From microscopic dynamics to macroscopic behavior in systems with two symmetric absorbing states

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Outline

- Spin systems with two symmetric absorbing states/examples.
- Langevin approach for macroscopic behavior on square lattices.
- Application/discussion on non-linear models.
- Non-equilibrium phase transitions/universality classes.
- Summary.

Spin systems with two symmetric absorbing states

- Systems with two symmetric (equivalent) states represented by S_i= -1,1 are called Z_2-symmetric.

- Absorbing state (AS): any state in a statistical system that has no microscopic fluctuations *[Hinrichsen 2000, Ódor 2003]*.
- Consequence: once the AS is reached, the system cannot scape from it (non-equilibrium).

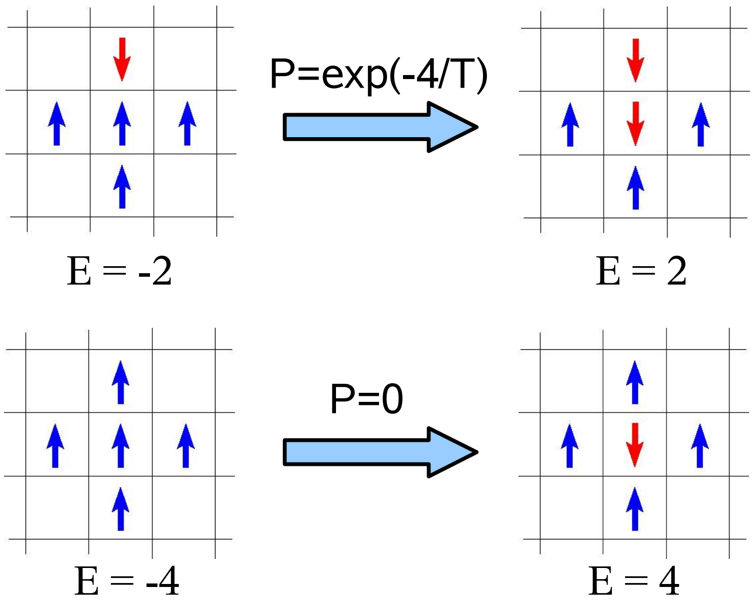
- Z_2 AS: Fully ordered states S_i= -1, 1 (i=1..N) are symmetric and absorbing.

[Dickman 1995, Dornic 2001, Droz 2003, Muñoz 2005]

Examples of Z_2 AS

- Absorbing Ising model *[Droz 2003]*:

$$P(\uparrow \rightarrow \downarrow) = \begin{cases} 1 & \text{if } \Delta E \leq 0 \\ \exp(-\Delta E/T) & \text{if } 0 < \Delta E < 8 \\ 0 & \text{if } \Delta E = 8 \end{cases}$$

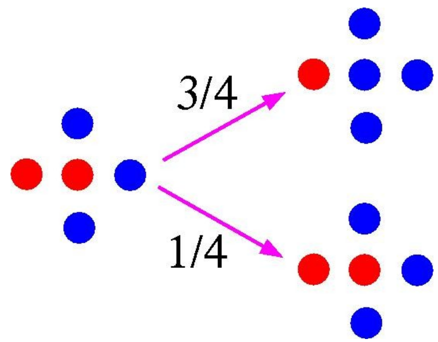


- The voter model *[Clifford 1973, Ligget 1975]*:

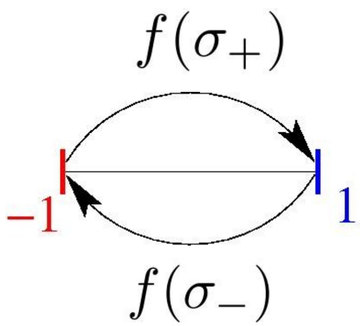
  Simplest IPS: Two possible positions $\sigma \in \{-1, 1\}$ on a political issue.

  Individuals ("voters") blindly adopt the position of a random neighbor.

$$P(\sigma_i \to -\sigma_i) = \frac{1}{2}\left(1 - \frac{\sigma_i}{z} \sum_{j \text{ NN } i} \sigma_j\right)$$



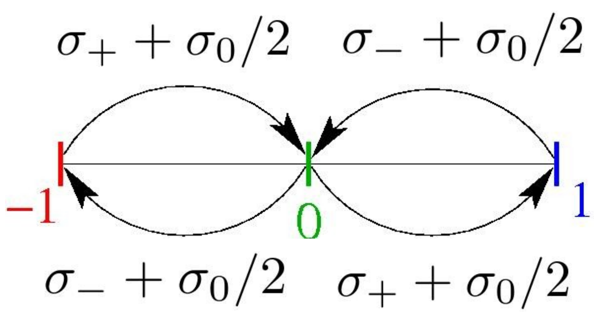
- Non-linear Voter Models for species competition *[Schweitzer 2008]*:



σ_- = density of species 1.

σ_+ = density of species 2.

- Models with intermediate states *[Castelló 2006, Baronchelli 2006, Dall'Asta 2008]:*



-1: speaking A

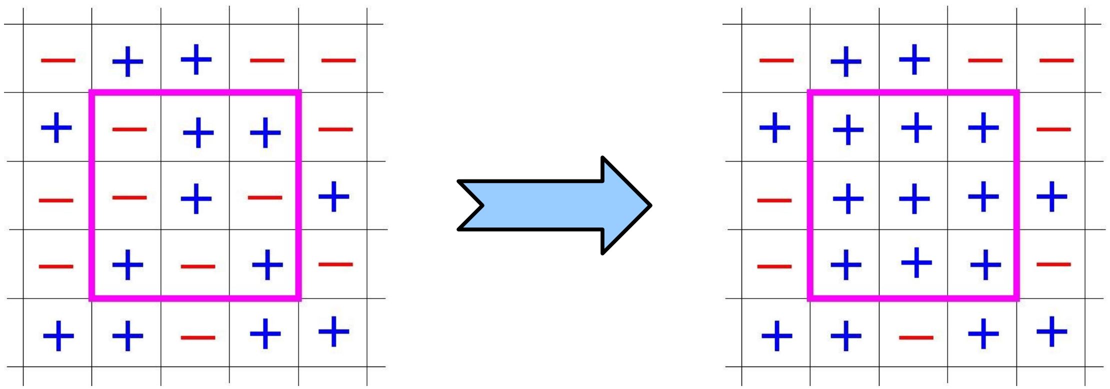
1: speaking B

0: speaking A and B

-1 and 1 are symmetric.

- Memory / inertia dynamics in 2-state VM *[Dall'Asta 2007, Stark 2008]:*
  - Spin flips after interacting n>1 times with opposite state (memory).
  - Flipping probability decreases with number of interactions (inertia).

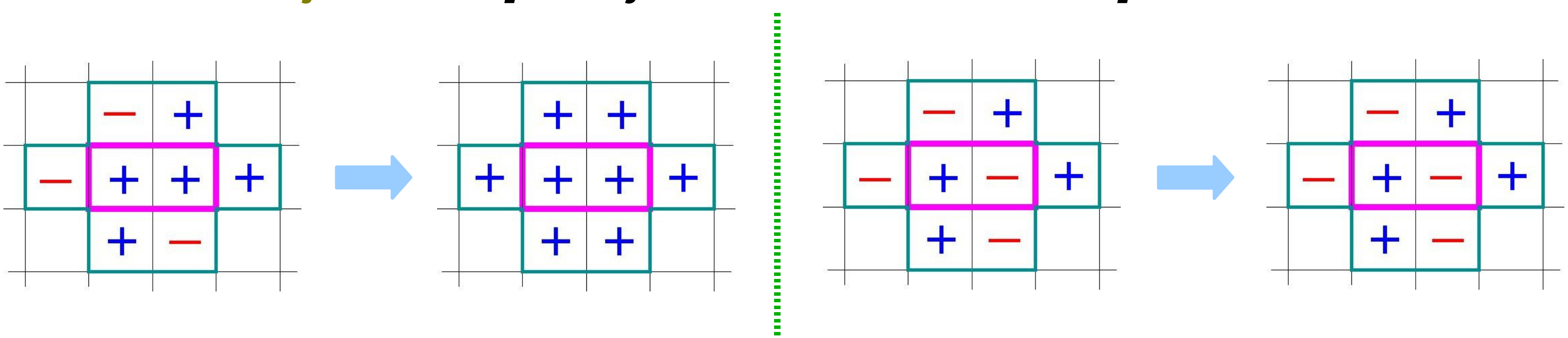
- Majority rule *[Galam 1999, Redner 2003]:*



+ = yes

– = no

- 2-d Sznajd model *[Sznajd 2000, Stauffer 2000]:*



- Universality:
  - Many models with different dynamical rules but the same macroscopic behavior (coarsening, critical exponents).

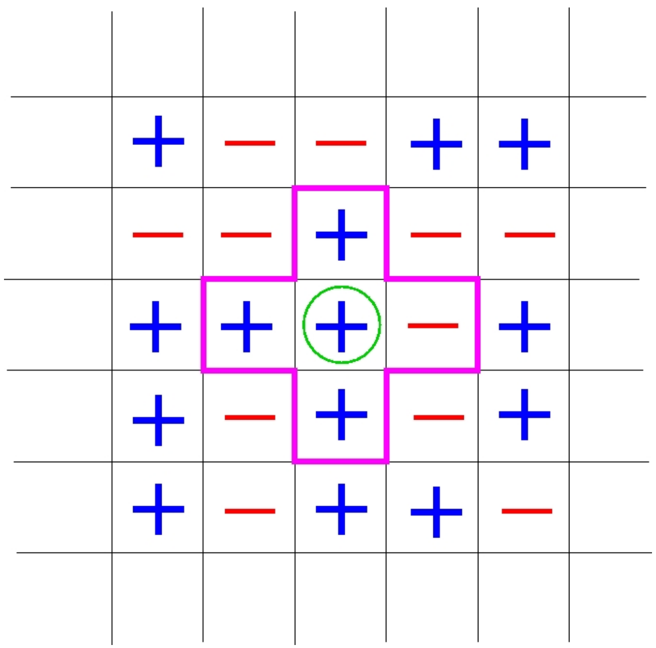
  - Three types of phase transitions.

- Question:

  Can we classify models by their microscopic dynamics?

Z_2 AS systems: a general approach

Generic lattice model:



- S_r = -1, 1 (spin at site r).

  r =(r_1, r_2, ..r_d), d = space dimension.

- $\psi_{\mathbf{r}} \equiv \frac{1}{z} \sum_{\mathbf{r}'/\mathbf{r}} S_{\mathbf{r}'}$ (local magnetization)

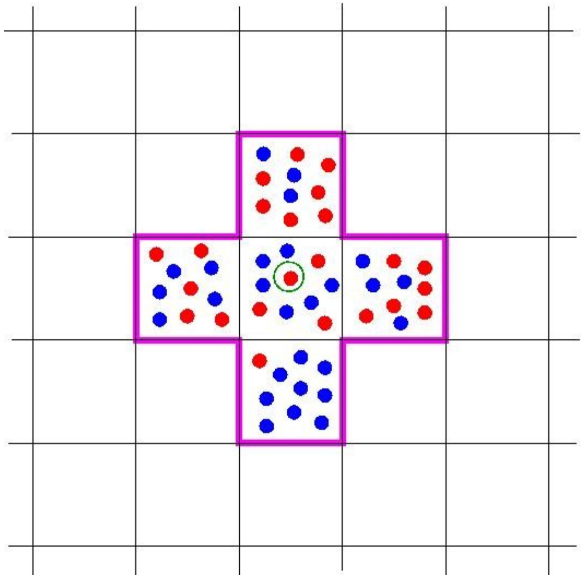
- $f(-S_r \psi_r)$ (spin-flip probability)

- $f(-1) = 0$ (absorbing condition)

Minimal conditions on f(ψ) to predict macro behavior?

Field approach [Dall'Asta and Galla]:

$\Phi_{\mathbf{r}}(t)$ = *magnetization field* at site **r** at time t (continuous spin)



$$\phi_{\mathbf{r}}(t) \rightarrow \frac{1}{\Omega}\sum_{j=1}^{\Omega} S_{\mathbf{r}}^{j} \qquad \psi_{\mathbf{r}} \rightarrow \frac{1}{z}\sum_{\mathbf{r}'/\mathbf{r}} \phi_{\mathbf{r}'}(t)$$

- Choose a site **x** at random.
- Choose one particle from **x** at random.
- Flip its spin S_x with probability $f(-S_x \psi_x)$.
- Repeat.

Transition rates:

$$W\left(\{\phi\} \to \ \{\phi\} \pm \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}}\right) \equiv W^{\pm}(\phi, \mathbf{x}, t) = \frac{1}{2}\left(1 \mp \phi_{\mathbf{x}}\right) f(\pm\psi_{\mathbf{x}})$$

Master equation for the probability distribution

$$\begin{aligned}
\frac{\partial \mathcal{P}(\{\phi\}, t)}{\partial t} \quad = \quad & \sum_{\mathbf{x}} \Bigg[W\left(\{\phi\} + \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}} \to \{\phi\}\right) \mathcal{P}\left(\{\phi\} + \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}}, t\right) \\
+ \quad & W\left(\{\phi\} - \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}} \to \{\phi\}\right) \mathcal{P}\left(\{\phi\} - \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}}, t\right) \\
- \quad & W\left(\{\phi\} \to \{\phi\} - \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}}\right) \mathcal{P}\left(\{\phi\}, t\right) \\
- \quad & W\left(\{\phi\} \to \{\phi\} + \frac{2}{\Omega}\delta_{\mathbf{x},\mathbf{r}}\right) \mathcal{P}\left(\{\phi\}, t\right) \Bigg]
\end{aligned}$$

Fokker-Planck equation

$$\frac{\partial}{\partial t}\mathcal{P}(\{\phi\},t) = \sum_{\mathbf{r}} -\frac{1}{\Omega}\frac{\partial}{\partial\phi}\Big\{2\left[W^{+}(\phi,\mathbf{r},t) - W^{-}(\phi,\mathbf{r},t)\right]\mathcal{P}(\{\phi\},t)\Big\}$$
$$+ \frac{1}{\Omega^2}\frac{\partial^2}{\partial\phi^2}\Big\{2\left[W^{+}(\phi,\mathbf{r},t) + W^{-}(\phi,\mathbf{r},t)\right]\mathcal{P}(\{\phi\},t)\Big\}$$

Langevin equation

$$\boxed{\frac{\partial\phi_{\mathbf{r}}(t)}{\partial t} = \left[1-\phi_{\mathbf{r}}(t)\right] f(\psi_{\mathbf{r}}) - \left[1+\phi_{\mathbf{r}}(t)\right] f(-\psi_{\mathbf{r}}) + \eta_{\mathbf{r}}(t)}$$

Noise:

$$\boxed{\langle\eta_{\mathbf{r}}(t)\eta_{\mathbf{r}'}(t')\rangle = \Big\{\left[1-\phi_{\mathbf{r}}(t)\right] f(\psi_{\mathbf{r}}) + \left[1+\phi_{\mathbf{r}}(t)\right] f(-\psi_{\mathbf{r}})\Big\}\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t')/\Omega^{1/2}}$$

Approximate equations

Expansion around $\psi_r = 0$, up to 4^{th} order.

$$f(\psi_{\mathbf{r}}) = \frac{1}{2}(1+\psi_{\mathbf{r}})\left(c + a\psi_{\mathbf{r}} + d\psi_{\mathbf{r}}^2 - b\psi_{\mathbf{r}}^3\right)$$

$$c \equiv 2f(0), \quad a \equiv 2f'(0) - c, \quad d \equiv f''(0) - a, \quad b \equiv -\frac{f'''(0)}{3} + d$$

$$\Delta\phi_{\mathbf{r}} \equiv \frac{1}{z}\sum_{\mathbf{r}'/\mathbf{r}}(\phi_{\mathbf{r}'} - \phi_{\mathbf{r}}) = \psi_{\mathbf{r}} - \phi_{\mathbf{r}}$$

(Laplacian operator)

Neglecting $(\Delta\Phi)^2$ terms:

Langevin equation for Φ

$$\boxed{\frac{\partial\phi_{\mathbf{r}}}{\partial t} = (1-\phi_{\mathbf{r}}^2)(a\phi_{\mathbf{r}} - b\phi_{\mathbf{r}}^3) + \left[a + c + (d - 2a - 3b)\phi_{\mathbf{r}}^2\right]\Delta\phi_{\mathbf{r}} + \eta_{\mathbf{r}}}$$

Noise:

$$\boxed{\langle \eta_{\mathbf{r}}(t)\eta_{\mathbf{r}'}(t')\rangle = \left\{(1-\phi_{\mathbf{r}}^2)(c + d\phi_{\mathbf{r}}^2) + (a - c + 2d)\phi_{\mathbf{r}}\Delta\phi_{\mathbf{r}}\right\}\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t')}$$

Phenomenological Langevin equation for magnetization field Φ

To get Generalized Voter, Ising and Directed Percolation transitions:

- Symmetric under *Φ* → *-Φ* reversal.
- Absorbing states for *Φ=-1* and *Φ=1*.
- Two odd terms (Ising-like symmetry breaking).

$$\frac{\partial \phi}{\partial t} = (1-\phi^2)(a\phi - b\phi^3) + D\nabla^2\phi + \sigma\sqrt{1-\phi^2}\,\eta$$

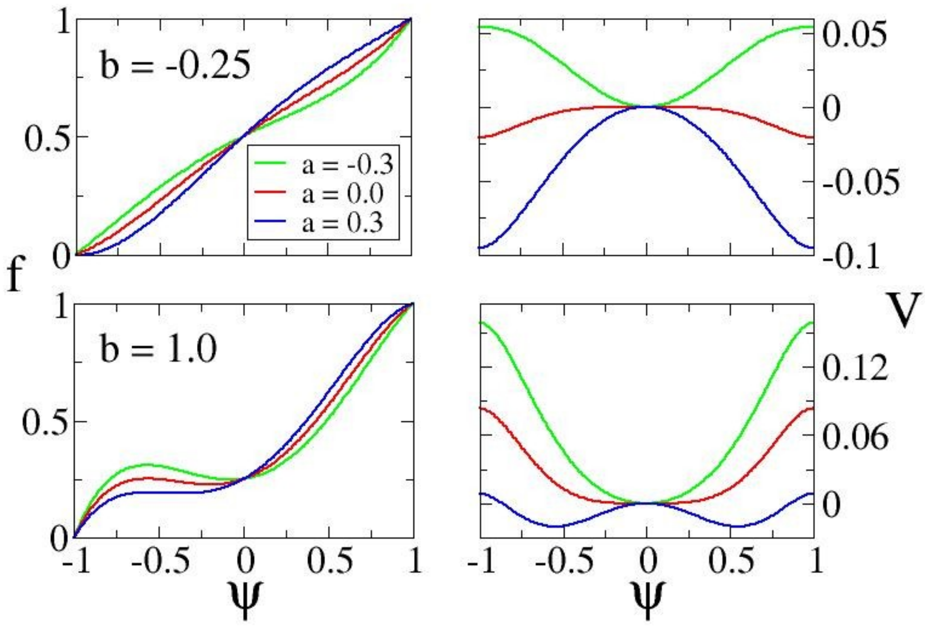
[Muñoz 2005]

Phase ordering

$$\frac{\partial \phi}{\partial t} = D\Delta\phi - \frac{\partial V}{\partial \phi}$$

Time-dependent Ginzburg-Landau equation with potential

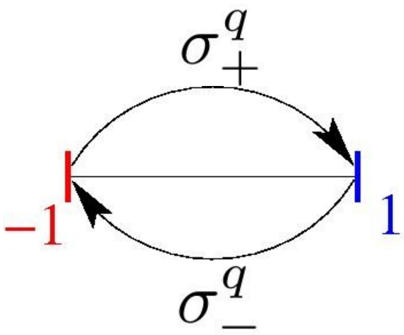
$$V(\phi) = -\frac{a}{2}\phi^2 + \frac{a+b}{4}\phi^4 - \frac{b}{6}\phi^6$$



f '(0) < f(0) : disordered active state

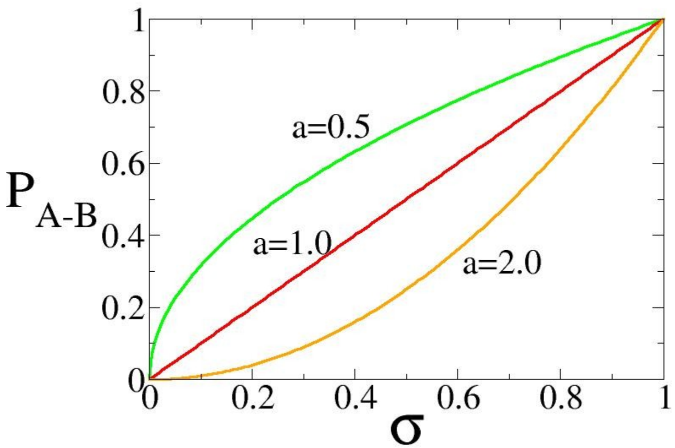
f '(0) > f(0) : coarsening by surface tension

Abrams-Strogatz model for language evolution: Non-linear flipping probability



-1 = Language A.

1 = Language B.



$$P(\mp \to \pm) = \sigma_\pm^q = \left(\frac{1 \pm \psi_{\mathbf{r}}}{2}\right)^q, \qquad q \geq 0$$

$$f(\psi) = \left(\frac{1+\psi}{2}\right)^q$$ (transition probability)

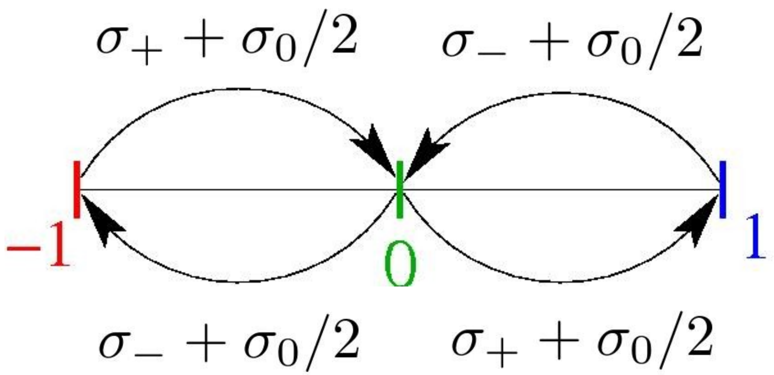
$$\boxed{\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{(q-1)}{3\times 2^q}(1-\phi^2)\left[6\phi + (q-2)(q-3)\phi^3\right] \\ &+ \frac{q}{2^q}\left[2+(q-1)(q-4)\phi^2\right]\Delta\phi + \eta \end{aligned}}$$

$$\begin{aligned} \langle \eta_{\mathbf{r}}(t)\eta_{\mathbf{r}'}(t')\rangle &= \left\{ \frac{1}{2^q}(1-\phi^2)\left[2+(q-1)(q-2)\phi^2\right] \right. \\ &+ \left. \frac{(q-2)}{3\times 2^q}\left[6q\phi + (q-1)(q-12)\right]\Delta\phi \right\}\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t') \end{aligned}$$

q=1 case (voter model) → $$\frac{\partial \phi}{\partial t} = \Delta\phi + \sqrt{1-\phi^2-\phi\Delta\phi}\,\eta$$ (Dickman '95)

q=2 case (3-state model) → $$\frac{\partial \phi}{\partial t} = \frac{1}{2}\left(\phi-\phi^3\right) + (1-\phi^2)\Delta\phi + \sqrt{1-\phi^2}\,\eta$$

(Dall'Asta and Galla '08)



3-state models:

(Minet-Wang '05, Castelló '06, Baronchelli '06.)

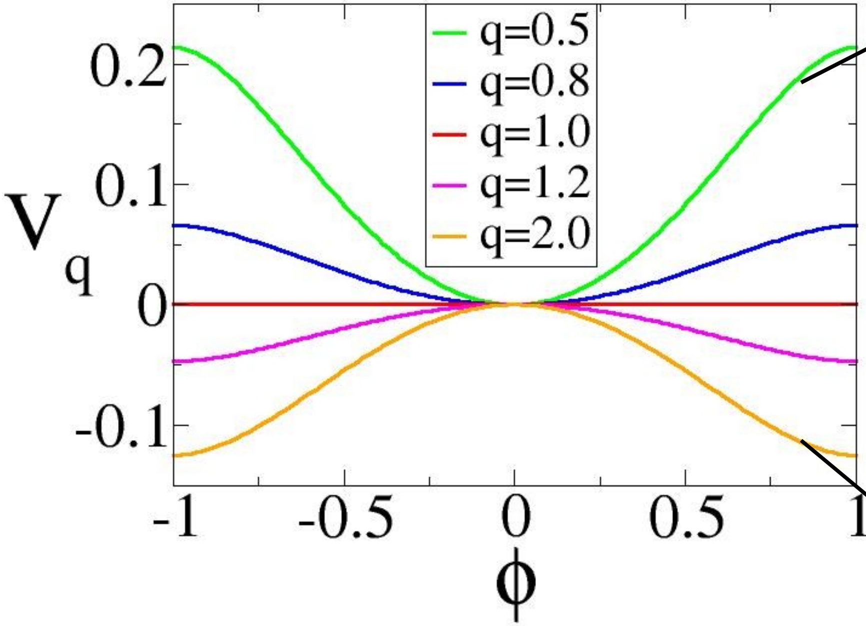
n-state models: Dall'Asta '08.

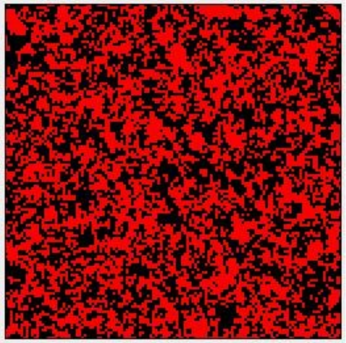
$$P_{eff}(\mp \to \pm) \simeq \left(\sigma_\pm + \frac{1}{2}\sigma_0\right)^2$$

Observations:

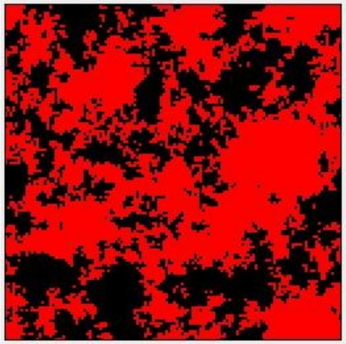
- System orders driven by curvature.
- Density of interfaces $\rho \sim t^{-0.45}$.

$$V_q(\phi) = -\frac{(q-1)}{3\times 2^q}\left\{3\phi^2 + [(q-2)(q-3)-6]\,\frac{\phi^4}{4} - (q-2)(q-3)\frac{\phi^6}{6}\right\}$$

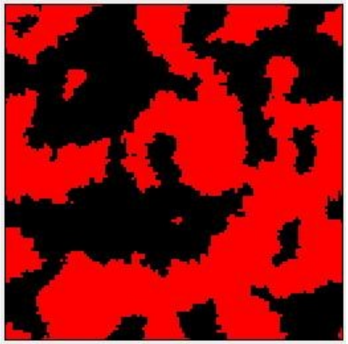




q = 0.5
Disordered active state.



q = 1.0
Ordering without surface tension.



q = 2.0
Ordering by surface tension.

■ down spins

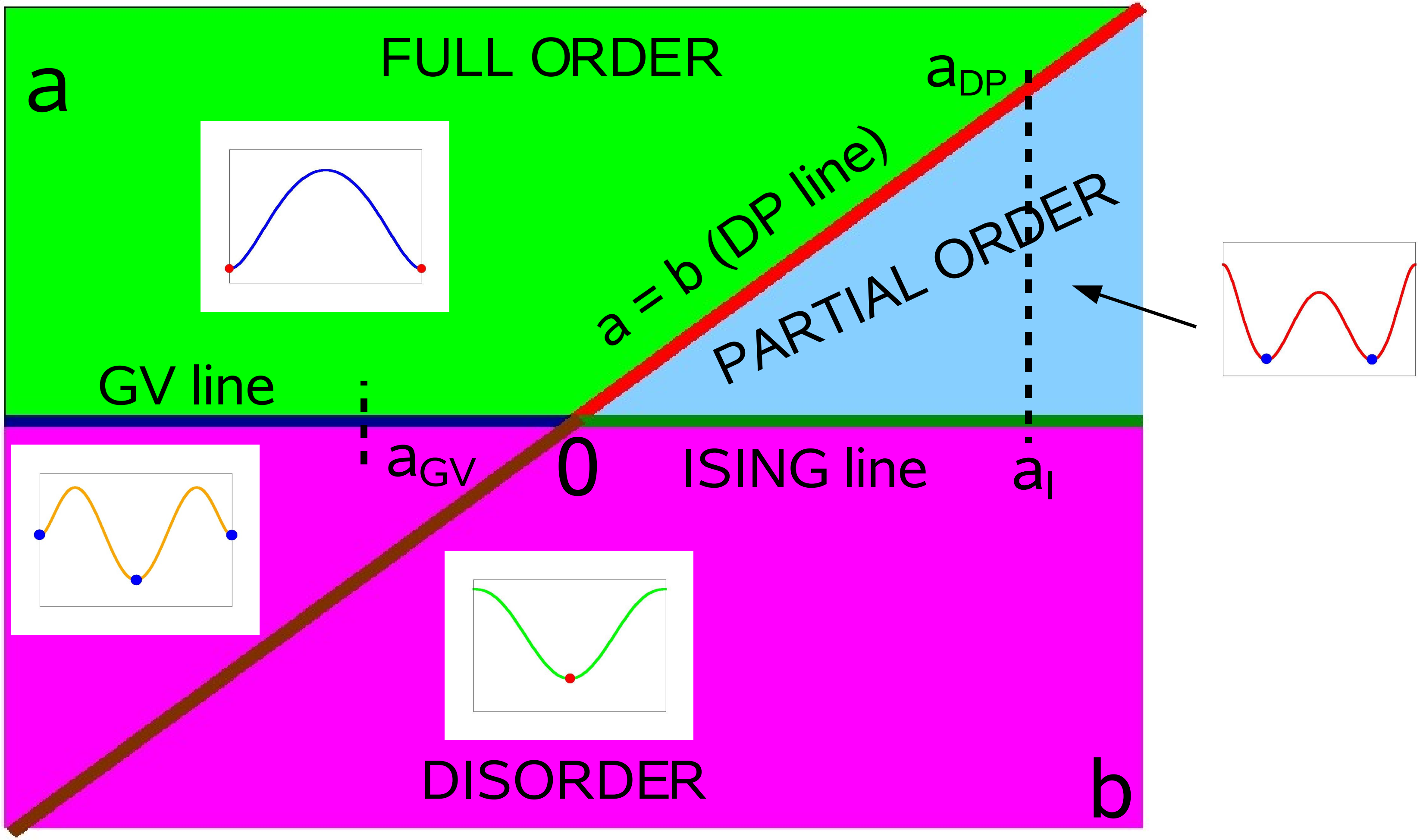
■ up spins

Classes of transitions:

$$V(\phi) = -\frac{a}{2}\phi^2 + \frac{a+b}{4}\phi^4 - \frac{b}{6}\phi^6$$

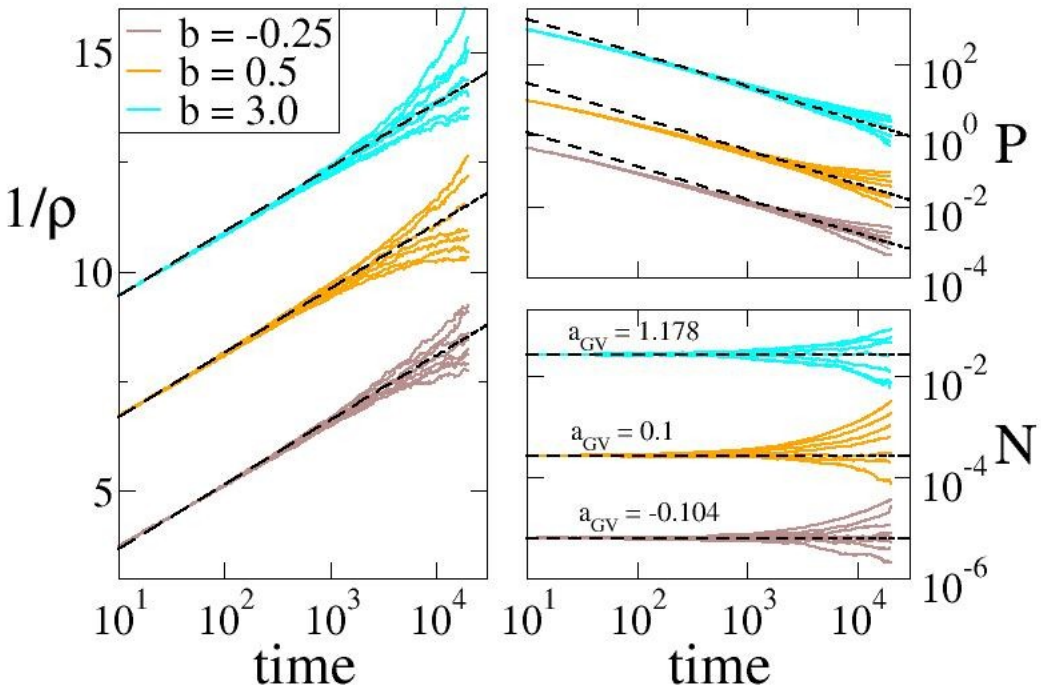
b ≤ 0: Generalized Voter

b > 0: Ising and Directed Percolation



Monte Carlo simulations on a 2-d square lattice

- *1st nearest-neighbors interactions (z=4):*



$\rho \sim \pi/[2\ln(t)]$ (interface density)

$P \sim t^{-1.0}$ (survival probability)

$N \sim t^{0}$ (density of + spins)

voter critical exponents
(Dickman '95)

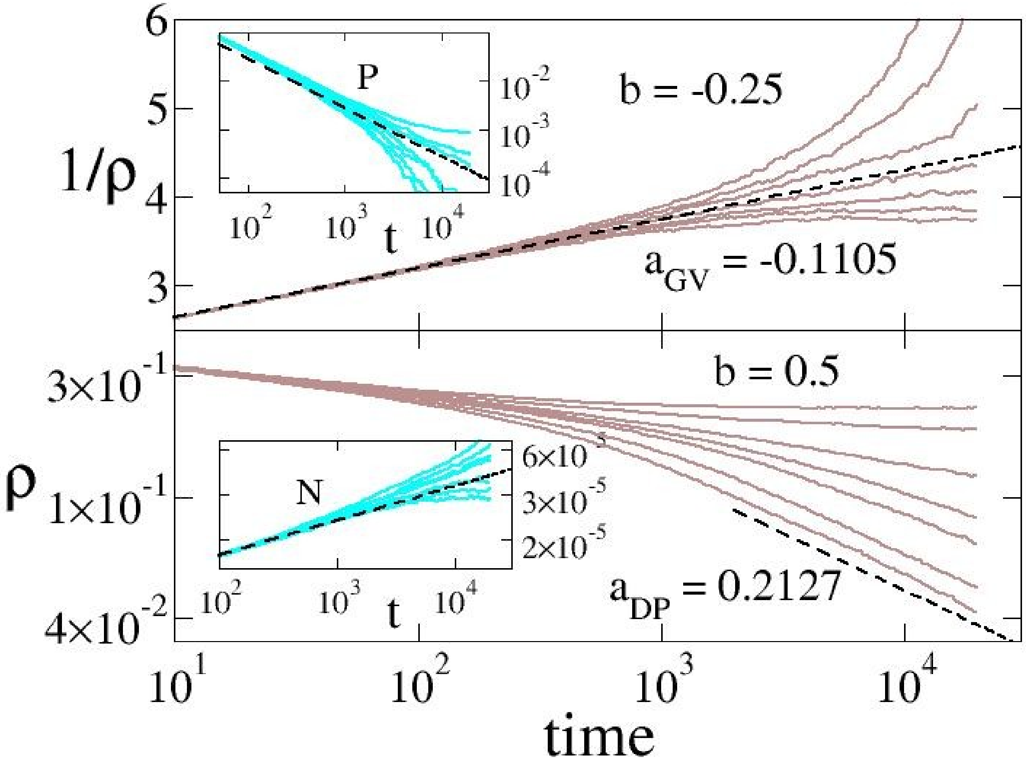
Only GV transition !

Dornic's conjecture:

Z_2 models without bulk noise

exhibit GV transitions.

- *3rd nearest-neighbors interactions (z=12):*



b = -0.25: GV transition

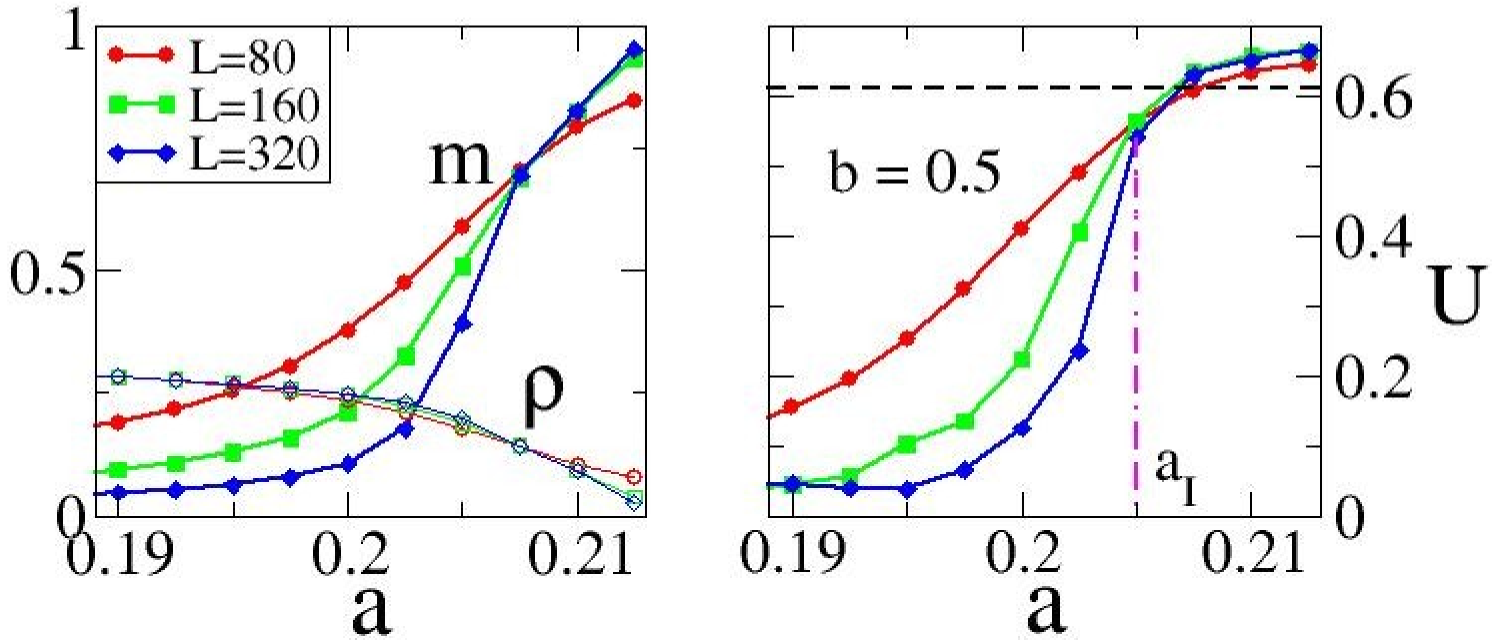
$\rho \sim \pi/[2\ln(t)]$, $P \sim t^{-1.0}$, $N \sim t^{0}$

b = 0.5: DP transition

$\rho \sim t^{-0.45}$, $P \sim t^{-0.45}$, $N \sim t^{0.2295}$

magnetization

cumulants: $1-m_4/3m_2^2$



b = 0.5: ISING transition at $a_I \approx 0.205$

Transition classes depend on the interaction range.

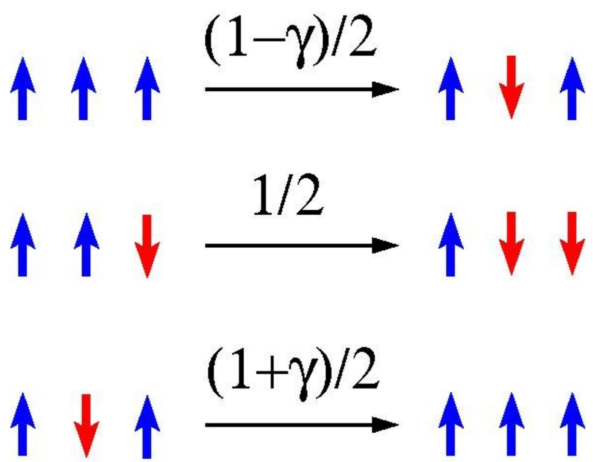
Summary

- Starting from the microscopic dynamics, we derived a Langevin equation for the macroscopic evolution of general spin systems with two symmetric absorbing states.
- The equation allows to predict the macroscopic behavior (ordering dynamics, critical properties) of models, by knowing the first derivatives of the transition probability.
- Open problem: more than two symmetric states ?

Examples

- **Probability Theory:**
  - Stochastic Ising Model *[Glauber 1963]*.

Model for magnetism. Each site of lattice a occupied by one atom with spin -1 or +1.

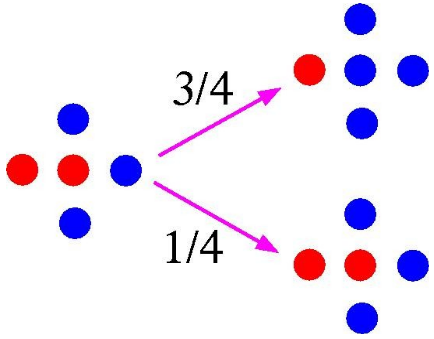


Evolution to thermodynamic equilibrium.

Detailed balance ⇒ $\gamma = \tanh(2\beta J)$

- The Voter Model.
  - Species invasion *[Clifford 1973]:* Each site of a lattice occupied by one of two species $\sigma \in \{0,1\}$. A site is invaded by its neighboring species.
  - Simplest IPS *[Liggett 1975]:* Two possible positions $\sigma \in \{-1,1\}$ on a political issue. Individuals ("voters") blindly adopt the position of a random neighbor.

$$P(\sigma_i \rightarrow -\sigma_i) = \frac{1}{2}\left(1 - \frac{\sigma_i}{z}\sum_{j \text{ NN } i} \sigma_j\right)$$



- Kinetics of catalytic reactions.

The dimer-dimer model *[Krapivsky 1992].* A and B particles adsorb into vacant sites of a surface. Neighboring A-B pairs react and desorb. Empty pair is replaced by an AA or BB dimer.

$$A + V \xrightarrow{k_A} A_V\,,$$

$$B + V \xrightarrow{k_B} B_V\,,$$

$$A_V + B_V \xrightarrow{k_r} AB\uparrow + 2V.$$

$$2V \rightarrow A_V A_V \text{ or } B_V B_V\,.$$

- The Contact Process.

Epidemics propagation *[Harris 1974]*. Individuals are either healthy $\eta=0$ or infected $\eta=1$. Infected individuals become healthy at rate 1. Healthy individuals are infected by their neighbors.

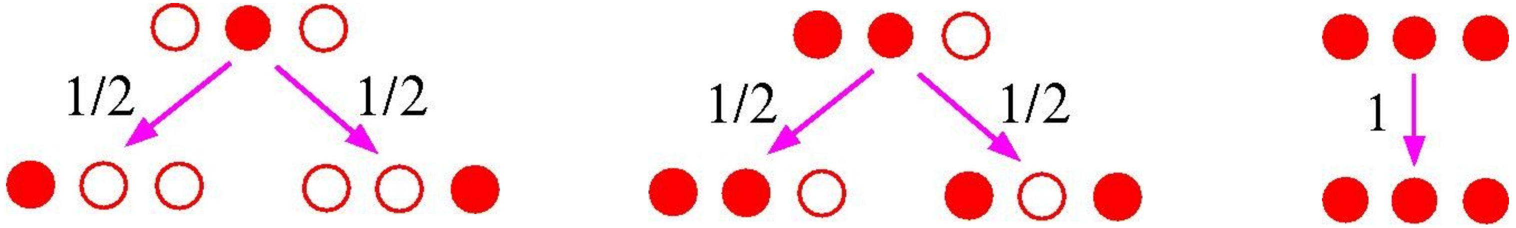
$$\eta(x) = 1 \rightarrow \eta(x) = 0 \quad \text{at rate } 1$$

$$\eta(x) = 0 \rightarrow \eta(x) = 1 \quad \text{at rate } \lambda \sum_{y \text{ NN } x} \eta(y)$$

- The Exclusion Process.

- Lattice gas at infinite temperature *[Spitzer 1970]*. Sites are either occupied by only one particle $\eta=1$ or empty $\eta=0$. Particles jump to empty neighboring sites.

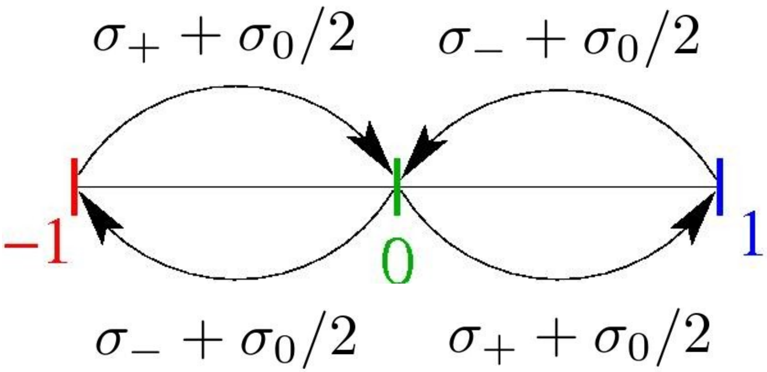
- Model for two species that swap territory *[Clifford 1973]*.



- **Probability theory:**
  - Stochastic Ising Model *[Glauber 1963]*.
  - The Voter Model.
  - The Contact Process.
  - The Exclusion Process.
- **Ecology:**
  - Invasion process.
  - Species competition.
  - Predator-prey models (Lotka Volterra).

- **Biology:**
  - Epidemic spreading (SIS, SIR).
  - Allele frequency (genetics).
  - Bacteria dynamics.
  - Tumor growth.
- **Social Science:**
  - Opinion dynamics (Deffuant, Galam, Snajd, Minority/Majority rule).
  - Cultural dynamics (Axelrod, Levine).
  - Language dynamics (Abrams-Strogatz, Minett-Wang).
- **Surface Physics/ Chemistry:**
  - Catalytic reactions.
  - Deposition/ reaction-diffusion/ aggregation.

Application: models with intermediate states and Z_2-symmetry



σ = local density of states.

3-state models:

(Minet-Wang '05, Castelló '06, Baronchelli '06.)

n-state models: Dall'Asta '08.

Observations:

- System orders driven by curvature.
- Density of interfaces $\rho \sim t^{-0.45}$.

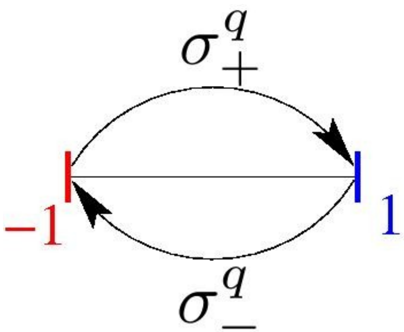
Field equation for 3-state models

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\left(\phi - \phi^3\right) + (1-\phi^2)\Delta\phi + \sqrt{1-\phi^2}\,\eta$$

$$P_{eff}(\mp \to \pm) \simeq \left(\sigma_\pm + \frac{1}{2}\sigma_0\right)^2$$ Effective transition probability

linear 3-state model ≡ 2-state model with quadratic transitions.

Abrams-Strogatz model for language evolution: Non-linear model.



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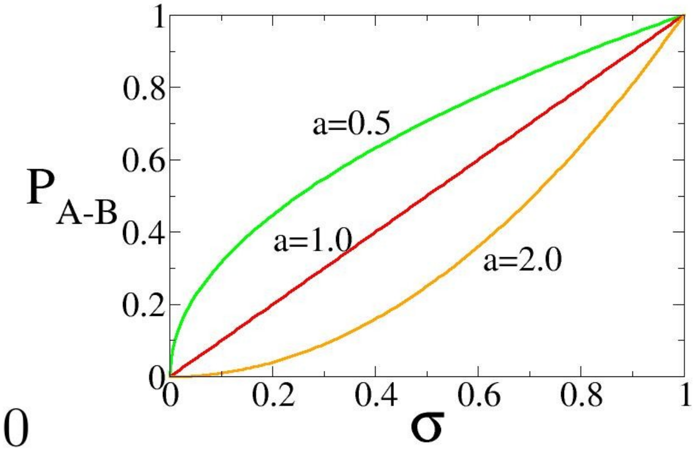
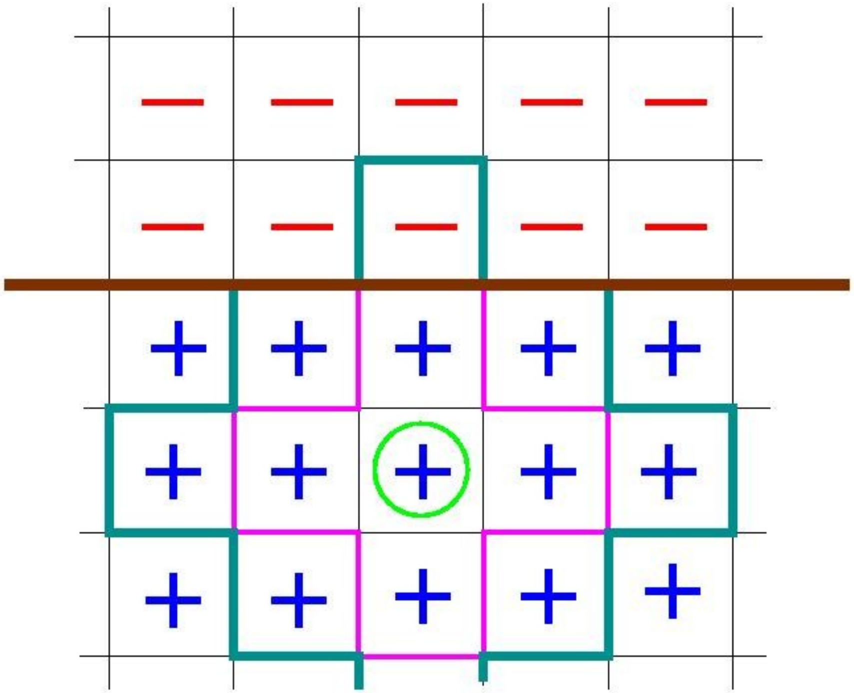


$$P(\mp \to \pm) = \sigma_\pm^q = \left(\frac{1 \pm \psi_\mathbf{r}}{2}\right)^q, \qquad q \geq 0$$

- Ising behavior: spin-flip inside domain (bulk noise).
- Voter behavior: spin-flip at domain interfaces only (1st NN interactions).



- Need interaction range $R \geq 2$.

Interacting particle systems

- IPS: system composed by particles whose states evolve in a coupled manner (interaction).
- Born in the 60's as a branch of probability theory (Spitzer, Dobrushin).
- Motivation (Statistical Mechanics): analyze time evolution of stochastic models.
- Goal: Better understanding of phase transitions.