Patterns Formed by Growing Sandpiles

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Work done with

S. Ostojic (2002)
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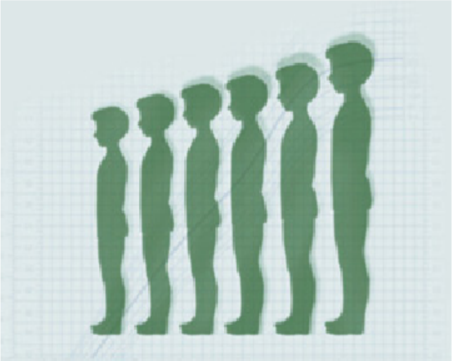
Outline

Proportionate Growth

- Animals grow in size, with different parts of body growing at roughly same rate.
- Proportionate growth requires regulation, and/or communication between different parts.
- Same food becomes different tissues in different parts of the body.
- Most existing models of growth in physics literature do not have this.
  DLA, Eden growth, KPZ growth, Invasion percolation ..
- Mechanism in our model not same as in biology

Proportionate Growth



Animals grow in size, with different parts of the body growing roughly at same rate.

Proportionate growth.

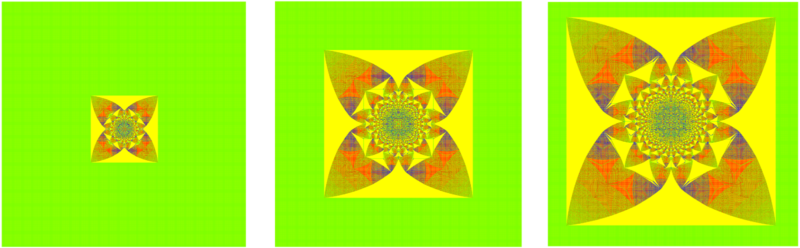
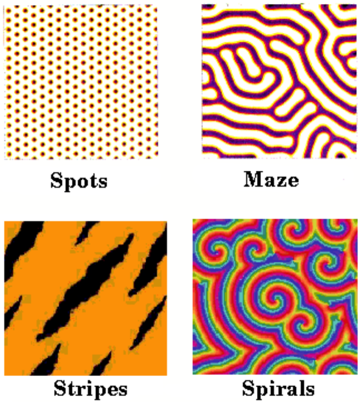


Figure: (a)$N = 4 \times 10^4$ (b)$N = 2 \times 10^5$ (c)$N = 4 \times 10^5$

Diameter $\sim \sqrt{N}$.

Pattern formation in growing sandpiles

Complex patterns in nature



Pattern formation

- Growing sandpiles give rise to beautiful complex patterns from simple local evolution rules
- Complete characterization of the asymptotic pattern in some cases
- Extra symmetry and robustness
- Effect of perturbations like boundaries

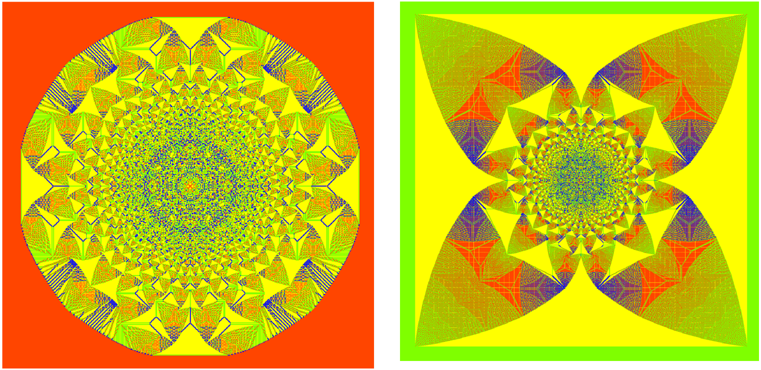


Figure: Patterns produced by adding 400000 particles at the origin, on a square lattice ASM, with initial state (a) all 0 (b) all 2.
Color code 0, 1, 2, 3 = R,B,G,Y

Discrete analytic functions and discrete approximants

Exact characterization of the pattern involves some interesting mathematics

Not fully understood.

- Solution of discrete Cauchy-Riemann equations on a graph.
- Eigenfunctions of the laplacian on discretized Riemann surface of many sheets
- A variational formulation using discrete piece-wise quadratic approximants?

Outline

Patterns formed by Growing Sandpiles

Definition of Abelian sandpiles

Non-negative intger height z_i at sites i of a lattice
Add rule: $z_i \rightarrow z_i + 1$
Relaxation rule : if $z_i > z_c$, topple, and move one grain to each neighbor.

Complex patterns in sandpile models.

Rule for forming patterns:

Add N particles at one site, and relax.

Deterministic patterns.

This is what we study here.

Outline

Characterizing the asymptotic pattern

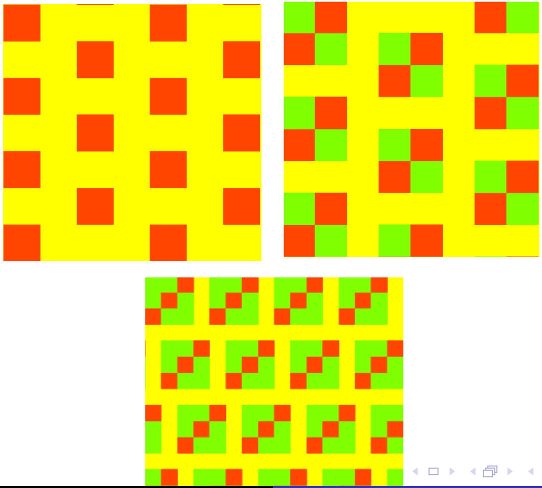
How do we characterize a complex pattern like this?

- A pixel by pixel description
- list of all patches and colors
- the rule for generating the pattern without doing topplings
- A Variational formulation?

The key observations S. Ostojic (2003).

- Diameter $\sim \sqrt{N}$
- Proportionate growth.
- Periodic height pattern in each patch. [ignoring Transients]
- Reduced coordinates $\xi = x/\sqrt{N}, \eta = y/\sqrt{N}$
  coarse-grained density $\rho(\xi, \eta)$ is constant within a patch.
- Define
  $\phi(\xi, \eta) = Lim_{N \to \infty}(1/N)[\ \#$ of topplings at $(\xi, \eta)]$
  Then,
  ϕ is a quadratic function of ξ, η in each patch.

Examples of periodic patterns in patches



Outline

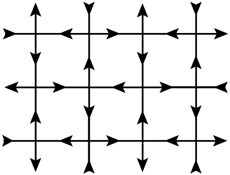
For the square lattice, the number of different patches is infinite, and not easy to characterize.

Other lattices, or backgrounds?

The F-Lattice.

Two arrows in and two out at each vertex.

Allowed stable ASM heights are 0 and 1.



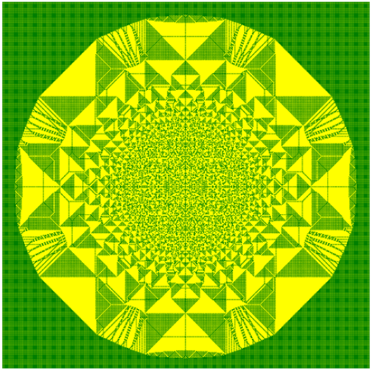


Figure: Pattern produced by adding 10^5 particles at the origin, on the F-lattice with initially empty lattice.

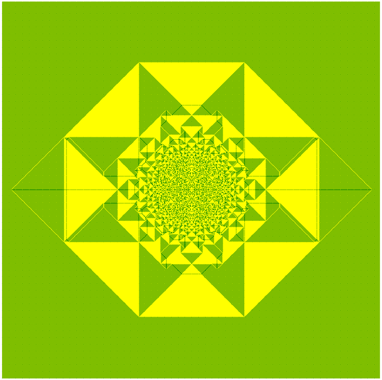


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard.

Characterizing the pattern on the F-lattice
Back-ground density 1/2

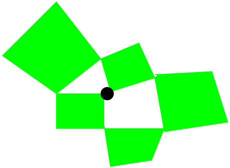
- Only two types of patches: densities 1/2 and 1.
- All boundaries are straight lines: slopes $0, \pm 1$, or ∞
- Each patch is 3- or 4- sided polygon

An electrostatic formulation

We have $\nabla\phi = +\overline{\delta z} - \delta(\xi,\eta)$

Positive point charge $+1$ at origin,
negative charge of areal density 1

Can we distribute the negative charge in such a way that the net potential is piecewise-quadratic, and exactly zero far away?



The answer, presumably unique, is the observed pattern.

The exact characterization involves four steps:

- Labelling patches using two integers (m, n). The adjacency graph is a discretized two-sheeted Riemann surface.
- Parameterize the potental in the (m, n) patch by

$$\phi_P(\xi, \eta) = \frac{1}{8}(m_P + 1)\xi^2 + \frac{1}{4} n_P \xi\eta + \frac{1}{8}(1 - m_P)\eta_P^2 + d_P \xi + e_P \eta + f_P$$

- Continuity of ϕ and derivatives implies that $d_{m,n}$ and $e_{m,n}$ both satisfy the equation

$$\psi_{m+1,n+1} + \psi_{m+1,n-1} + \psi_{m-1,n+1} + \psi_{m-1,n-1} - 4\psi_{m,n} = 0,$$

- Solve equations numerically on a large grid, to get the exact boundaries of patches

Most easily seen by $1/z^2$ transform of the picture.
Patches are assigned integer labels (m, n).
Graph is bipartite. Patch is dense, if $m + n =$ odd.

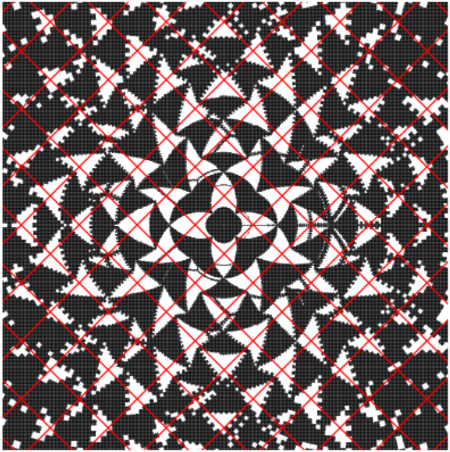
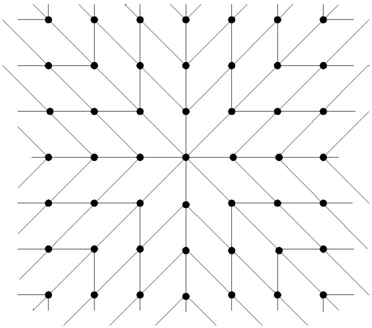


Figure: $z' = 1/z^2$ transform of original figure.



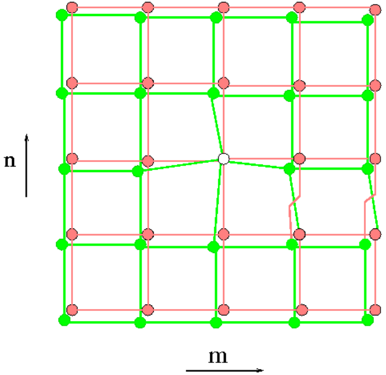


Figure: (a) Adjacency graph of the pattern. (b) representation as a square lattice wedge of wedge angle 4π.

Outline

The arguments only depend on the existence of only two types of patches, and stright line boundaries.
These can be found (by trial and error) in other cases also.
Then the asymptotic pattern is identical.
Some examples:
Other backgrounds on the F- lattice [e.g. alternate rows filled and empty]
F-lattice with density $5/8$.

Initially all sites (i, j) with $i + j = 0(mod 2)$
or $(i, j) = (0, 1)(mod 4)$ or $(i, j) = (2, 3)(mod 4)$ occupied.
F-lattice with density $1/2 +$ small noise in the initial conditions
Manhattan lattice with density $1/2$

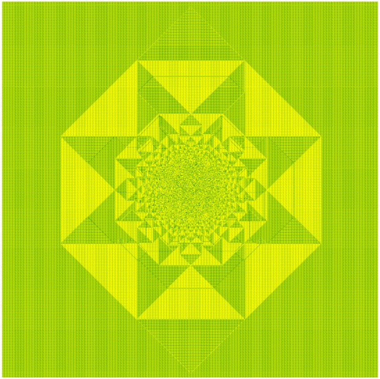


Figure: F-lattice with background density 5/8

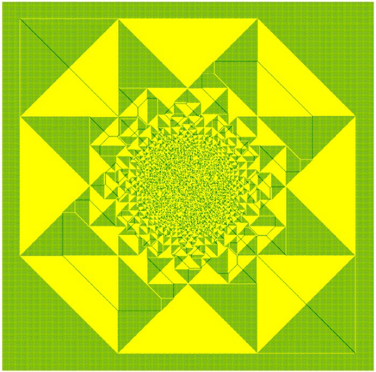


Figure: Manhattan lattice, with initial density $1/2$, and $120,000$ particles

Robustness.

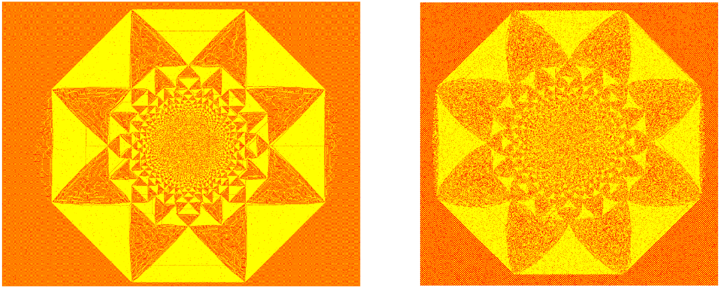


Figure: (a) 1% noise (b) 10%

Noise in the initial particle distribution.

Robustness.

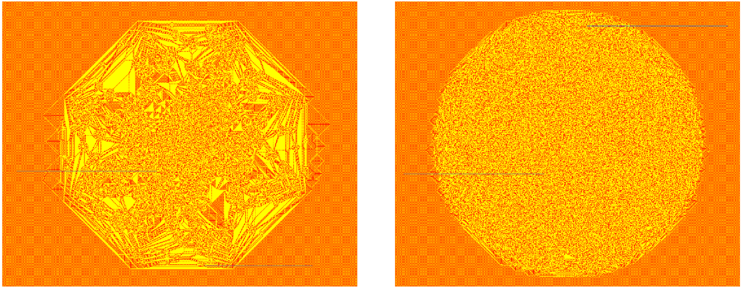


Figure: (a) 0.1% noise (b) 1%

Noise in the relaxation rule.

Outline

Near a line of sink sites, the law growth of diameter of pattern with N changes.

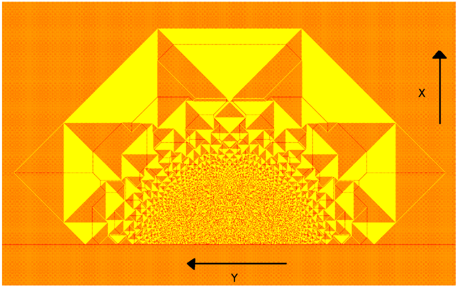


Figure: Growth on the F-lattice near a line of sink sites, with initial density $1/2$, and 14336000 particles

In the electrostaic analogy, the sink line is a grounded conducting wire.

Then, for a pattern of size Λ, number of particles present $\sim \Lambda^2$.
Number of particles absorbed$\sim \Lambda^2 \int d\xi \frac{\partial}{\partial \eta} \phi(\xi, 0)$

But $\phi \approx \phi_{dipole} \sim \frac{cos\theta}{r}$
then integrand$\sim 1/\xi^2$. Integral $\sim \frac{1}{\xi_{min}} \sim \Lambda$
Hence we expect $a\Lambda^3 + b\Lambda^2 = N$

Hence for large N,$\Lambda \sim N^{1/3}$
This equation also gives corrections to scaling .
In fact, the inhomogenous scaling equation holds to unexpected accuracy.

For $a = .1853$, and $b = .528$, the solution of this equation differs from the actual $\Lambda(N)$ by at most 1, for all $N, 100 < N < 3x10^6$.

For a wedge of angle θ, exponent is $\frac{1}{2+\pi/\theta}$.
Also generalizable to higher dimensions. Similar accuracy.

For exact characterization, the adjacency graph now has a three-sheeted Riemann surface structure.

In general when the background density is low enough, one gets compact clusters.

$$\Lambda \sim N^{1/d} \tag{1}$$

If background density is too high, one gets infinite avalanches.
We have been able to find a interesting case at the threshold of instability, where the cluster remains bounded for any finite N, but diameter $\Lambda \sim N$.
Directed triangular lattice. Background density $4/3$.
Interestingly, in this case, the adjacency graph is a hexagonal lattice.

Exactly characterized.

Directed triangular lattice.

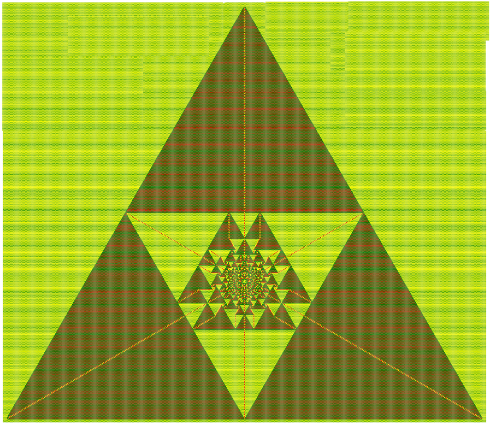


Figure: $N = 1500$. Color code: red=0, green=1, blue=2

Outline

Discrete Quadratic Approximants

Example of discrete approximants:

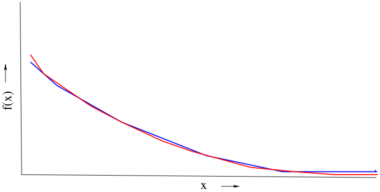


Figure: Approximate $f(x)$ by piece-wise linear functions with integer slopes

A variational formulation?

Start with a trial pattern. Determine the corresponding $\phi(\xi, \eta)$

We can determine the "best" piece-wise quadratic approximants to $\phi(\xi, \eta)$ only using a given set of quadratic functions ϕ_P.

$$\phi_P(\xi, \eta) = \frac{1}{8}(m_P + 1)\xi^2 + \frac{1}{4} n_P \xi\eta + \frac{1}{8}(1 - m_P)\eta_P^2 + d_P\xi + e_P\eta + f_P$$

The correspond charge density is piece-wise constant. Remove singularities at boundaries.

Determine corresponding potential $\phi^{(1)}(\xi, \eta)$.

Determine corresponding best quadratic approximant.

Iterate.

If the process converges, we get a piecewise-quadratic potential function, with continuous derivatives.
The background and lattice determine the allowed tiling functions.

Outline

- We can fully characterize quantitatively patterns when only two types of patches allowed.
- The pattern has additional (8-fold rotational) symmetry, and robustness to small noise in initial background.
- Pattern in the presence of a lines of sinks can be quanitatively determined
- Extension to more general patterns?
- Criteria for what periodic patterns are allowed ?
- Theory of approximating functions with piecewise parabolic approximants?

Thank You.

References

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