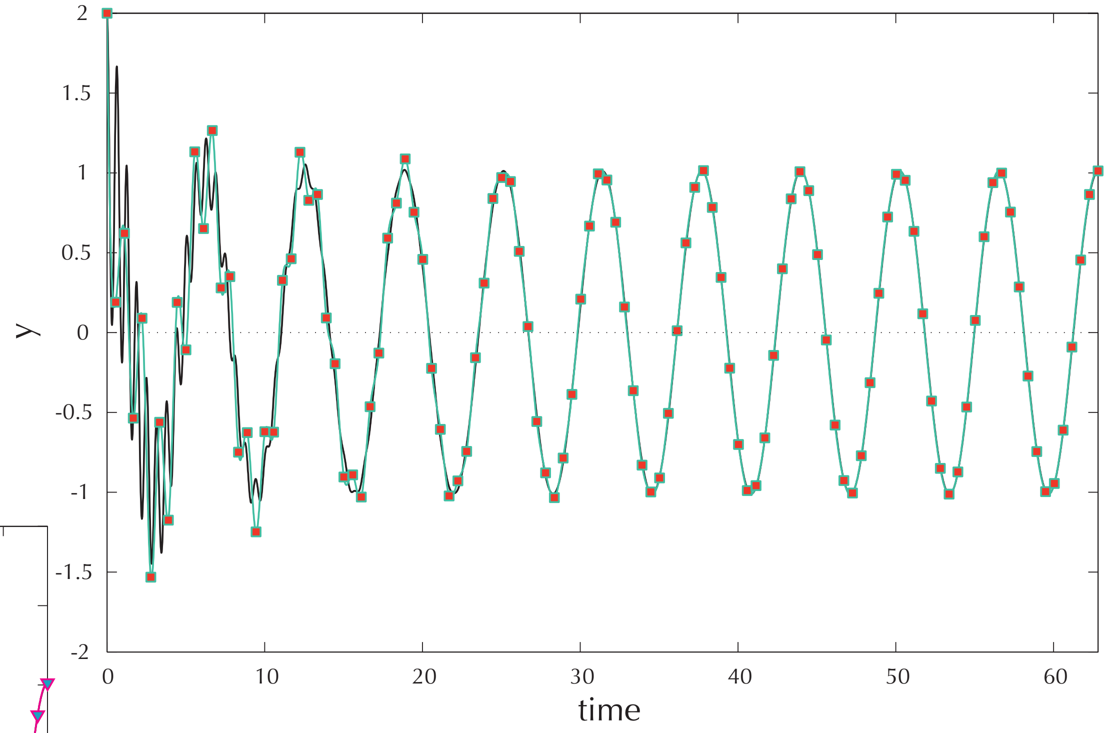
$\Delta t = 5.55\sqrt{\varepsilon}$ Super-implicit scheme

Numerics

Blended scheme $\Delta t = 5.55\sqrt{\varepsilon}$

$$\Delta y\big|_{\mathrm{BL}} = \eta\,\Delta y\big|_{\mathrm{IMP}} + (1-\eta)\,\Delta y\big|_{\mathrm{SU}}$$





$\Delta t = 5.55\sqrt{\varepsilon}$ BDF2 – for comparison

Numerics

Compressible flow equations:

$$\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \pi = -\rho g \boldsymbol{k}$$

$$\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$P = p^{\frac{1}{\gamma}} = \rho \theta \,, \qquad \pi = p / \Gamma P \,, \qquad \Gamma = c_p / R$$

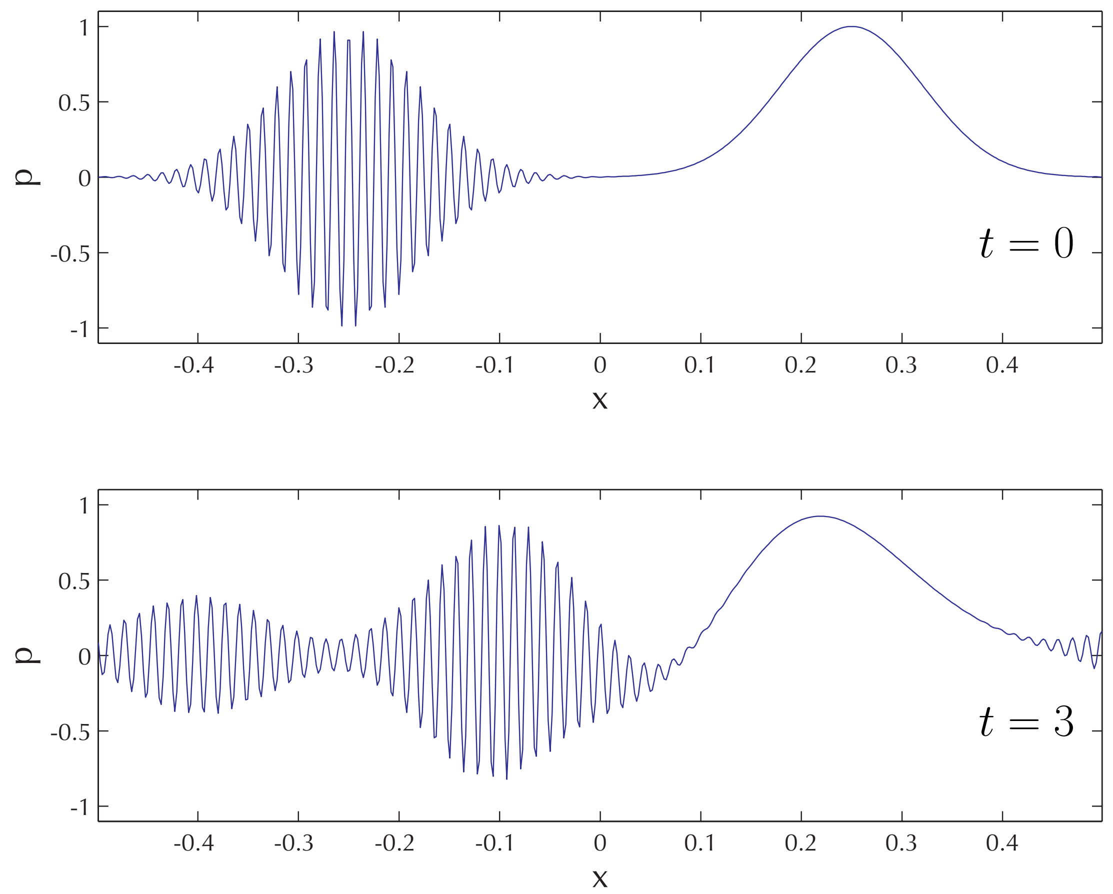
Numerics

1D Linear acoustics:

$$u_t + p_x = 0$$
$$p_t + c^2 u_x = 0$$

Desired:

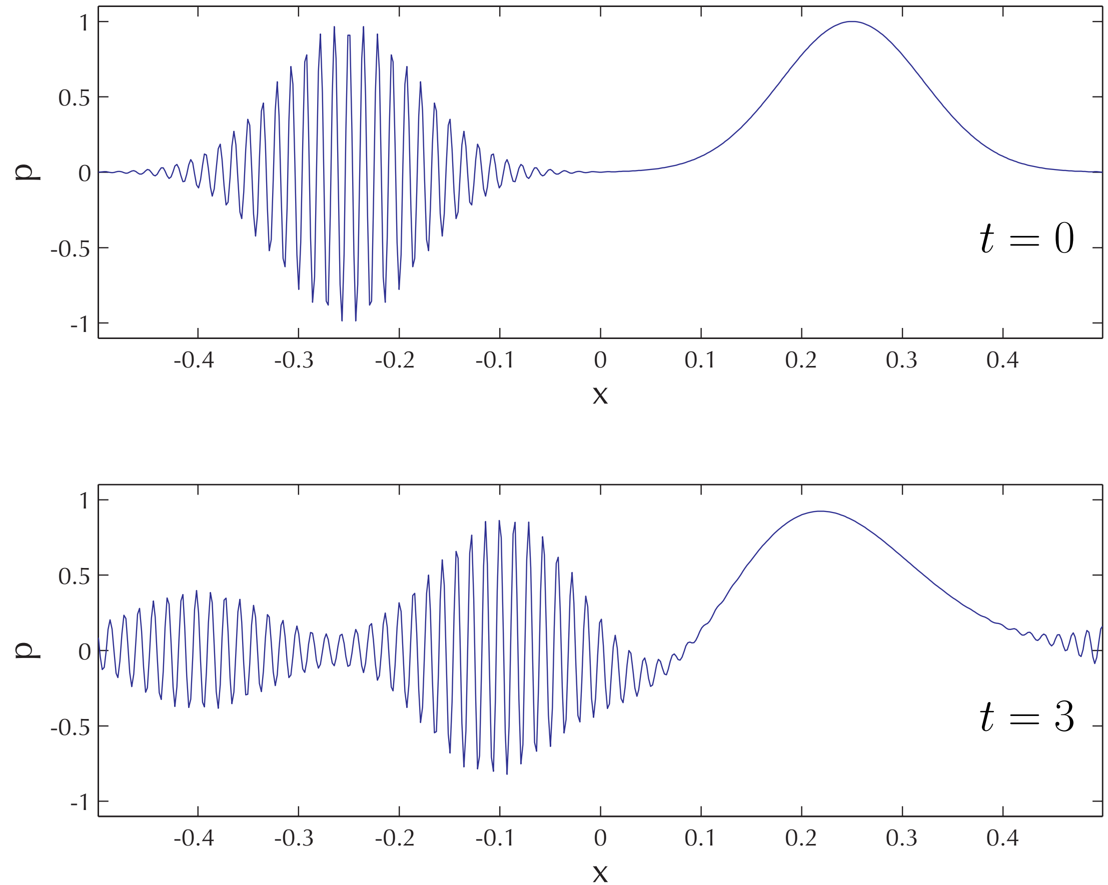
- remove underresolved modes
- minimize dispersion for marginally resolved modes



Numerics

1D Linear acoustics:

$$u_t + p_x = 0$$
$$p_t + c^2 u_x = 0$$



Desired:

- remove underresolved modes
- minimize dispersion for marginally resolved modes

Strategy:

scale-dependent IMP-SU-Blended scheme via **multi grid**

Numerics

Implicit mid-point rule for linear acoustics

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial}{\partial x} p^{n+\frac{1}{2}} = 0\,, \qquad \frac{p^{n+1} - p^n}{\Delta t} + c^2 \frac{\partial}{\partial x} u^{n+\frac{1}{2}} = 0$$

with

$$X^{n+\frac{1}{2}} = \frac{1}{2}\left(X^{n+1} + X^n\right)$$

Implicit problem for half-time fluxes

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x} p^{n+\frac{1}{2}}\,, \qquad p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2}\frac{\partial}{\partial x} u^{n+\frac{1}{2}}$$

Eliminate $u^{n+\frac{1}{2}}$

$$\left(1 - \frac{c^2 \Delta t^2}{4}\frac{\partial^2}{\partial x^2}\right) p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2}\frac{\partial}{\partial x} u^n$$

Numerics

Implicit mid-point rule for linear acoustics

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial}{\partial x} p^{n+\frac{1}{2}} = 0\,, \qquad \frac{p^{n+1} - p^n}{\Delta t} + c^2 \frac{\partial}{\partial x} u^{n+\frac{1}{2}} = 0$$

with

$$X^{n+\frac{1}{2}} = \frac{1}{2}\left(X^{n+1} + X^n\right) \qquad \text{or} \qquad \underline{X^{n+1} = 2X^{n+\frac{1}{2}} - X^n}$$

Implicit problem for half-time fluxes

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}\,, \qquad p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}}$$

Eliminate $u^{n+\frac{1}{2}}$

$$\left(1 - \frac{c^2 \Delta t^2}{4} \frac{\partial^2}{\partial x^2}\right) p^{n+\frac{1}{2}} = p^n - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^n$$

Numerics

Implicit mid-point rule $\Rightarrow$ **super-implicit**

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$\underline{p^{n+\frac{1}{2}}} = \underline{p^n} - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}}$$

step 1:

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$= \quad - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}} - \frac{\Delta t}{2}\left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}}$$

Pressure **"projection"** equation

$$\frac{c^2\Delta t}{2}\frac{\partial^2}{\partial x^2}p^{n+\frac{1}{2}} = c^2\frac{\partial}{\partial x}u^n + \left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}}$$

Numerics

Implicit mid-point rule $\Rightarrow$ **super-implicit**

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$\underline{p^{n+\frac{1}{2}}} = \underline{p^n} - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}}$$

step 1:

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$= - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}} - \frac{\Delta t}{2}\left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}}$$

step 2:

$$p^{n+1} = 2p^{n+\frac{1}{2}} - \underline{p^n}$$

Numerics

Implicit mid-point rule $\Rightarrow$ super-implicit

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$\underline{p^{n+\frac{1}{2}}} = \underline{p^n} - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}}$$

step 1:

$$\begin{aligned} u^{n+\frac{1}{2}} &= u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}} \\ &= - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}} - \frac{\Delta t}{2}\left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}} \end{aligned}$$

step 2:

$$p^{n+1} = 2p^{n+\frac{1}{2}} - p^n \quad \Rightarrow \quad p^{n+1} = 2p^{n+\frac{1}{2}} - \frac{1}{2}\left(p^{n+\frac{1}{2}} + p^{n-\frac{1}{2}}\right)$$

Numerics

Scale-dependence via multi-grid

$$p = \sum_{j=1}^{J} p^{(j)}$$

where

$$p^{(j)} = (1 - P \circ R)\ R^{j-1} p \qquad \text{with} \qquad \begin{array}{ll} R & : \text{ MG restriction} \\ P & : \text{ MG prolongation} \end{array}$$

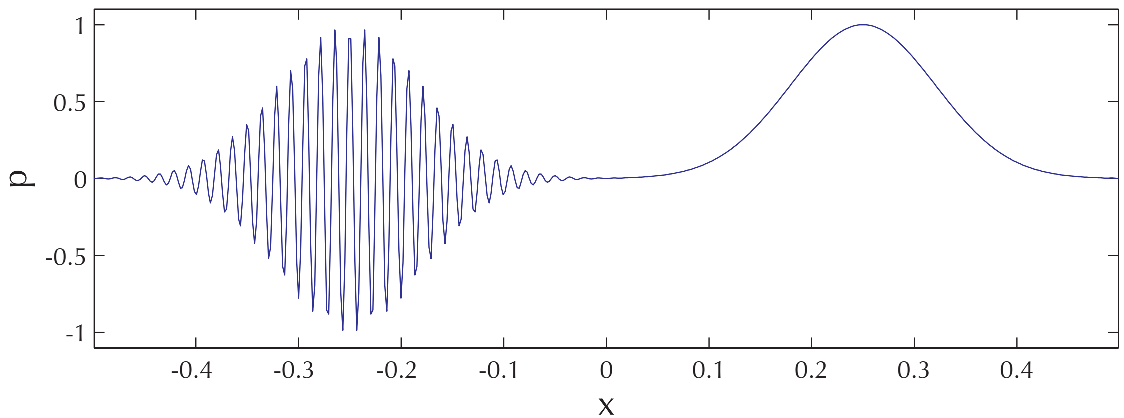
scale-dependent blending

$$u^{n+\frac{1}{2}} = u^{n} - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}$$

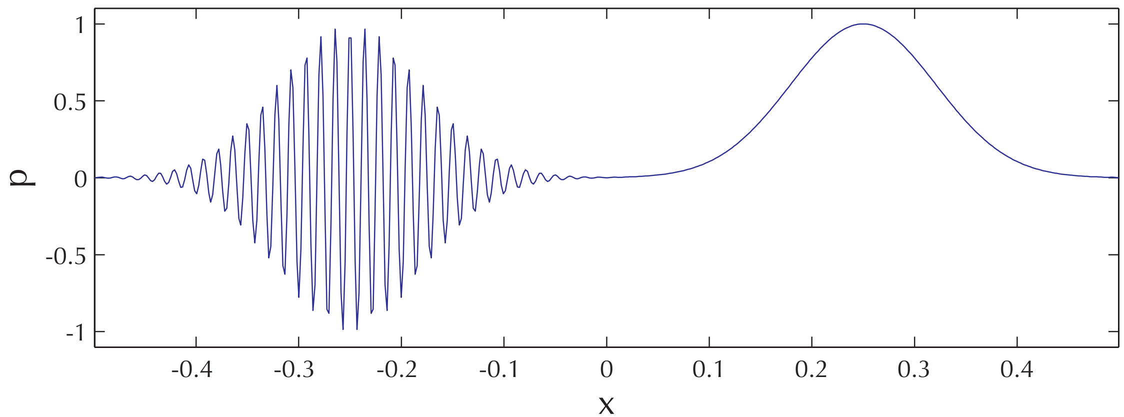
$$\sum_j \boldsymbol{\eta}^{(j)} p^{(j)^{n+\frac{1}{2}}} = \sum_j \boldsymbol{\eta}^{(j)} p^{(j)^{n}} - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}} - \sum_j (1 - \boldsymbol{\eta}^{(j)}) \frac{\Delta t}{2} \left(\frac{\partial p^{(j)}}{\partial t} \right)^{\mathbf{BD}, n+\frac{1}{2}}$$

Numerics

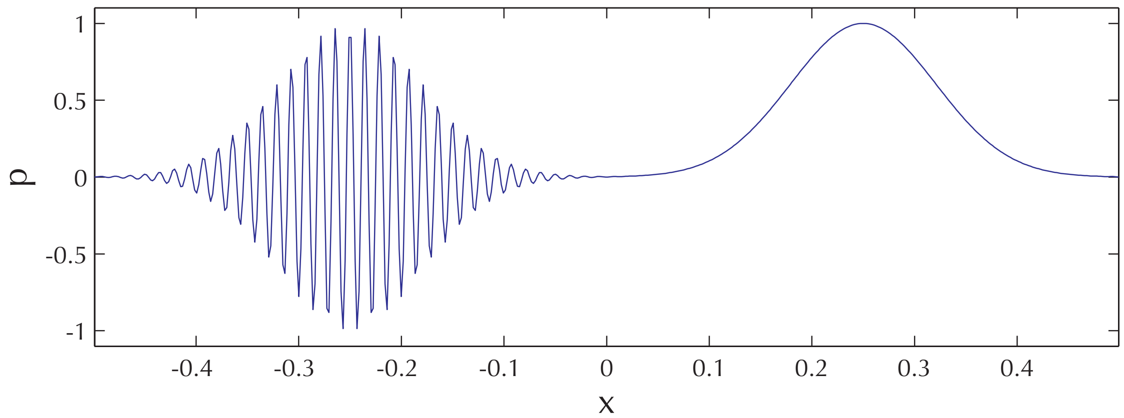
implicit midpoint



new scheme



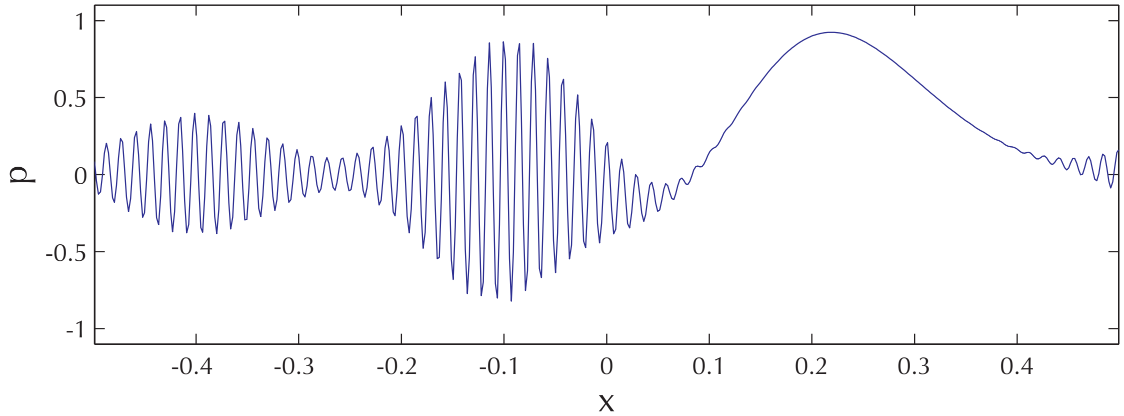
BDF2



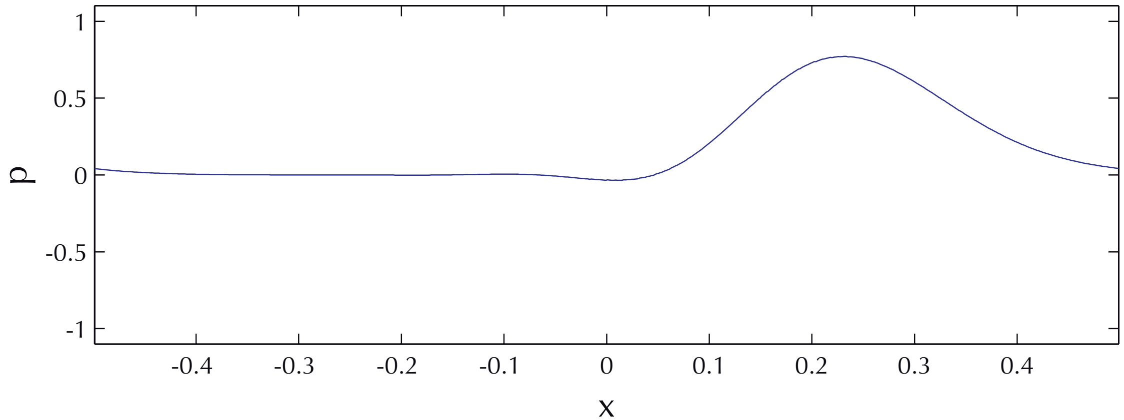
$t = 0$

Numerics

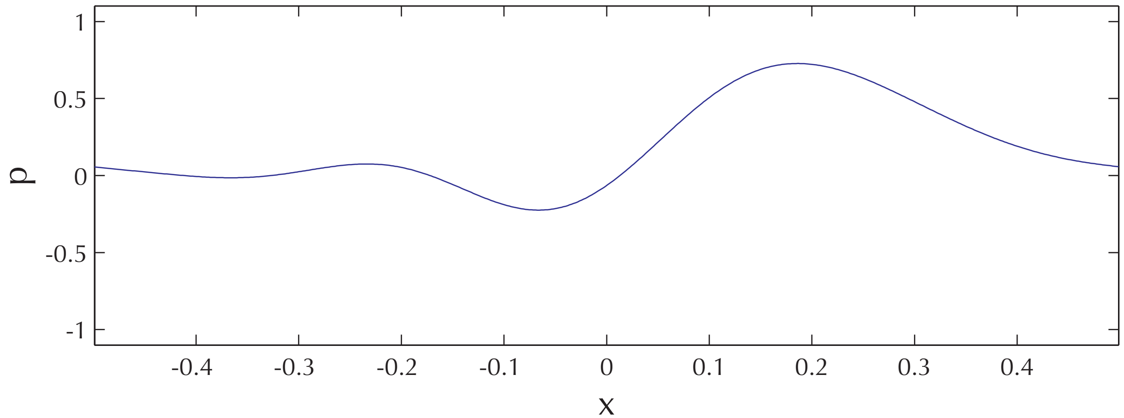
implicit midpoint



new scheme



BDF2



$t = 3$

Motivation

Asymptotics

Two-Scale Models

Numerics

Conclusions