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Substitution operators

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Abstract

In many areas of science we deal with long sequences of *letters*, that is elements of a finite set called *alphabet*. These sequences undergo various local random perturbations. Let us emphasize that these local perturbations may change the lengths of those sequences to which they are applied. For example, in molecular biology the terms *insertions* and *deletions* mean certain perturbations which increase or decrease by one the length of a sequence of nucleotides.

On the other hand, mathematicians, motivated by problems of statistical physics, study random processes with local iterations, whose states are infinite sequences of letters. For example, we may have only two letters: 0 which means an empty place and 1 which means a particle. Transitions from 0 to 1 and back may be interpreted as birth and death of a particle, but the places which these particles may occupy do not appear or disappear.

Our purpose is to combine these two approaches: we want our configurations to be infinite and at the same time we want to include in our study such local random changes, which include some generalizations of insertions and deletions. Thus we study translation-invariant measures on infinite sequences of letters and their random transformations, which we call *substitutions*. To describe them, let us call by a *word* any finite sequence of letters. The number of letters in a word is called its *length*. A substitution operator is determined by two words G and H (with some restrictions) and a real number $r \in [0,1]$. Informally speaking, a substitution operator substitutes any occurence of the word G by the word H with a probability r (and leaves G unchanged with a probability $1-r$).

One difficulty in dealing with substitution operators is that they are generally non-linear. However, they have another property, almost as good as linearity. Let us denote by M the set of translation-invariant measures on our configuration space. For any two measures $\mu,\ \nu$ we denote

$$convex(\mu,\ \nu) = \{c \cdot \mu + (1-c) \cdot \nu | 0 \le c \le 1\}.$$

Let us call a map $P : M \to M$ *fine* if

$$\forall\ \mu,\ \nu, \lambda : \lambda \in convex(\mu,\ \nu) \Rightarrow P(\lambda) \in convex(P(\mu),\ P(\nu)).$$

One of our main results: all substitution operators are fine.