State estimation of long-range correlated non-equilibrium systems: media estimation

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Background

- Long-ranged correlations in non-equilibrium
- Nonlocal likelihoods for fluctuation fields (density, temperature, ...)

Local information on fluctuations has
implications on the global structure

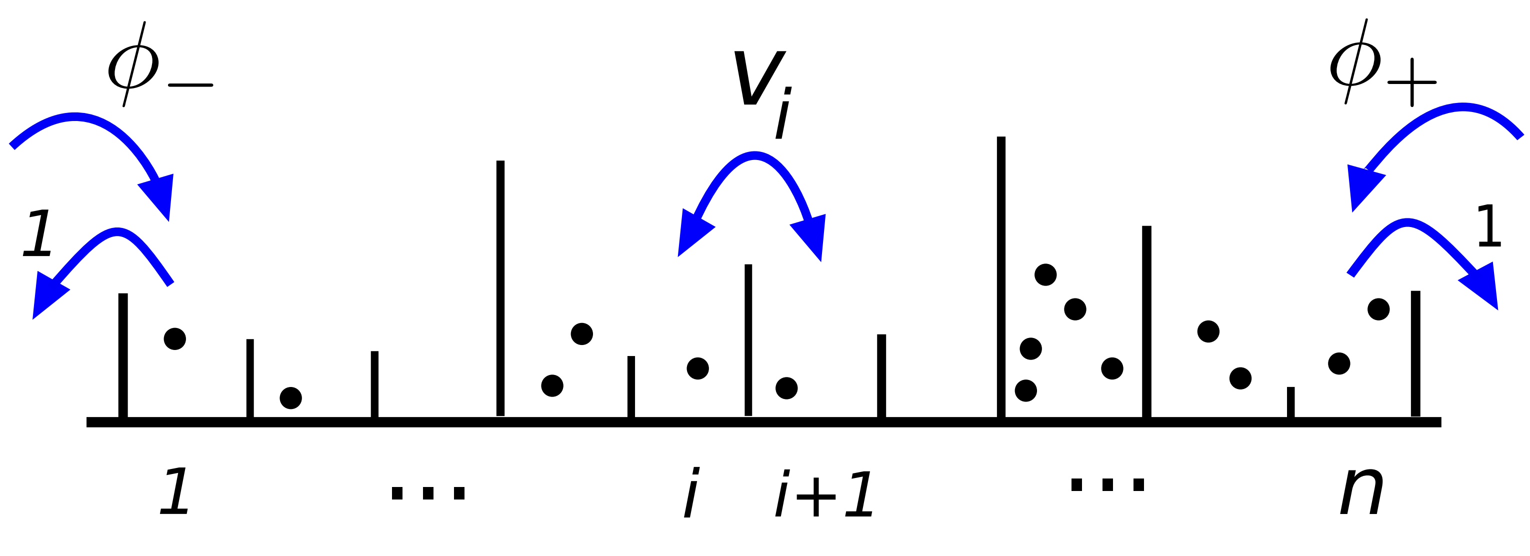
⇒ Successful analysis of inverse problems
based on finite data possible

Transport in random media

- Non-interacting particles in symmetric random media
- Product Poisson measures $\mu(\eta) = \prod_1^n \frac{\phi_j^{\eta_j}}{\eta_j!} \mathrm{e}^{-\phi_j}$
- Tied down by the fugacity

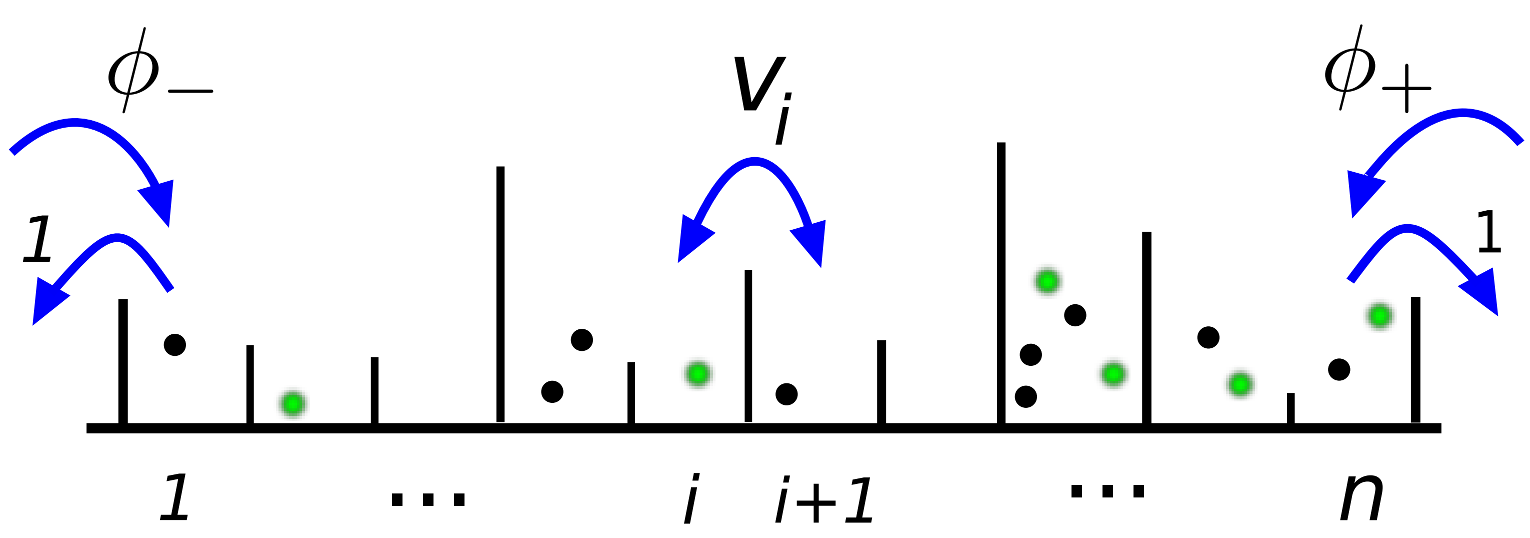
$$\phi_j = \phi_- \left(1 - \frac{R_j}{1 + R_n}\right) + \phi_+ \frac{R_j}{1 + R_n}, \quad j = 1, \ldots, n,$$

$$R_j = 1 + \sum_{i=1}^{j-1} v_i^{-1}$$



Estimation of random media

- Reservoirs diluted with a marker
- Try to estimate the structure of the underlying medium given marker observations at $(x_1, x_2, \ldots, x_k)$



Markers: scaling limit

$$P(N(A)=k)=\frac{\left(\int_A \phi(x)\,\mathrm{d}x\right)^k}{k!}\mathrm{e}^{-\int_A \phi(x)\,\mathrm{d}x}$$

- Weak disorder: $Ev_i^{-1}<\infty$

$$\phi(x)=\phi_-(1-x)+\phi_+x$$

Poisson point process

Trivial intensity structure

- Strong disorder: $Ev_i^{-1}=\infty$

$$\phi(x)=\phi_-\left(1-\frac{R(x)}{R(1)}\right)+\phi_+\frac{R(x)}{R(1)},$$

R is an α-stable increasing Lévy process

$\alpha\in(0,1)$

Cox process!

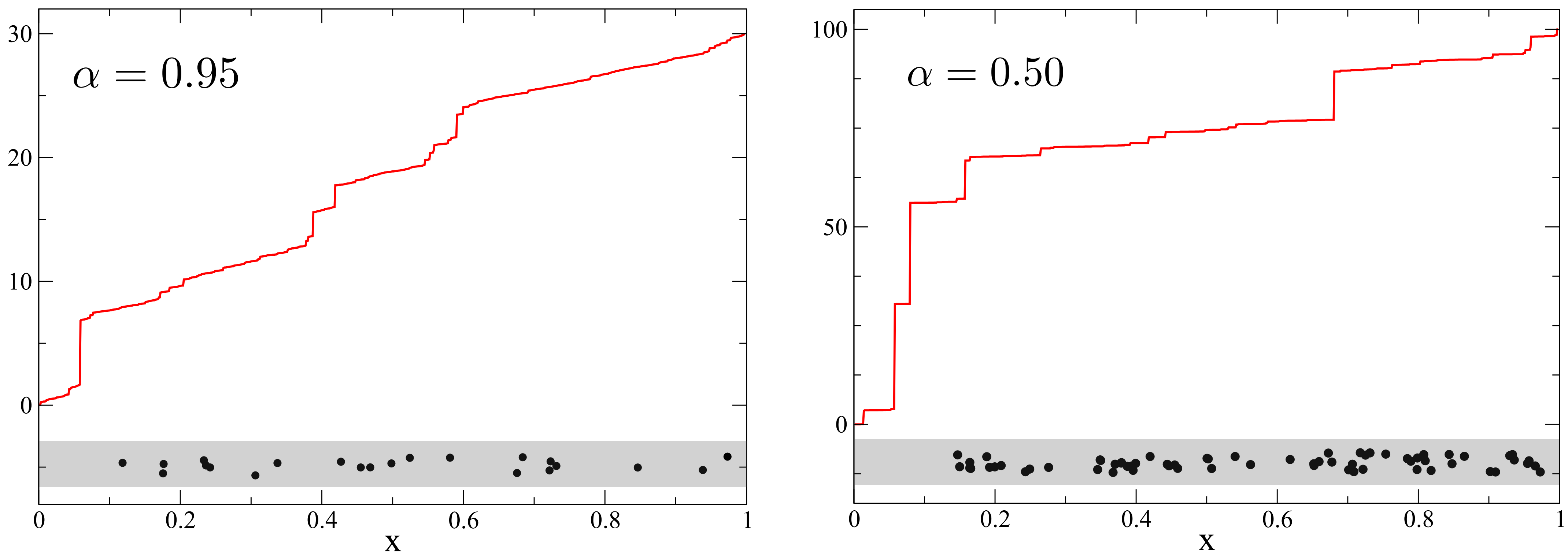
Markers: scaling limit

$$P(N(A)=k)=\frac{\left(\int_A \phi(x)\,\mathrm{d}x\right)^k}{k!}\mathrm{e}^{-\int_A \phi(x)\,\mathrm{d}x}$$

- Strong disorder: $Ev_i^{-1}=\infty$

$$\rho(x):=E\phi(x)=E\left[\phi_-\left(1-\frac{R(x)}{R(1)}\right)+\phi_+\frac{R(x)}{R(1)}\right]=\phi_-(1-x)+\phi_+x$$

$$C(x,y):=\mathrm{Cov}(\phi(x),\phi(y))=(1-\alpha)(\phi_+-\phi_-)^2x(1-y)$$

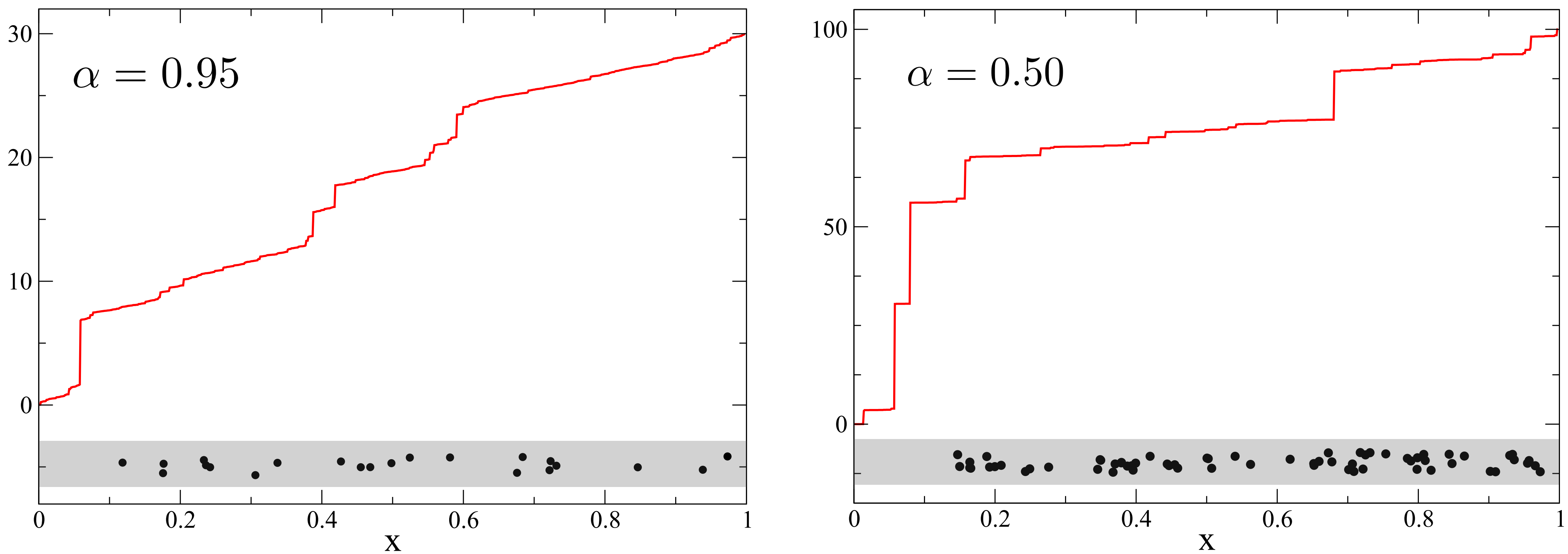


Estimation of random media

- Resistances from the fugacity jumps: $r(a, b; \phi) = \dfrac{\phi(b) - \phi(a)}{\phi_+ - \phi_-}$
- Minimum mean square error estimator for the fugacity

$$\hat{\phi}(x) = \frac{E\phi(x)\,\mathrm{e}^{-\int_\Lambda \phi(y)\mathrm{d}y} \prod_{i=1}^{k} \phi(x_i)}{E\mathrm{e}^{-\int_\Lambda \phi(y)\mathrm{d}y} \prod_{i=1}^{k} \phi(x_i)}$$

Need approximative estimators!

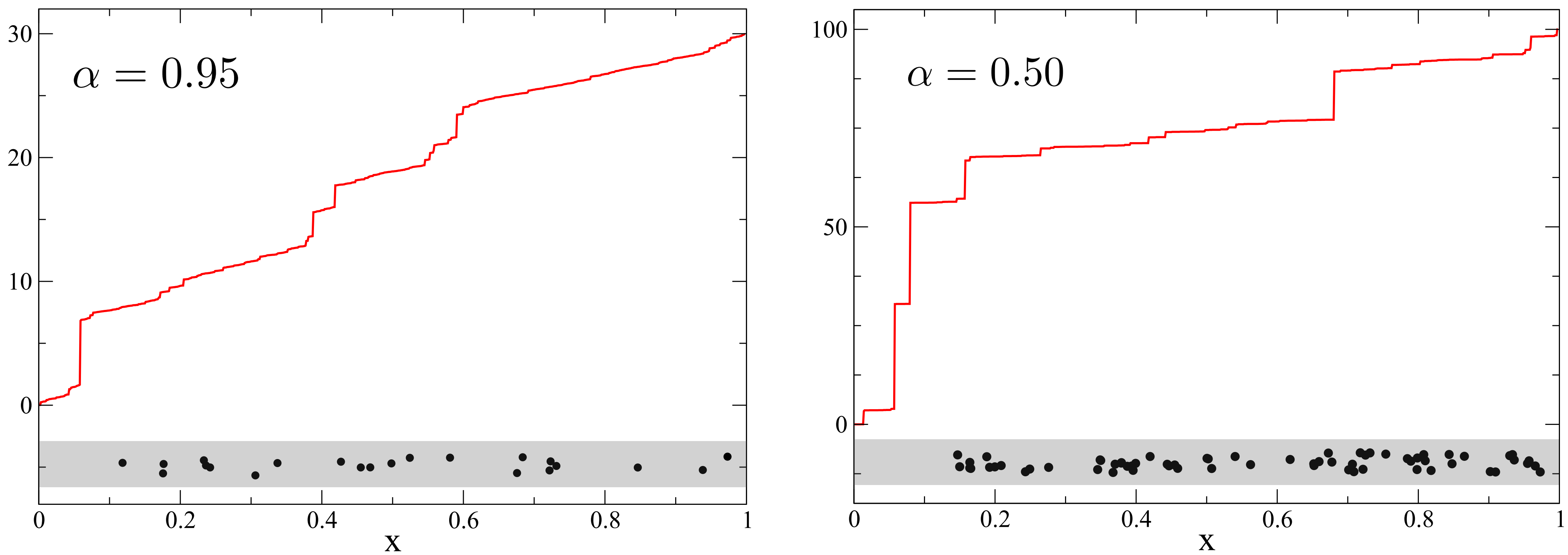


MMSE linear estimator

- Find a kernel *K* that translates each point to density fluctuations separately
- Grandell's theorem (1971):

$$\hat{\phi}_{\mathrm{L}}(x) = \rho(x) + \int_{[0,1]} K(x,y)[\,\mathrm{d}N(y) - \rho(y)\,\mathrm{d}y]$$

$$K(x,y)\rho(y) + \int_{[0,1]} K(x,z)C(z,y)\,\mathrm{d}z = C(x,y).$$



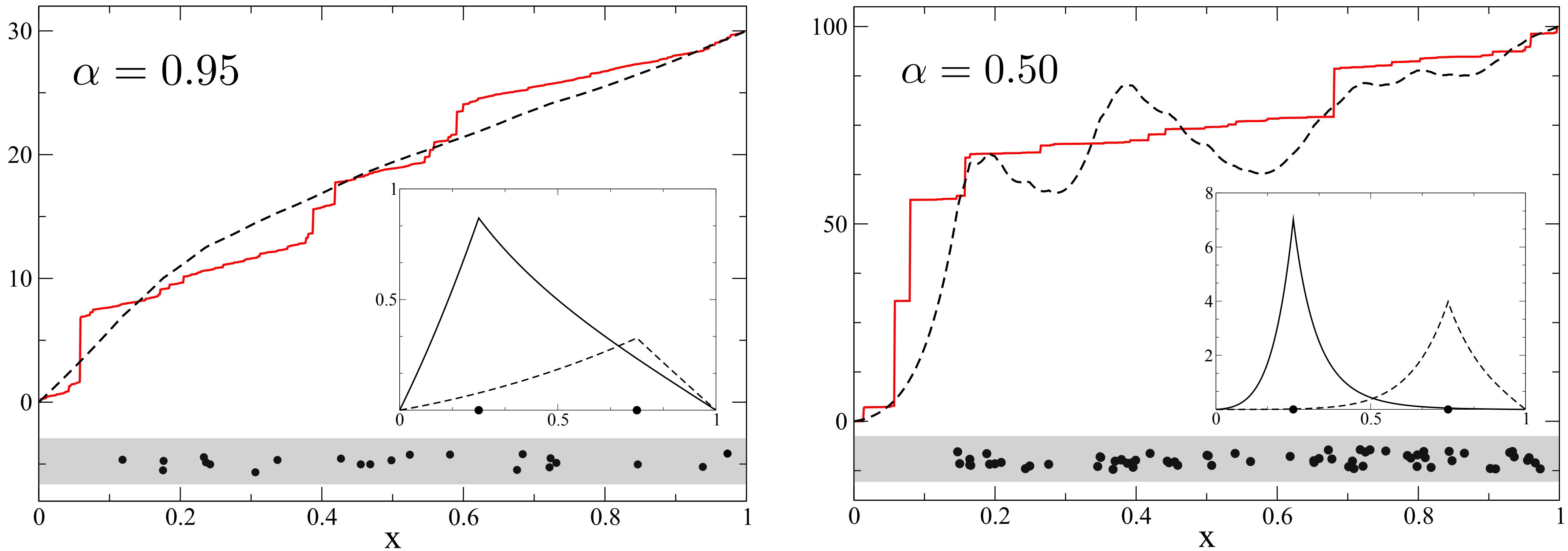
MMSE linear estimator

- Piecewise linear covariance: explicit solution

$$\hat{\phi}_{\rm L}(x) = \rho(x) + \int_{[0,1]} K(x,y)[\,{\rm d}N(y) - \rho(y)\,{\rm d}y]$$

$$K(x,y) = \frac{h(x,y)C(x,x)}{h(x,x)\rho(x) + \int_0^1 h(x,z)C(z,x)\,{\rm d}z}\,, \quad h(x,y) = g_-(x \wedge y) g_+(x \vee y)\,,$$

$$g_{\pm}(x) = \frac{2}{\sqrt{\rho(x)}}\left\{ I_1\left(2\sqrt{(1-\alpha)\rho(x)}\right) - \frac{I_1(2\sqrt{(1-\alpha)\phi_{\pm}})}{K_1(2\sqrt{(1-\alpha)\phi_{\pm}})} K_1\left(2\sqrt{(1-\alpha)\rho(x)}\right)\right\}$$



Maximum likelihood estimator

- Seek a *monotone* function that maximizes the likelihood

$$L(\phi, (x_i)) = \mathrm{e}^{-\int_{[0,1]} \phi(x)\,\mathrm{d}x} \prod_{i=1}^{k} \phi(x_i)$$

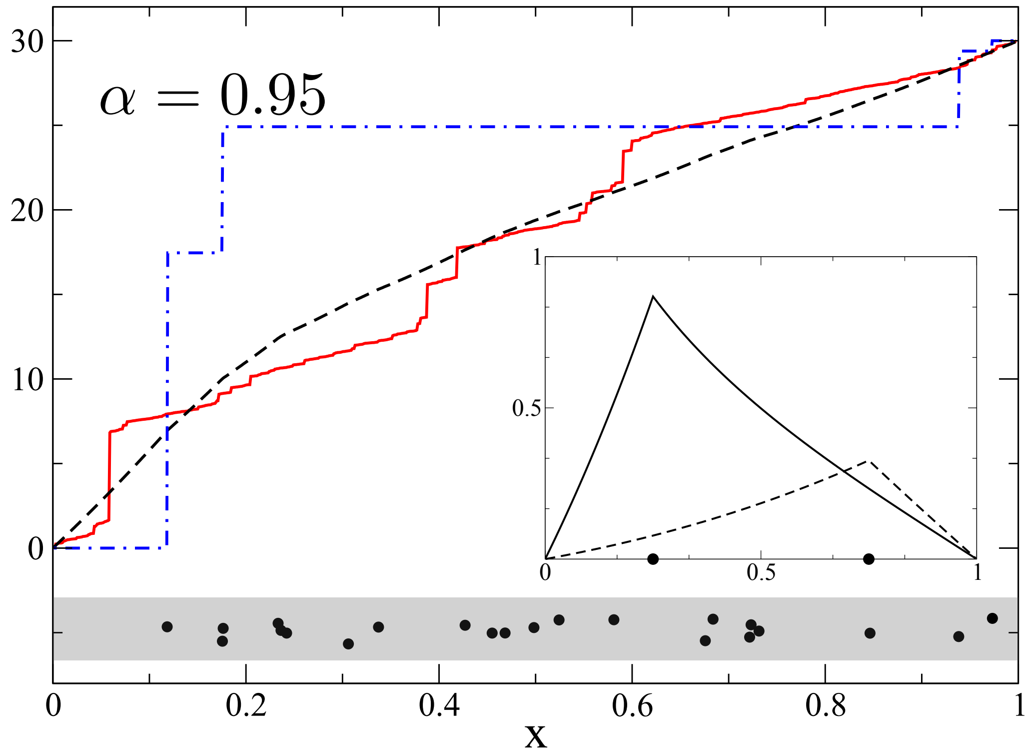
- Neglects the Lévy-process structure
- Solution based on [Brunk (1955), Boswell (1966)]:

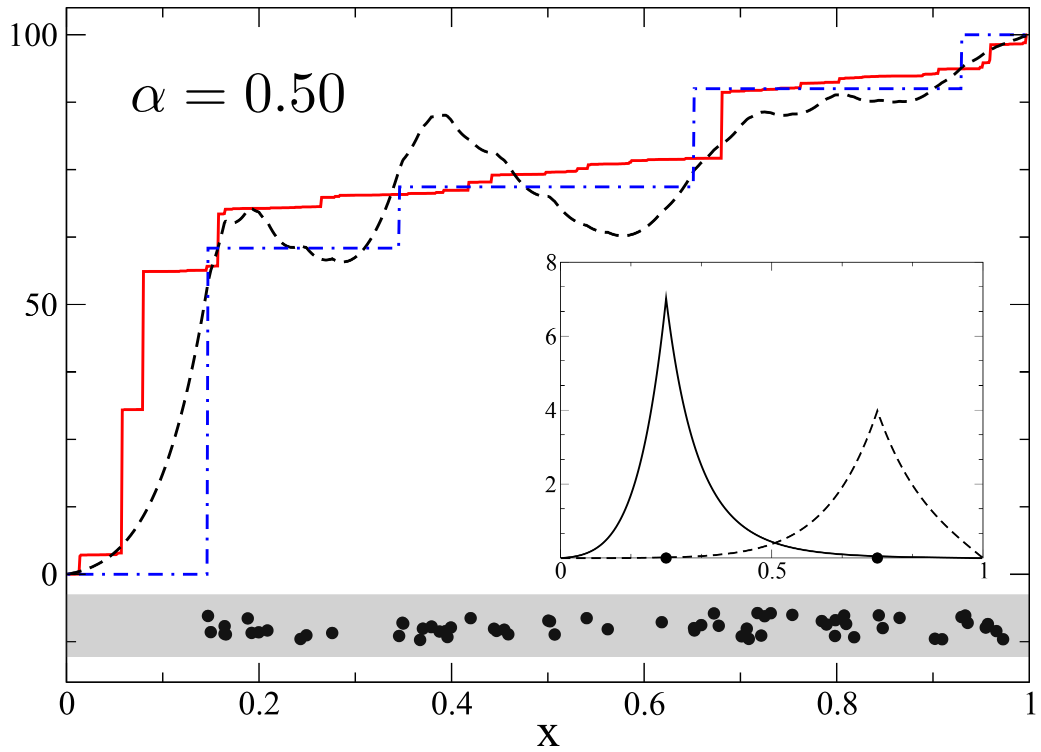
$$\hat{\phi}_{\mathrm{ML}}(x) = \begin{cases} \phi_- & \text{for } 0 = x_0 \le x \le x_1\,, \\ \phi_- \vee \psi_i \wedge \phi_+ & \text{for } x_i \le x \le x_{i+1}, \quad i = 1\ldots,k\,, \\ \phi_+ & \text{for } x = x_{k+1} = 1\,. \end{cases}$$

$$\psi_i = \max_{1\le j\le i} \min_{i\le m\le k} \frac{m+1-j}{x_{m+1}-x_j}$$

Maximum likelihood estimator

- More flexible than the linear estimator: good for small α
- Biased: outperformed by the linear estimator at low marker densities



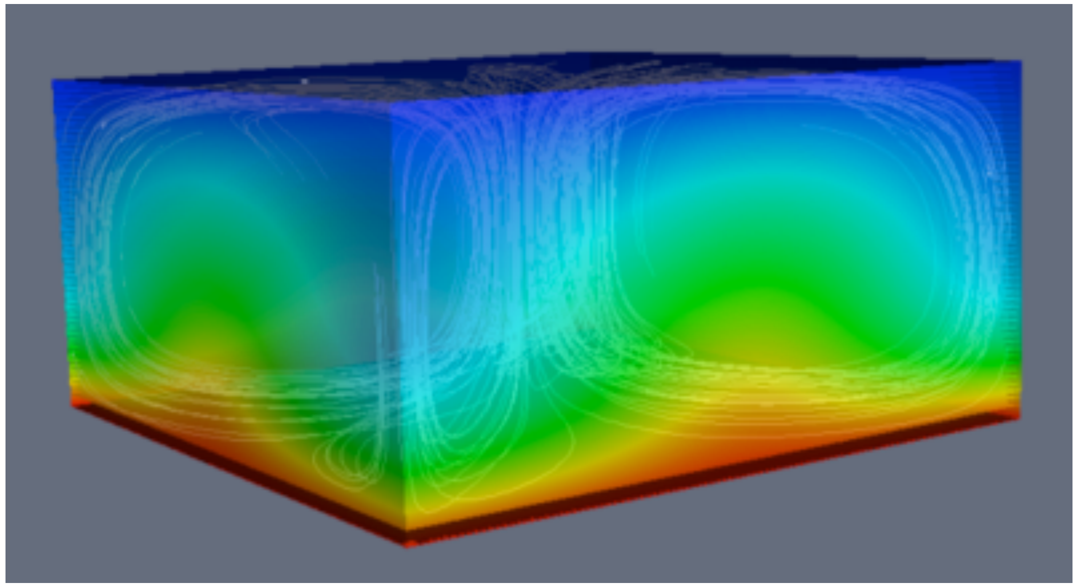


Other systems

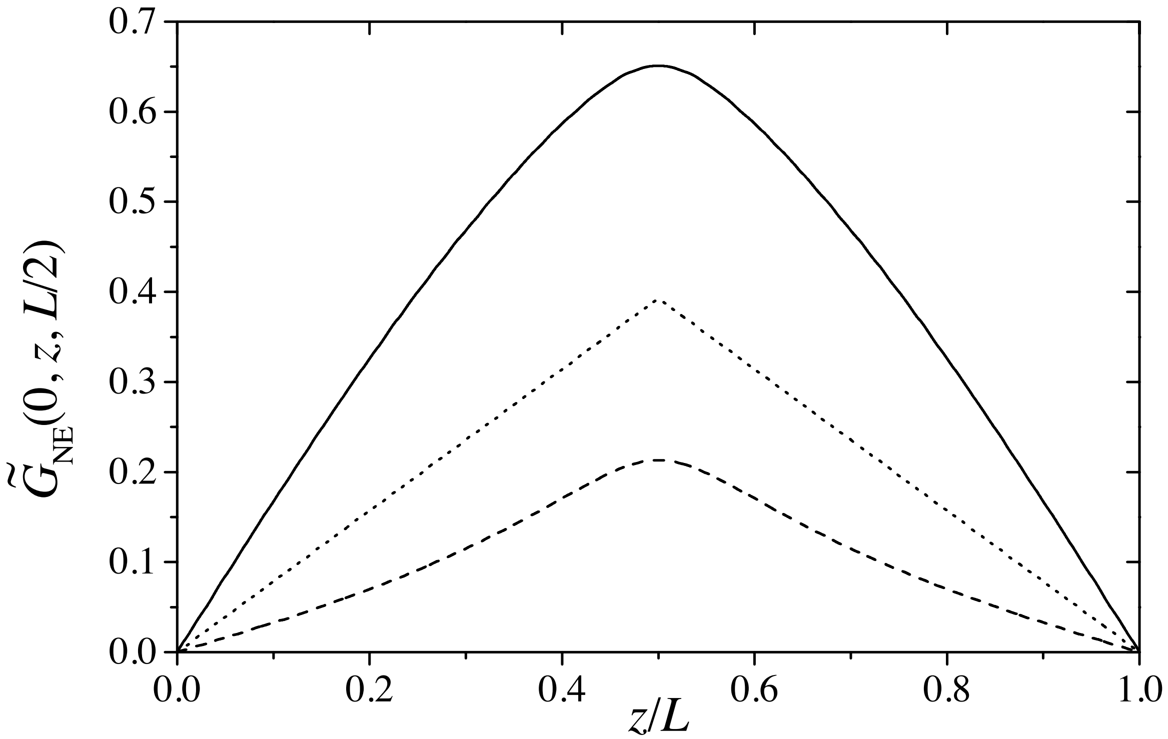
- Symmetric exclusion, KMP: *weak* piecewise linear correlations

$$\mathrm{Cov}_{\mathrm{NE}}(\phi(x), \phi(y)) \propto \frac{c}{n} x(1-y)$$

- Rayleigh-Bénard system: approximately piecewise linear for $R<R_c$

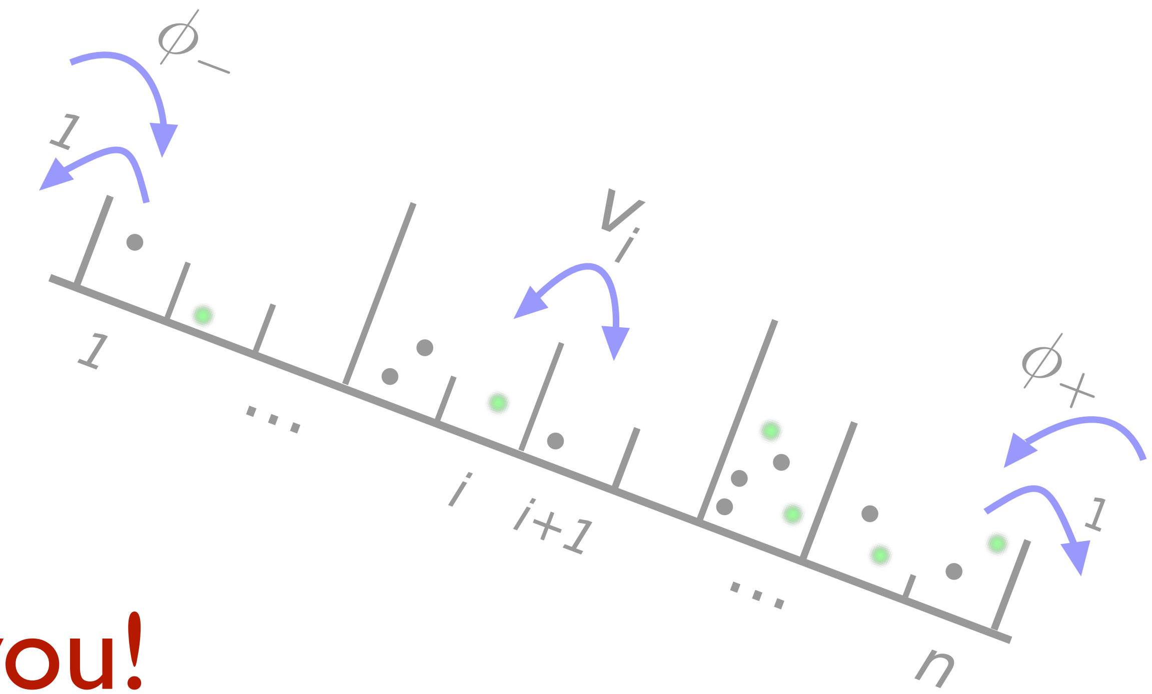


picture from wikipedia (author unknown)



Ortiz de Zarate and Sengers (2001)

Want to know more?



- ArXiv: 0912.0714
- Talk at Queen Mary on Thursday

Thank you!