Local scale-invariance and ageing phenomena: where do we stand ?

Malte Henkel

Groupe de Physique Statistique,
Départment de Physique de la Matière et des Matériaux,
Institut Jean Lamour, CNRS – Nancy Université, France

collaborators:

R. Cherniha, S.B. Dutta, X. Durang, J.Y. Fortin,
M. Pleimling, S. Stoimenov

Workshop NEQ,
University of Warwick, the 11$^{\mathrm{th}}$ of January, 2010

Contents :

Reviews : MH, J. Phys. Cond. Matt. **19**, 065101 (2007)
MH & Baumann, J. Stat. Mech. P07015 (2007)
MH & Pleimling, *Non-equilibrium phase transitions 2* (2010)

I. Why local dynamical scaling ?

- non-equilibrium systems naturally display dynamical scaling
- a common example : **ageing** phenomena
  1. slow relaxation (non-exponential)
  2. breaking of time-translation-invariance
  3. dynamical scaling
- which (reversible) microscopic processes lead to such macroscopic effects ?
- physical ageing known since (pre-)historical times, but systematic studies first in glassy systems STRUIK 78
  - *a priori*, behaviour prehistory-dependent
  - **but** evidence for reproducible and **universal** behaviour
- for better conceptual understanding : study ageing in simpler systems without disorder (i.e. ferromagnets)

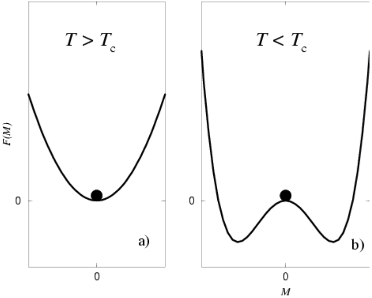
Question : what is the current evidence for larger,

local scaling symmetries ?

for symmetry analysis : simple ageing systems without disorder

consider a simple magnet (ferromagnet, i.e. Ising model)

1. prepare system initially at high temperature $T \gg T_c > 0$
2. quench to temperature $T < T_c$ (or $T = T_c$)
   $\rightarrow$ non-equilibrium state
3. fix T and observe dynamics



competition :

at least 2 equivalent ground states

local fields lead to rapid local ordering

no global order, relaxation time ∞

formation of ordered domains, of linear size $L = L(t) \sim t^{1/z}$

dynamical exponent z

Scaling behaviour & exponents

single relevant time-dependent length scale $L(t) \sim t^{1/z}$

Bray 94, Janssen et al. 92, Cugliandolo & Kurchan 90s, Godrèche & Luck 00, ...

$\phi(t,\mathbf{r})$ – space-time-dependent order-parameter (magnetisation)

$$\text{correlator } C(t,s;\mathbf{r}) := \langle \phi(t,\mathbf{r})\phi(s,\mathbf{0})\rangle = s^{-b} f_C\left(t/s, |\mathbf{r}|^z/(t-s)\right)$$

$$\text{response } R(t,s;\mathbf{r}) := \left.\frac{\delta\langle\phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{0})}\right|_{h=0} = s^{-1-a} f_R\left(t/s, |\mathbf{r}|^z/(t-s)\right)$$

No fluctuation-dissipation theorem : $\boxed{R(t,s;\mathbf{r}) \neq T\, \partial C(t,s;\mathbf{r})/\partial s}$

values of exponents : equilibrium correlator → classes **S** and **L**

$$C_{\rm eq}(\mathbf{r}) \sim \left\{ \begin{array}{l} \exp(-|\mathbf{r}|/\xi) \\ |\mathbf{r}|^{-(d-2+\eta)} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \text{class } \mathbf{S} \\ \text{class } \mathbf{L} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} a = 1/z \\ a = (d-2+\eta)/z \end{array} \right.$$

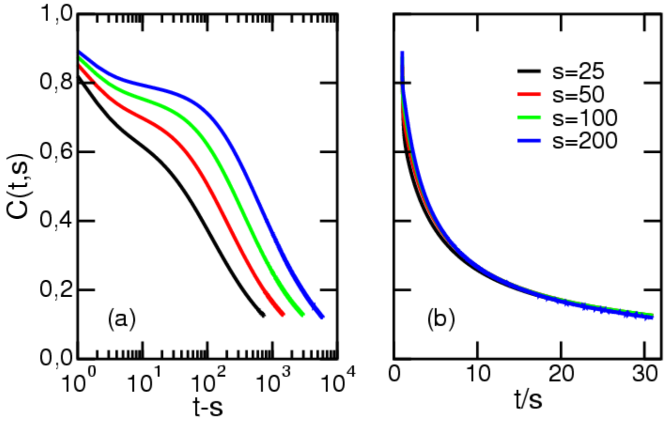
if $T < T_c$: $z = 2$ and $b = 0$

for $y \to \infty$: $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$,

if $T = T_c$: $z = z_c$ and $b = a$

$\lambda_{C,R}$ independent exponents

Test of dynamical scaling : $3D$ Ising model, $T < T_c$



no time-translation invariance

dynamical scaling

$C(t,s)$: autocorrelation function, quenched to $T < T_c$

scaling regime : $t, s \gg \tau_{\mathrm{micro}}$ and $t - s \gg \tau_{\mathrm{micro}}$

Question : how to find the scaling functions $f_R(y)$ and $f_C(y)$?

How to understand these scaling forms → mean-field

Langevin eq. for order parameter $m(t)$

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = 3\lambda^2 m(t) - m(t)^3 + \eta(t) \;\;,\;\; \langle \eta(t)\eta(s)\rangle = 2T\delta(t-s)$$

contrôle parameter λ^2 :

(1) $\lambda^2 > 0$: $T < T_c$, (2) $\lambda^2 = 0$: $T = T_c$, (3) $\lambda^2 < 0$: $T > T_c$

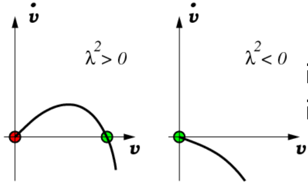
two-time observables : **response** $R(t,s)$, **correlation** $C(t,s)$

$$R(t,s) = \left.\frac{\delta\langle m(t)\rangle}{\delta h(s)}\right|_{h=0} = \frac{1}{2T}\langle m(t)\eta(s)\rangle \;\;,\;\; C(t,s) = \langle m(t)m(s)\rangle$$

mean-field equation of motion (cumulants neglected) :

$$\begin{aligned} \partial_t R(t,s) &= 3\left(\lambda^2 - v(t)\right) R(t,s) + \delta(t-s) \\ \partial_s C(t,s) &= 3\left(\lambda^2 - v(s)\right) C(t,s) + 2TR(t,s) \end{aligned}$$

with variance $v(t) = \langle m(t)^2\rangle$, $\dot{v}(t) = 6\big(\lambda^2 - v(t)\big)v(t)$



if $\lambda^2 \geq 0$: **fluctuations** persist
if $\lambda^2 < 0$: **fluctuations** disappear

in the long-time limit $t, s \to \infty$: $(t > s)$

$$R(t,s) \simeq \begin{cases} 1 \\ \sqrt{s/t} \\ e^{-3|\lambda^2|(t-s)} \end{cases} ; \; C(t,s) \simeq T \begin{cases} 2\min(t,s) & ; \; \lambda^2 > 0 \\ s\sqrt{s/t} & ; \; \lambda^2 = 0 \\ \frac{1}{(3|\lambda^2|)} e^{-3|\lambda^2|\,|t-s|} & ; \; \lambda^2 < 0 \end{cases}$$

fluctuation-dissipation ratio measures distance from equilibrium

$$X(t,s) = \frac{TR(t,s)}{\partial_s C(t,s)} \simeq \begin{cases} 1/2 + \mathrm{O}(e^{-6\lambda^2 s}) & ; \; \lambda^2 > 0 \\ 2/3 & ; \; \lambda^2 = 0 \\ 1 + \mathrm{O}(e^{-|\lambda^2|\,|t-s|}) & ; \; \lambda^2 < 0 \end{cases}$$

relaxation far from equilibrium, when $X \neq 1$, if $\lambda^2 \geq 0$ $(T \leq T_c)$

Consequences :

If $\lambda^2 > 0$: free random walk,
the system never reaches equilibrium !
If $\lambda^2 = 0$: slow relaxation, because of critical fluctuations

In both situations : observe

1. slow dynamics (non-exponential relaxation)
2. time-translation-invariance **broken**
3. **dynamical scaling behaviour**

$\longrightarrow$ the conditions for **physical ageing** are **all satisfied** if $T \leq T_c$
$\longrightarrow$ the system remains **out of equilibrium**

If $\lambda^2 < 0$: rapid relaxation, with finite relaxation time $\tau_{\rm rel} \sim 1/|\lambda^2|$, towards unique equilibrium state

II. Local scale-invariance for $z \neq 2$

Extend known cases $z = 1, 2 \Longrightarrow$ **axioms of LSI** :

MH 97/02, Baumann & MH 07

1. Möbius transformations in time (generator X_n)

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta} \;\; ; \;\; \alpha\delta - \beta\gamma = 1$$

   require commutator : $[X_n, X_{n'}] = (n - n')X_{n+n'}$
2. Dilatation generator : $X_0 = -t\partial_t - \frac{1}{z}\mathbf{r}\cdot\partial_{\mathbf{r}} - \frac{x}{z}$
   Implies simple power-law scaling $L(t) \sim t^{1/z}$ (no glasses !).
3. Spatial translation-invariance $\to$ 2^e family Y_m of generators.
4. X_n contain phase terms from the scaling dimension $x = x_\phi$
5. X_n, Y_m contain further 'mass terms' (**Galilei** !)
6. finite number of independent conditions for n-point functions.

Theorem : LSI without 'masses'

Commutators $[X_n, X_{n'}] = (n-n')X_{n+n'}$, $[X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$ with $n, n' \in \mathbb{Z}$ and $m \in \mathbb{Z} - 1/z$ have **only** the realisations :

z	X_n	$=$	$-t^{n+1}\partial_t - \frac{n+1}{z} t^n r \partial_r - \frac{(n+1)x}{z} t^n - \frac{n(n+1)}{2} B_{10} t^{n-1} r^z$
	$Y_{k-1/z}$	$=$	$-t^k \partial_r - \frac{z^2}{2} k B_{10} t^{k-1} r^{-1+z}$
2	X_n	$=$	$-t^{n+1}\partial_t - \frac{1}{2}(n+1) t^n r \partial_r - \frac{1}{2}(n+1) x t^n$
			$-\frac{n(n+1)}{2} B_{10} t^{n-1} r^2 - \frac{(n^2-1)n}{6} B_{20} t^{n-2} r^4$
	$Y_{k-1/2}$	$=$	$-t^k \partial_r - 2k B_{10} t^{k-1} r - \frac{4}{3} k(k-1) B_{20} t^{k-2} r^3$
1	X_n	$=$	$-t^{n+1}\partial_t - A_{10}^{-1}[(t + A_{10} r)^{n+1} - t^{n+1}]\partial_r$
			$-(n+1)xt^n - \frac{n+1}{2} \frac{B_{10}}{A_{10}} [(t + A_{10} r)^n - t^n]$
	Y_{k-1}	$=$	$-(t + A_{10} r)^k \partial_r - \frac{k}{2} B_{10} (t + A_{10} r)^{k-1}$

free parameters (two in each case) : z, A_{10}, B_{10}, B_{20}

similar classification from a geometric point of view DUVAL & HORVATHY 09

1. generic z and $B_{10} = 0 : \Longrightarrow [Y_m, Y_{m'}] = 0.$ MH 97

2. $z = 2$. Find infinite-dimensional extension of $\mathfrak{sch}_1$:

$Z_n^{(0)} := -2t^n$, $Z_m^{(1)} := -2t^{m-1/2}r$, $Z_n^{(2)} := -nt^{n-1}r^2$ and

$$\begin{aligned}
[Y_m, Y_{m'}] &= (m-m')\left(4B_{20}Z^{(2)}_{m+m'} + B_{10}Z^{(0)}_{m+m'}\right) \\
[X_n, Z^{(0,2)}_{n'}] &= -n'Z^{(0,2)}_{n+n'} \;\; , \;\; [X_n, Z^{(1)}_m] = -(n/2-m)Z^{(1)}_{n+n'} \\
[Y_m, Z^{(1)}_{m'}] &= -Z^{(0)}_{m+m'} \;\; , \;\; [Y_m, Z^{(2)}_n] = -nZ^{(1)}_{m+n}
\end{aligned}$$

For $B_{20} = 0$ and $B_{10} = \mathcal{M}/2$ one is back to $\mathfrak{sv}_1 \supset \mathfrak{sch}_1$.

3. $z = 1$. Then $[Y_n, Y_{n'}] = A_{10}(n-n')Y_{n+n'}$, in $d = 1$ dimensions.

If $A_{10} \neq 0$, isomorphic to $\mathfrak{vect}(S^1) \times \mathfrak{vect}(S^1)$.

In the limit $A_{10} \to 0$, contraction to $\mathfrak{av}_1 \supset \mathfrak{alt}_1 = \mathrm{CGA}(1)$; $(\gamma \in \mathbb{R})$

$$\begin{aligned}
X_n &= -t^{n+1}\partial_t - (n+1)t^n r\partial_r - (n+1)t^n x - n(n+1)\gamma t^{n-1} r \\
Y_n &= -t^{n+1}\partial_r - (n+1)\gamma t^n
\end{aligned}$$

two Virasoro-like **independent** central charges OVSIENKO & ROGER 98

For $d = 2$ so-called **exotic** central extension of $\mathfrak{alt}_2$, but incompatible with ∞-dim. extension $\mathfrak{alt}_2 \subset \mathfrak{av}_2$ LUKIERSKI, STICHEL, ZAKREWSKI 06/07 consider z arbitrary, set $B_{10} = 0$.

For the case $z = 1$, see HAVAS & PLEBANSKI 78, NEGRO ET AL 97, MH 97 & 02 ; ...09-10.

Extend to $z \neq 1, 2$ by generators with mass terms, for $d = 1$:

$$
\begin{aligned}
Y_{1-1/z} &:= -t\partial_r - \mu z r \nabla_{\mathbf{r}}^{2-z} - \gamma z(2-z)\partial_r \nabla_{\mathbf{r}}^{-z} && \text{Galilei} \\
X_1 &:= -t^2\partial_t - \frac{2}{z} tr\partial_r - \frac{2(x+\xi)}{z} t - \mu r^2 \nabla_{\mathbf{r}}^{2-z} && \text{special} \\
&\quad -2\gamma(2-z) r\partial_r \nabla_{\mathbf{r}}^{-z} - \gamma(2-z)(1-z)\nabla_{\mathbf{r}}^{-z}
\end{aligned}
$$

- depend on two parameters γ, μ and on two dimensions x, ξ
- contains fractional derivative ($\widehat{f}$: Fourier transform)

$$\nabla_{\mathbf{r}}^{\alpha} f(\mathbf{r}) := \mathrm{i}^{\alpha} \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{\alpha} e^{\mathrm{i}\mathbf{r}\cdot\mathbf{k}} \widehat{f}(\mathbf{k})$$

- some properties : $\nabla_{\mathbf{r}}^{\alpha} \nabla_{\mathbf{r}}^{\beta} = \nabla_{\mathbf{r}}^{\alpha+\beta}$, $[\nabla_{\mathbf{r}}^{\alpha}, r_i] = \alpha \partial_{r_i} \nabla_{\mathbf{r}}^{\alpha-2}$
  $\nabla_{\mathbf{r}}^{\alpha} \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r}) = \mathrm{i}^{\alpha} |\mathbf{q}|^{\alpha} \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r})$

Fact 1 : simple algebraic structure :

$$[X_n, X_{n'}] = (n - n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$$

$\rightarrow$ Generate Y_m from $Y_{-1/z} = -\partial_r$.

Fact 2 : LSI-invariant Schrödinger operator :

$$\mathcal{S} := -\mu\partial_t + z^{-2}\nabla_{\mathbf{r}}^{z}$$

Let $x_0 + \xi = 1 - 2/z + (2-z)\gamma/\mu$. Then $[\mathcal{S}, Y_m] = 0$ and

$$[\mathcal{S}, X_0] = -\mathcal{S} \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + \frac{2\mu}{z}(x - x_0)$$

$\Longrightarrow$ $\boxed{\mathcal{S}\phi = 0}$ is **lsi-invariant** equation, if $\underline{x_\phi = x_0}$.

Physical assumption (hidden & approximate) : equations of motion remain of first order in ∂_t, even after renormalisation.

Fact 3 : non-trivial conservation laws :
iterated commutator with $G := Y_{1-1/z}$, $\mathrm{ad}_G . = [., G]$

$$M_\ell := (\mathrm{ad}_G)^{2\ell+1}\, Y_{-1/z} = a_\ell \mu^{2\ell+1} \nabla_{\mathbf{r}}^{(2\ell+1)(1-z)+1}$$

For $z = 2$, $a_\ell = 0$ if $\ell \geq 1$. For a n-point function $F^{(n)} = \langle \phi_1 \ldots \phi_n \rangle$, $M_\ell F^{(n)} = 0$ gives in momentum space

$$\begin{aligned} \left(\sum_{i=1}^{n} \mu_i^{2\ell-1} |\mathbf{k}_i|^{2\ell-(2\ell-1)z} \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) &= 0 \\ \left(\sum_{i=1}^{n} \mathbf{k}_i \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) &= 0 \end{aligned}$$

$\Longrightarrow$ momentum conservation & conservation of $|\mathbf{k}|^\alpha$!
analogous to relativistic factorisable scattering — Zamolodchikov^2 79, 89
equil. analogy : $2D$ Ising model at $T = T_c$ in magnetic field

Consequence : a $\mathfrak{lsi}$-covariant $2n$-point function $F^{(2n)}$ is only non-zero, if the 'masses' μ_i can be arranged in pairs $(\mu_i, \mu_{\sigma(i)})$ with $i = 1, \ldots, n$ such that $\boxed{\mu_i = -\mu_{\sigma(i)}}$.

generalised Galilei-invariance with $z \neq 2 \Longrightarrow$ integrability

Corollary 1 : Bargman rule : $\boxed{\langle \phi_1 \ldots \phi_n \widetilde{\phi}_1 \ldots \widetilde{\phi}_m \rangle_{\mathbf{0}} \sim \delta_{n,m}}$

Corollary 2 : derive reduction formulæ for averages :

go to stochastic field-theory, action JANSSEN 92, DE DOMINICIS,...

$$\mathcal{J}[\phi, \widetilde{\phi}] = \mathcal{J}_{\mathbf{0}}[\phi, \widetilde{\phi}] \underbrace{-T \int \widetilde{\phi}^2 - \int \widetilde{\phi}_{t=0} C_{init} \widetilde{\phi}_{t=0}}_{+\, \mathcal{J}_b[\widetilde{\phi}] \;:\; \text{noise}}$$

$\widetilde{\phi}$: response field ; $\boxed{C(t,s) = \langle \phi(t)\phi(s) \rangle, \; R(t,s) = \langle \phi(t)\widetilde{\phi}(s) \rangle}$

averages : $\langle A \rangle_{\mathbf{0}} := \int \mathcal{D}\phi \mathcal{D}\widetilde{\phi}\, A[\phi, \widetilde{\phi}] \exp(-\mathcal{J}_{\mathbf{0}}[\phi, \widetilde{\phi}])$

identify masses (generalised Bargman rule) :

$$\boxed{\mu_\phi = -\mu_{\widetilde{\phi}}}$$

application to the response

$$
\begin{aligned}
R(t,s) &= \left\langle \phi(t)\widetilde{\phi}(s)\right\rangle = \left\langle \phi(t)\widetilde{\phi}(s) e^{-\mathcal{J}_b[\widetilde{\phi}]}\right\rangle_{\mathbf{0}} \\
&= \left\langle \phi(t)\widetilde{\phi}(s)\right\rangle_{\mathbf{0}} = R_{\mathbf{0}}(t,s)
\end{aligned}
$$

Bargman rule $\Longrightarrow$ response function independent of noise !
left side : computed in stochastic models
right side : **local scale-symmetry** of **deterministic** equation

Corollary 3 : response function noise-independent

$$
\begin{aligned}
R(t,s;\mathbf{r}) &= R(t,s)\mathcal{F}^{(\mu_1,\gamma_1)}(|\mathbf{r}|(t-s)^{-1/z}) \\
R(t,s) &= r_0\, s^{-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s}-1\right)^{-1-a'} \\
\mathcal{F}^{(\mu,\gamma)}(\mathbf{u}) &= \int_{\mathbb{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{\gamma} \exp\left(\mathrm{i}\mathbf{u}\cdot\mathbf{k} - \mu|\mathbf{k}|^z\right)
\end{aligned}
$$

choice of the (quasi-)primary operators ?

Finite transformation (spatial part here for $z = 2$) :

$t = \beta(t')$, $\mathbf{r} = \mathbf{r}'\sqrt{\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'}}$ and $\beta(0) = 0$

$$\phi(t,\mathbf{r}) = \dot{\beta}(t')^{-x/2} \underbrace{\left(\frac{\mathrm{d}\ln\beta(t')}{\mathrm{d}\ln t'}\right)^{-\xi}}_{\text{extra transformation}} \underbrace{\exp\left[-\frac{\mathcal{M}\mathbf{r}'^2}{4}\frac{\mathrm{d}\ln\dot{\beta}(t')}{\mathrm{d}t'}\right]}_{\text{mass term}} \phi'(t',\mathbf{r}')$$

reduce to usual lsi-primary operator $\Phi(t,\mathbf{r}) := t^{-2\xi/z}\phi(t,\mathbf{r})$.

Then $\boxed{\Phi(t) = \dot{\beta}(t')^{-(x+2\xi)/z}\Phi'(t')}$, transforms as a primary.

a) mean-field equation $\partial_t m = \Delta m + 3(\lambda^2 - v(t))m$ reduces to diffusion equation $\partial_t\Phi = \Delta\Phi$ via

$$m(t,\mathbf{r}) = \Phi(t,\mathbf{r})\exp\int_0^t \mathrm{d}\tau\, 3\big(\lambda^2 - v(\tau)\big)$$

$$\text{two cases}: \begin{cases} \text{if } T = T_c \Leftrightarrow \lambda^2 = 0: & \Phi(t) \sim t^{1/2}m(t) \\ \text{if } T < T_c \Leftrightarrow \lambda^2 > 0: & \Phi(t) \sim 1\cdot m(t) \end{cases}$$

$\Rightarrow$ **magnetisation $m(t)$ and primary operator $\Phi(t)$ distinct**

b) kinetic spherical model equation

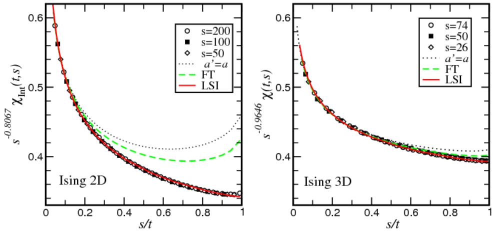
$$\partial_t \phi(t) = \Delta\phi(t) - v(t)\phi(t) + \text{noise} \quad , \quad v(t) \sim t^{-1}$$

gauge transformation $\Phi(t) = \phi(t)\exp\left[-\int_0^t \mathrm{d}\tau\, v(\tau)\right]$,
gives diffusion eq. for Φ

c) kinetic Glauber-Ising model $T = T_c$

MH, Enss, Pleimling 06

$1D \quad a' - a = -\frac{1}{2}$
$2D \quad a' - a \simeq -0.17(2)$
$3D \quad a' - a \simeq -0.022(5)$



$*$ $2^{\rm nd}$-order ε-expansion disagrees with lattice data Pleimling & Gambassi 05

$*$ $a' - a < 0$ required to match LSI with lattice data, but still disagrees with FT

$\Rightarrow$ resum ε-expansion to be able to compare with lattice data ?

Some known values of a, a' and λ_R/z at $T = T_c$.

model	d	a	$a' - a$	λ_R/z	Réf.
Ising	1	0	$-1/2$	$1/2$	Godrèche & Luck 00
	2	0.115	$-0.17(2)$	$0.732(5)$	H & P 03
	3	0.506	$-0.022(5)$	$1.36(2)$	H & P 03
EA spin glass	3	$0.060(4)$	$-0.76(3)$	$0.38(2)$	H & P 05
FA	1	1	$-3/2$	2	Mayer et al 06
	> 2	$1 + d/2$	-2	$2 + d/2$	Mayer et al 06
contact proc.	1	-0.681	$0.270(10)$	$1.76(5)$	H, E & P 06
NEKIM	1	$-0.430(2)$	$0.00(1)$	$1.9(2)$	Odor 06
OJK model	≥ 2	$(d-1)/2$	$-1/2$	$d/4$	Mazenko 04

$\Longrightarrow$: $a \neq a'$ should be the generic case.

$\Longrightarrow$: order-parameter $m(t)$ does in general **not** transform in the most simple way !

Corollary 4 :

Correlators obtained from factorised 4-point responses :

$$C(t,s) = \langle \phi(t)\phi(s) \rangle = \langle \phi(t)\phi(s) e^{-\mathcal{J}_b[\widetilde{\phi}]} \rangle_{\mathbf{0}}$$

example : contribution of 'initial' noise at time u :

$$C_{\rm init}(t,s;\mathbf{r}) = \int_{\mathbb{R}^{2d}} \mathrm{d}\mathbf{R}\mathrm{d}\mathbf{R}' \underbrace{F^{(4)}(t,s,u;\mathbf{r},\mathbf{R},\mathbf{R}')}_{\text{4-pt function}} \quad \underbrace{\mathbf{C}(u,\mathbf{R}-\mathbf{R}')}_{\text{'initial' correlator}}$$

$$= c_0\,(ts)^{2\xi/z+F}\, s^{4\widetilde{x}/z-2F}\,(t-s)^{-2(2\xi+x)/z} \times \int_{\mathbb{R}^d} \mathrm{d}\mathbf{k}\, |\mathbf{k}|^{2\beta} \exp\left[\mathrm{i}\mathbf{r}\cdot\mathbf{k} - \alpha|\mathbf{k}|^z(t-s)\right] \widehat{\mathbf{C}}(s,\mathbf{k})$$

where we have also sent $u \to s$.

Relevant, e.g. for **phase-ordering kinetics** $\to z = 2$ Bray & Rutenberg 94

Ising model, more precise 'initial' correlator : Ohta, Jasnow, Kawasaki '82

$$\mathbf{C}(t;\mathbf{r}) = \frac{2}{\pi} \arcsin\left(\exp\left[-\frac{\mathbf{r}^2}{L(t)^2} \right] \right)$$

III. How to test the foundations of LSI

theory is built on :

a) simple scaling – domain sizes $L(t) \sim t^{1/z}$

b) invariance under Möbius transformation $t \mapsto t/(\gamma t + \delta)$

c) Galilei-invariance generalised to $z \neq 2$

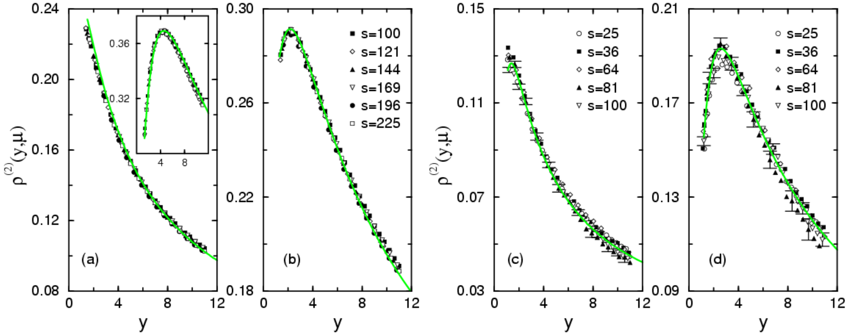
together with spatial translation-invariance

$\Longrightarrow$ extended Bargman rules

$\Longrightarrow$ factorisation of $2n$-point functions

Möbius transformation	autoresponse $R(t,s)$
generalised Galilei-invariance	space-time response $R(t,s;\mathbf{r})$
factorisation	two-time correlation function

Example : Ising model, space-time behaviour **(parameter-free !)** :



spatio-temporally integrated response Ising model $T < T_c$

(a,b) $2D$; $\mu = 1, 2, 4$ (c,d) $3D$; $\mu = 1, 2,$

$$\int_0^s \mathrm{d}u \int_0^{\sqrt{\mu s}} \mathrm{d}r\, r^{d-1} R(t, u; \mathbf{r}) = s^{d/2-a} \rho^{(2)}(t/s, \mu)$$

MH & M. Pleimling, Phys. Rev. **E68**, 065101(R) (2003)

analogous results in the q-states $2D$ Potts model, $T < T_c$

E. Lorenz & W. Janke, Europhys. Lett. **77**, 10003 (2007)

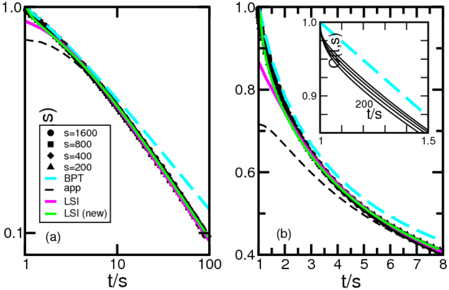
$2D$ Ising model, $T < T_c$: autocorrelator in the scaling limit

$$C(ys,s) \;=\; C_0 y^{\rho}(y-1)^{-\rho-\lambda_C/z}\int_0^\infty \mathrm{d}x\, e^{-x} f_\nu\left(\sqrt{\frac{x}{y-1}}\right)$$

$$f_\nu(\sqrt{u}) \;=\; \int_0^\infty \mathrm{d}v\ \arcsin\left(e^{-\nu v}\right) J_0(\sqrt{uv}\,)$$

parameters to be fitted : ρ, ν.

$z = 2$



of practical importance :
'good' choice of 'initial' correlations
$C_{\rm ini}(\mathbf{r}) = c_0\delta(\mathbf{r})$ not sufficient

BAUMANN & MH 10

$\Longrightarrow$ for the first time, a theoretical calculation for $C(t,s)$ reproduces the simulations for **all** t/s !

Tests of LSI for $z \neq 2$:

- spherical model with conserved order-parameter, $T = T_c$, $z = 4$ — Baumann & MH 06
- Mullins-Herring model for surface growth, $z = 4$ — Röthlein, Baumann, Pleimling 06
- spherical model with long-ranged interactions, $T \leq T_c$, $0 < z = \sigma < 2$ — Cannas et al. 01 ; Baumann, Dutta, MH 07 ; Dutta 08
- ferromagnets at their critical point (Ising, XY), $z \approx 2.0 - 2.2$ — MH, Enss, Pleimling 06 ; Abriet & Karevski 04
- critical particle-reaction models (DP ?, NEKIM), $z \approx 1.6 - 2$ — Ódor 06
- particle-reaction models with Lévy-flight transport, $0 < z = \eta < 2$ — Durang & MH 09

important : consideration of invariant differential equation

NB : **all** of the exactly solved models in this list are **markovian** !

What tests of LSI have been achieved ?

1. $\mathbf{R}(t,s)$:
   - $T < T_c, d = 2$: Ising, Potts, spherical (A&B), disord. Ising
   - $T < T_c, d = 3$: Ising, XY, spherical (A&B)
   - $T = T_c, d \leq 2$: Ising, spherical (A&B), HvL, DP ?, NEKIM
   - $T = T_c, d = 3$: Ising, spherical (A&B), BCPD/L, BPCPD
   - growth : Edwards-Wilkinson, Mullins-Herring
2. $R(t,s;\mathbf{r})$:
   - $T < T_c$: Ising, Potts-3 & 8, spherical (A&B)
   - $T = T_c$: Ising $1D$, spherical (A&B), BCPD/L, BPCDP
   - growth : Edwards-Wilkinson, Mullins-Herring

Difficulty : oscillating dependence on $|\mathbf{r}|$

3. $C(t,s)$:
   - $T < T_c$: Ising $2D$, Potts $2D$, spherical (A&B)
   - $T = T_c$: Ising $1D$, spherical (A&B), BCPD/L, BPCPD
   - growth : Family, Edwards-Wilkinson, Mullins-Herring

Required : precise single-time correlator $C(t,\mathbf{r})$

IV. Conclusions

1. look for extensions of dynamical scaling in ageing systems

   recently, scaling derived for phase-ordering ARENZON ET AL. 07

2. here : **hypothesis** of generalised Galilei-invariance
3. leads to Bargman rule if $z = 2$
   and further to 'integrability' if $z \neq 1, 2$.
4. **hidden** dynamical symmetry of deterministic part of (linear & first-order !) Langevin equations
5. Tests : derive two-time response and correlation functions
6. LSI exactly proven for linear Langevin equations
   very good numerical evidence for non-linear systems

Some questions (the list could/should be extended) :

- how to physically justify Galilei-invariance ?
- how to extend to non-linear equations ?
- **non**-markovian effects ? choice of fractional derivative ?
- what is the algebraic (non-Lie !) structure of LSI ?
- treatment of master equations with LSI ?

Vol. 2 – co-author M. **Pleimling** – will treat ageing phenomena in simple magnets and LSI (to appear still in 2010)