On the gradient of potential vorticity

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In collaboration with Darryl Holm

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Summary of this lecture

1. Some elementary introductory remarks on the 3D incompressible Euler equations & vortex stretching ;
2. Vortex stretching in **3D incompressible, stratified, rotating Euler equns ;**
3. **Ertel's Theorem** & its consequences in GFD ;
4. **The theme is the role of potential vorticity :** (θ is potential temperature)

$$q = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\theta$$

& the dynamics of ∇q (about which little is known) in the context of

- the 3D incompressible, stratified, rotating Euler equations ;
- the 3D incompressible Navier-Stokes/Boussinesq equations ;
- the hydrostatic Primitive equations of the oceans & atmosphere.

5. The use of these ideas to suggest a new diagnostic for the relative accuracy of Euler codes (in collaboration with Charlie Doering).
6. **Finally :** Do these ideas formally apply to the compressible Euler equations?

Vortex stretching in Euler

For an incompressible fluid ($\text{div}\,\boldsymbol{u} = 0$), the Euler equations are

$$\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p \qquad \text{with} \qquad \frac{D}{Dt} = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$$

With the vorticity as $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$, an alternative is

$$\partial_t \boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{\omega} = -\boldsymbol{\nabla}\left(p - \tfrac{1}{2}u^2\right)$$

$$\partial_t\,\boldsymbol{\omega} = \text{curl}\,(\boldsymbol{u}\times\boldsymbol{\omega})\,, \qquad \text{or} \qquad \frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}\,.$$

The vortex stretching term $\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ can be written as

$$\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} = S\boldsymbol{\omega}$$

where the strain matrix is $S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$.

Vortex "stretching & folding" in Euler

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega}\,, \qquad\qquad \mathrm{div}\,\boldsymbol{u} = 0\,.$$

For short periods the alignment of $\boldsymbol{\omega}$ with eigenvectors of S may lead to exponential stretching/collapse depending on the signs of the eigenvalues $\lambda_S(\boldsymbol{x},\,t)$. This growth/collapse process produces the *fine-scale "crinkles" in the vorticity field*, which is driven down to deeper scales & might end as a finite time singularity.

Need a local math-formulation for the dynamics of higher derivatives of ω such as $\nabla\omega$ – a difficult problem! What do we have?

Global existence of solutions (BKM Theorem 1984) :

There exists a global solution of the 3D Euler equations $\boldsymbol{u} \in C([0,\,\infty];H^s) \cap C^1([0,\,\infty];H^{s-1})$ *for* $s \geq 3$ *if*

$$\int_0^t \|\boldsymbol{\omega}(\cdot\,,\tau)\|_\infty\, d\tau < \infty\,, \qquad\qquad \text{for every } t > 0\,.$$

3D incompressible, stratified, rotating Euler equations

The 3D incompressible Euler equations for an incompressible, stratified, rotating flow ($\boldsymbol{\Omega} = \hat{\boldsymbol{k}}\,\Omega$) in terms of the velocity field $\boldsymbol{u}(\boldsymbol{x},\,t)$ and temperature θ are

$$\frac{D\boldsymbol{u}}{Dt} + \underbrace{2\,(\boldsymbol{\Omega}\times\boldsymbol{u})}_{rotation} + \underbrace{\hat{\boldsymbol{k}}\,\theta}_{buoyancy} = -\boldsymbol{\nabla} p\,,$$

and where the temperature $\theta(\boldsymbol{x},\,t)$ evolves passively according to

$$\frac{D\theta}{Dt} = 0\,.$$

Information about $\boldsymbol{\nabla}\theta$ is needed to determine how $\theta(x,\,t)$ might accumulate into large local concentrations.

Now consider the vorticity $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$ for which $\boldsymbol{\omega}_{rot} = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$ satisfies

$$\frac{D\,\boldsymbol{\omega}_{rot}}{Dt} = \boldsymbol{\omega}_{rot}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^{\perp}\theta \qquad\qquad \boldsymbol{\nabla}^{\perp} = (-\partial_y,\,\partial_x,\,0)$$

The $3D$ Euler equations and Ertel's Theorem

Ertel's Theorem (1942) : If $\boldsymbol{\omega}_{rot}(\boldsymbol{x},\, t)$ satisfies the 3D incompressible, rotating Euler equations then any arbitrary differentiable $\mu(\boldsymbol{x},\, t)$ satisfies

$$\frac{D}{Dt}(\boldsymbol{\omega}_{rot}\cdot\nabla\mu) = \boldsymbol{\omega}_{rot}\cdot\nabla\left(\frac{D\mu}{Dt}\right).$$

The operations $\left[\frac{D}{Dt},\, \boldsymbol{\omega}_{rot}\cdot\nabla\right] = 0$ commute. Thus $\boldsymbol{\omega}_{rot}\cdot\nabla(t) = \boldsymbol{\omega}_{rot}\cdot\nabla(0)$ is a Lagrangian invariant & is "frozen in" (Cauchy 1859).

Proof :

$$\frac{D}{Dt}(\boldsymbol{\omega}_{rot}\cdot\nabla\mu) = \left(\frac{D\,\boldsymbol{\omega}_{rot}}{Dt} - \boldsymbol{\omega}_{rot}\cdot\boldsymbol{\nabla u}\right)\cdot\boldsymbol{\nabla}\mu + \boldsymbol{\omega}_{rot}\cdot\nabla\left(\frac{D\mu}{Dt}\right)$$

Ertel (1942); Truesdell & Toupin (1960); Ohkitani (1993); Kuznetsov & Zakharov (1997); Viudez (2001); Bauer (2000).

Potential Vorticity for rotating stratified Euler

Potential Vorticity is defined as

$$q = \boldsymbol{\omega}_{rot}\cdot\boldsymbol{\nabla}\theta \qquad \text{with} \qquad \frac{D\theta}{Dt} = 0\,.$$

PV is very important in GFD : – see Hoskins, McIntyre, & Robertson (1985).

Take $\mu(\boldsymbol{x},\,t) = \theta$, and thus

$$\begin{aligned}\frac{Dq}{Dt} &= \left(\frac{D\boldsymbol{\omega}_{rot}}{Dt} - \boldsymbol{\omega}_{rot}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)\cdot\boldsymbol{\nabla}\theta + \boldsymbol{\omega}_{rot}\cdot\boldsymbol{\nabla}\left(\frac{D\theta}{Dt}\right)\\ &= \boldsymbol{\nabla}^{\perp}\theta\cdot\boldsymbol{\nabla}\theta = 0\,.\end{aligned}$$

Because $Dq/Dt = 0$, q is a materially conserved quantity.

Thus we have two materially conserved quantities q and θ.

Evolution of the $\mathcal{B}$-field

The vector $\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} q \times \boldsymbol{\nabla}\theta$ satisfies

$$\partial_t \boldsymbol{\mathcal{B}} = \mathrm{curl}\,(\boldsymbol{u} \times \boldsymbol{\mathcal{B}}) \quad \Rightarrow \quad \frac{D\boldsymbol{\mathcal{B}}}{Dt} = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla}\boldsymbol{u}$$

Appears in Kurgansky & Tatarskaya (1987), Kurgansky & Pisnichenko (2000) & Kurgansky (2002) *"Adiabatic Invariants in large-scale atmospheric dynamics"*

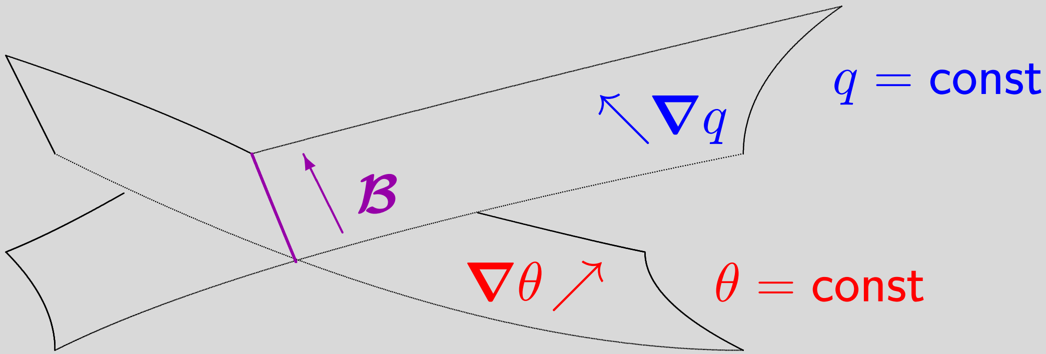
Proof:

$$\begin{aligned}
\partial_t \boldsymbol{\mathcal{B}} &= \partial_t(\boldsymbol{\nabla} q) \times (\boldsymbol{\nabla}\theta) + (\boldsymbol{\nabla} q) \times \partial_t(\boldsymbol{\nabla}\theta) \\
&= -\boldsymbol{\nabla}\big(\boldsymbol{u} \cdot \boldsymbol{\nabla} q\big) \times (\boldsymbol{\nabla}\theta) - (\boldsymbol{\nabla} q) \times [\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{\nabla}\theta)] \\
&= -\{\boldsymbol{u} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} q) + (\boldsymbol{\nabla} q) \cdot \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla} q) \times \boldsymbol{\omega}\} \times (\boldsymbol{\nabla}\theta) \\
&\quad -(\boldsymbol{\nabla} q) \times \{\boldsymbol{u} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla}\theta) + (\boldsymbol{\nabla}\theta) \cdot \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\theta) \times \boldsymbol{\omega}\} \\
&= -\boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{\mathcal{B}} + (\boldsymbol{\nabla} q)(\boldsymbol{\omega} \cdot \boldsymbol{\nabla}\theta) - (\boldsymbol{\nabla}\theta)(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} q) \\
&\quad +(\boldsymbol{\nabla}\theta) \times (\boldsymbol{\nabla} q \cdot \boldsymbol{\nabla}\boldsymbol{u}) - (\boldsymbol{\nabla} q) \times (\boldsymbol{\nabla}\theta \cdot \nabla\boldsymbol{u}) \\
&= \mathrm{curl}\,(\boldsymbol{u} \times \boldsymbol{\mathcal{B}})
\end{aligned}$$

Why are we not surprised?

Consider $\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} q \times \boldsymbol{\nabla}\theta$ where

$$\frac{Dq}{Dt} = 0 \qquad \text{and} \qquad \frac{D\theta}{Dt} = 0$$



$\boldsymbol{\mathcal{B}}$ is tangent to the curve defined by the intersection of $q = \text{const}$ and $\theta = \text{const}$

$$\frac{D\boldsymbol{\mathcal{B}}}{Dt} = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

which is also the equation for the stretching of a line-element $\boldsymbol{\mathcal{B}} \equiv \delta\ell$.

Stretching & folding in the $\mathcal{B}$-field

Because div $\boldsymbol{u} = 0$ and div $\boldsymbol{\mathcal{B}} = 0$ we have

$$\text{curl}\,(\boldsymbol{u} \times \boldsymbol{\mathcal{B}}) = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\mathcal{B}}$$

$$\partial_t \boldsymbol{\mathcal{B}} = \text{curl}\,(\boldsymbol{u} \times \boldsymbol{\mathcal{B}}) \qquad \text{or} \qquad \frac{D\boldsymbol{\mathcal{B}}}{Dt} = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla} \boldsymbol{u}\,,$$

The same as that for $\boldsymbol{\omega}$ & also for the magnetic $\boldsymbol{B}$-field in MHD (Moffatt 1978).

(i) **Thus all the "stretching & folding" properties associated with ω or magnetic field-lines lift over to $\mathcal{B}$ even though $\mathcal{B}$ contains ω, $\boldsymbol{\nabla}\omega$, $\boldsymbol{\nabla}\theta$ and $\boldsymbol{\nabla}^2\theta$ in various forms of projection.**

(ii) Moreover, for any surface $S(\boldsymbol{u})$ moving with the flow $\boldsymbol{u}$, one finds

$$\frac{d}{dt} \int_{S(\boldsymbol{u})} \boldsymbol{\mathcal{B}} \cdot dS = 0\,.$$

Helicity in the $\mathcal{B}$-field

Now define the vector potential $\boldsymbol{\mathcal{A}}$ such that $\boldsymbol{\mathcal{B}} = \text{curl}\,\boldsymbol{\mathcal{A}}$ where

$$\boldsymbol{\mathcal{A}} = \tfrac{1}{2}\big(q\boldsymbol{\nabla}\theta - \theta\boldsymbol{\nabla}q\big) + \boldsymbol{\nabla}\psi\,.$$

The helicity H that results from this definition,

$$H = \int_V \boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{B}}\,dV = \int_V \text{div}\,\big(\psi\boldsymbol{\mathcal{B}}\big)\,dV = \oint_{\partial V}\psi\boldsymbol{\mathcal{B}}\cdot\hat{\boldsymbol{n}}\,dS\,,$$

measures the knottedness of the $\boldsymbol{\mathcal{B}}$ field-lines. $H = 0$ for homogeneous BCs but if realistic topographies were taken into account then there exists the possibility that $H \neq 0$. The boundaries may therefore be an important generating source for helicity, thus allowing the formation of knots and linkages in the $\boldsymbol{\mathcal{B}}$-field.

See **Ohkitani (2007)** for a discussion of helicity-free vorticity fields

$$\boldsymbol{\omega} = \boldsymbol{\nabla}f \times \boldsymbol{\nabla}g$$

. with $Df/Dt = 0$ and $Dg/Dt = 0$.

A remark about higher derivatives

Define the set of scalars q_n as

$$q_n = \boldsymbol{\mathcal{B}}_{n-1} \cdot \boldsymbol{\nabla}\theta\,.$$

Take $q_0 = \theta$ and $\boldsymbol{\mathcal{B}}_0 = \boldsymbol{\omega}$ as the starting point. Also, for $n \geq 1$, define the sequence of vectors

$$\boldsymbol{\mathcal{B}}_n = \boldsymbol{\nabla} q_n \times \boldsymbol{\nabla} q_{n-1}\,.$$

Thus $q_1 = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\theta$ and $\boldsymbol{\mathcal{B}}_1 = \boldsymbol{\nabla} q_1 \times \boldsymbol{\nabla}\theta$, and all the $\boldsymbol{\mathcal{B}}_n$ obey

$$\frac{D\boldsymbol{\mathcal{B}}_n}{Dt} = \boldsymbol{\mathcal{B}}_n \cdot \boldsymbol{\nabla}\boldsymbol{u}\,.$$

Thus all the $\boldsymbol{\mathcal{B}}_n$ have the same stretching equation as $\boldsymbol{\omega}$.

Aside: Is there a connection with the 2D surface QG equations?

To extract a 2D result let $q = z =$ const and $\theta =$ const be material surfaces:

$$\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} z \times \boldsymbol{\nabla}\theta = \hat{\boldsymbol{k}} \times \boldsymbol{\nabla}\theta = -\boldsymbol{\nabla}^{\perp}\theta \,.$$

In $\mathbb{R}^2$ if $\boldsymbol{u}$ is chosen as

$$\boldsymbol{u} = \boldsymbol{\nabla}^{\perp}\psi \qquad \text{with} \qquad \theta = -(-\Delta)^{1/2}\psi$$

then

$$\frac{D\boldsymbol{\mathcal{B}}}{Dt} = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla}\boldsymbol{u} \qquad\qquad \boldsymbol{\mathcal{B}} = -\boldsymbol{\nabla}^{\perp}\theta$$

are the **2D surface quasi-geostrophic (QG) equations discussed by Constantin, Majda & Tabak (1994) who conjectured that strong fronts in numerical calculations might be finite time singularities.**

See Ohkitani & Yamada (1997); Constantin, Nie & Schorghofer (1998); Cordoba (1998); Cordoba, Fefferman & Rodrigo (2004) & Rodrigo (2004).

Does the stretching & folding in the $\mathcal{B}$-field survive dissipation?

Consider the NS–equations coupled to the θ-field

$$\frac{D\boldsymbol{u}}{Dt} + \theta\,\hat{\boldsymbol{k}} = Re^{-1}\Delta\boldsymbol{u} - \boldsymbol{\nabla}p\,, \qquad\qquad \frac{D\theta}{Dt} = \left(\sigma Re\right)^{-1}\Delta\theta\,.$$

then the PV $q = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\theta$ evolves according to

$$\begin{aligned}\frac{Dq}{Dt} &= \left(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)\cdot\boldsymbol{\nabla}\theta + \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\left(\frac{D\theta}{Dt}\right)\\ &= \left(Re^{-1}\Delta\boldsymbol{\omega} - \boldsymbol{\nabla}^{\perp}\theta\right)\cdot\boldsymbol{\nabla}\theta + \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\left((\sigma Re)^{-1}\Delta\theta\right)\\ &= \mathrm{div}\left(Re^{-1}\Delta\boldsymbol{u}\times\boldsymbol{\nabla}\theta + (\sigma Re)^{-1}\boldsymbol{\omega}\Delta\theta\right),\end{aligned}$$

The material property is destroyed but the trick of Haynes & McItyre 1987 gives

$$\partial_t q + \mathrm{div}\,(q\boldsymbol{\mathcal{U}}) = 0\,.$$

$$q\left(\boldsymbol{\mathcal{U}} - \boldsymbol{u}\right) = -Re^{-1}\left(\Delta\boldsymbol{u}\times\boldsymbol{\nabla}\theta + \sigma^{-1}\boldsymbol{\omega}\Delta\theta\right)$$

Remarks on the transport velocity $\boldsymbol{\mathcal{U}}$

Note that from

$$(\partial_t + \boldsymbol{\mathcal{U}}\cdot\boldsymbol{\nabla})q = -q\,\mathrm{div}\,\boldsymbol{\mathcal{U}}$$

and

$$q\big(\boldsymbol{\mathcal{U}} - \boldsymbol{u}\big) = -Re^{-1}\big(\Delta\boldsymbol{u}\times\boldsymbol{\nabla}\theta + \sigma^{-1}\boldsymbol{\omega}\Delta\theta\big)$$

1. q is the PV *density* ;
2. div$\boldsymbol{\mathcal{U}} \neq 0$ but nevertheless div$\boldsymbol{\mathcal{U}} = Re^{-1}$div $\left[q^{-1}\left(\;\ldots\;\right)\right]$;
3. Strictly speaking $\boldsymbol{\mathcal{U}}$ is not a physical velocity (Danielsen 1990), but $\boldsymbol{\mathcal{U}}$ can still be considered as a transport velocity .
4. **What of θ?**

$$\begin{aligned}\partial_t\theta + \boldsymbol{\mathcal{U}}\cdot\boldsymbol{\nabla}\theta &= \partial_t\theta + \boldsymbol{u}\cdot\boldsymbol{\nabla}\theta - Re^{-1}q^{-1}\left\{\Delta\boldsymbol{u}\times\boldsymbol{\nabla}\theta + \sigma^{-1}\boldsymbol{\omega}\Delta\theta\right\}\cdot\boldsymbol{\nabla}\theta\\ &= \partial_t\theta + \boldsymbol{u}\cdot\boldsymbol{\nabla}\theta - \big(\sigma Re\big)^{-1}\Delta\theta\\ &= \mathbf{0}\,.\end{aligned}$$

Formal result for the $\mathcal{B}$-field

$$\partial_t q + \text{div}\,\big(q\boldsymbol{\mathcal{U}}\big) = 0\,, \qquad \partial_t\theta + \boldsymbol{\mathcal{U}}\cdot\boldsymbol{\nabla}\theta = 0\,,$$

and $\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} Q(q) \times \boldsymbol{\nabla}\theta$ satisfies the stretching relation

$$\partial_t\boldsymbol{\mathcal{B}} - \text{curl}\,(\boldsymbol{\mathcal{U}}\times\boldsymbol{\mathcal{B}}) = \boldsymbol{\mathcal{D}}\,.$$

where the divergence-less vector $\boldsymbol{\mathcal{D}}$ is given by

$$\boldsymbol{\mathcal{D}} = -\nabla(qQ'\,\text{div}\,\boldsymbol{\mathcal{U}})\times\nabla\theta\,.$$

and $\boldsymbol{\mathcal{U}}$ is defined as

$$q(\boldsymbol{\mathcal{U}} - \boldsymbol{u}) = -Re^{-1}\left\{\Delta\boldsymbol{u}\times\boldsymbol{\nabla}\theta + \sigma^{-1}\boldsymbol{\omega}\Delta\theta\right\}\,, \qquad q \neq 0\,.$$

Moreover, for any surface $\boldsymbol{S}(\boldsymbol{\mathcal{U}})$ moving with the flow $\boldsymbol{\mathcal{U}}$, one finds

$$\frac{d}{dt}\int_{\boldsymbol{S}(\mathcal{U})}\boldsymbol{\mathcal{B}}\cdot d\boldsymbol{S} = \int_{\boldsymbol{S}(\mathcal{U})}\boldsymbol{\mathcal{D}}\cdot d\boldsymbol{S}\,.$$

Proof of the $\mathcal{B}$-equation

Consider $\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} Q(q) \times \boldsymbol{\nabla}\theta$, then

$$
\begin{aligned}
\partial_t \boldsymbol{\mathcal{B}} &= \partial_t(\nabla Q) \times (\nabla\theta) + (\nabla Q) \times \partial_t(\nabla\theta) \\
&= -\nabla\left[(qQ'\,\mathrm{div}\boldsymbol{\mathcal{U}}) + \boldsymbol{\mathcal{U}}\cdot\nabla Q)\right] \times (\nabla\theta) - (\nabla Q) \times [\nabla(\boldsymbol{\mathcal{U}}\cdot\nabla\theta)] \\
&= -\{\nabla(qQ'\,\mathrm{div}\boldsymbol{\mathcal{U}}) + \boldsymbol{\mathcal{U}}\cdot\nabla(\nabla Q) + (\nabla Q)\cdot\nabla\boldsymbol{\mathcal{U}} + (\nabla Q)\times\boldsymbol{\omega}_U\} \times (\nabla\theta) \\
&\quad - (\nabla Q) \times \{\boldsymbol{\mathcal{U}}\cdot\nabla(\nabla\theta) + (\nabla\theta)\cdot\nabla\boldsymbol{\mathcal{U}} + (\nabla\theta)\times\boldsymbol{\omega}_U\} \\
&= -\nabla(qQ'\,\mathrm{div}\boldsymbol{\mathcal{U}}) \times \nabla\theta - \boldsymbol{\mathcal{U}}\cdot\nabla\boldsymbol{\mathcal{B}} + (\nabla Q)(\boldsymbol{\omega}_U\cdot\nabla\theta) - (\nabla\theta)(\boldsymbol{\omega}_U\cdot\nabla Q) \\
&\quad + (\nabla\theta)\times(\nabla Q\cdot\nabla\boldsymbol{\mathcal{U}}) - (\nabla Q)\times(\nabla\theta\cdot\nabla\boldsymbol{\mathcal{U}}) \\
&= \mathrm{curl}\,(\boldsymbol{\mathcal{U}}\times\boldsymbol{\mathcal{B}}) - \nabla(qQ'\,\mathrm{div}\boldsymbol{\mathcal{U}}) \times \nabla\theta\,.
\end{aligned}
$$

The dynamics of the gradient of potential vorticity

(JDG and D. D. Holm),

J. Phys. A: Math. Theor. **43**, (2010) 172001.

Remarks on re-connection

Because div $\boldsymbol{\mathcal{U}} \neq 0$ but div $\boldsymbol{\mathcal{B}} = 0$ we have

$$\partial_t \boldsymbol{\mathcal{B}} - \text{curl}\,(\boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{B}}) = \boldsymbol{\mathcal{D}}$$

we could write $\boldsymbol{\mathcal{U}} = \boldsymbol{u} + \boldsymbol{v}$ where $\boldsymbol{u}$ is an Euler solution (div $\boldsymbol{u} = 0$), with

$$\boldsymbol{v} = -Re^{-1}\left\{q^{-1}\left[\Delta\boldsymbol{u} \times \boldsymbol{\nabla}\theta + \sigma^{-1}\boldsymbol{\omega}\Delta\theta\right]\right\}$$

where

$$\underbrace{\partial_t \boldsymbol{\mathcal{B}} - \text{curl}\,(\boldsymbol{u} \times \boldsymbol{\mathcal{B}})}_{\text{Euler}} = \underbrace{\boldsymbol{\mathcal{D}} + \text{curl}\,(\boldsymbol{v} \times \boldsymbol{\mathcal{B}})}_{\text{? re-connection ?}} \sim O\Big(Re^{-1}\Big)\{\ \cdot\ \}\,.$$

For numerical calculations on re-connection see: Herring, Kerr and Rotunno, *Ertel's PV in unstratified turbulence*, JAS, **51**, 35 (1994).

The hydrostatic primitive equations

Many simulations of weather, climate and ocean circulation employ a hydrostatic version of the primitive equations (denoted HPE). **The major difference of HPE from the NS-equations lies in the exclusion of the vertical velocity component $w(x, y, z, t)$ in the hydrostatic velocity field :**

$$\boldsymbol{v}(x, y, z, t) = (u,\, v,\, 0)\,.$$

However, w does appear in the full velocity field $\boldsymbol{V}$

$$\boldsymbol{V} = (u,\, v,\, \varepsilon w)$$

where ε is the Rossby number. The vertical velocity w has no evolution equation; it appears only in $\boldsymbol{V} \cdot \boldsymbol{\nabla}$. The z-derivative of the pressure field p and the dimensionless temperature Θ enter the problem through the hydrostatic equation

$$a_0\Theta + p_z = 0\,,$$

The hydrostatic approximation

To address the influence of the material derivative in $(\alpha_a = H/L)$

$$\alpha_a^2\varepsilon^2\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_3\right)w + a_0\Theta + p_z = \varepsilon\mathrm{Re}^{-1}\Delta_3 w\,.$$

Typical values of $\alpha_a^2\varepsilon^2$ for mid-latitude synoptic weather & climate systems are :

$$\begin{aligned}\alpha_a &= H/L \approx 10^4 m/10^6 m \approx 10^{-2}\\ W/U &\approx 10^{-2} m s^{-1}/10 m s^{-1} \approx 10^{-3}\\ \varepsilon &= U/(f_0 L) \approx 10 m s^{-1}/(10^{-4} s^{-1} 10^6 m) \approx 10^{-1}\,.\end{aligned}$$

For mid-latitude large-scale ocean circulation, the corresponding numbers are :

$$\begin{aligned}\alpha_a &= H/L \approx 10^3 m/10^5 m \approx 10^{-2}\\ W/U &\approx 10^{-3} m s^{-1}/10^{-1} m s^{-1} \approx 10^{-2}\\ \varepsilon &= U/(f_0 L) \approx 10^{-1} m s^{-1}/(10^{-4} s^{-1} 10^5 m) \approx 10^{-2}\,.\end{aligned}$$

Thus $\alpha_a^2\varepsilon^2 \approx 10^{-8} - 10^{-6} \ll 1$ **so the hydrostatic approxn is good.**

The velocity field $\boldsymbol{v} = (u,\, v,\, 0)$ obeys the motion equation

$$\varepsilon\big(\partial_t + \boldsymbol{V}\cdot\boldsymbol{\nabla}\big)\boldsymbol{v} + \hat{\boldsymbol{k}}\times\boldsymbol{v} + a_0\hat{\boldsymbol{k}}\Theta = \varepsilon\mathrm{Re}^{-1}\Delta\,\boldsymbol{v} - \boldsymbol{\nabla}p\,,$$

$a_0 = \varepsilon\sigma\alpha_a^{-3}R_a\mathrm{Re}^{-2}$; the aspect ratio $\alpha_a \ll 1$ & R_a is the Rayleigh no.

$$\boldsymbol{V}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{V}\times\boldsymbol{\zeta} + \tfrac{1}{2}\boldsymbol{\nabla}\big(u^2 + v^2\big)\,.$$

The vorticity equation for $\boldsymbol{\zeta} = \mathrm{curl}\,\boldsymbol{v}$ & the dimensionless temperature Θ satisfy

$$\big(\partial_t + \boldsymbol{V}\cdot\boldsymbol{\nabla}\big)\boldsymbol{\zeta} = (\sigma\mathrm{Re})^{-1}\Delta\boldsymbol{\zeta} + \boldsymbol{\zeta}\cdot\boldsymbol{\nabla}\boldsymbol{V} + \mathrm{curl}\,\boldsymbol{f}\,,$$

$$\big(\partial_t + \boldsymbol{V}\cdot\boldsymbol{\nabla}\big)\Theta = (\sigma\mathrm{Re})^{-1}\Delta\Theta + h\,,$$

$$\mathrm{div}\,\boldsymbol{V} = \mathrm{div}\,\boldsymbol{v} + \varepsilon w_z = 0$$

where $\boldsymbol{f} = -\varepsilon^{-1}\big(\hat{\boldsymbol{k}}\times\boldsymbol{v} + a_0\hat{\boldsymbol{k}}\Theta\big)$. The existence and uniqueness of strong solutions of HPE has been proved by **Cao and Titi (2007).** For earlier work see Lions, Temam & Wang (1992, 1995) & Lewandowski (2001).

The equation for B

$$\mathsf{q} = \boldsymbol{\zeta}\cdot\boldsymbol{\nabla}\Theta \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla}\mathsf{Q}\times\boldsymbol{\nabla}\Theta\,,$$

where $\mathsf{Q}(\mathsf{q})$ can be chosen as any smooth function of the potential vorticity q,

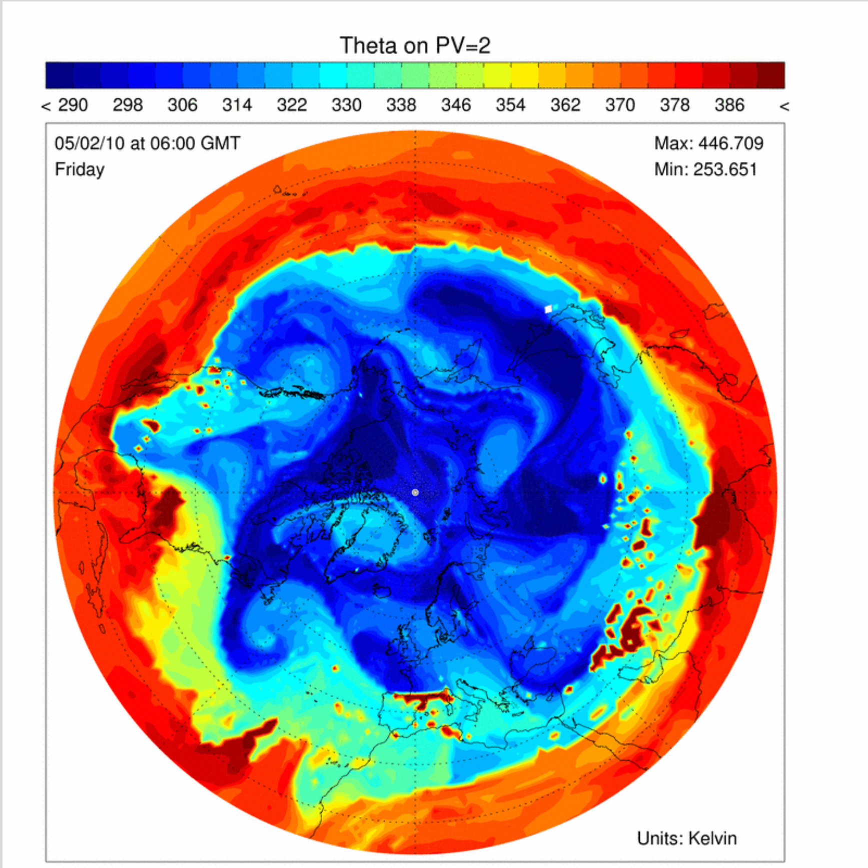
$$\partial_t\,\mathsf{q} + \mathrm{div}\,\big(\mathsf{q}\mathbf{U}\big) = 0\,, \qquad \mathsf{q}\,\big(\partial_t + \mathbf{U}\cdot\boldsymbol{\nabla}\big)\Theta = 0\,.$$

$$\mathsf{q}\,(\mathbf{U} - \boldsymbol{V}) = -\Big\{\big[\mathrm{Re}^{-1}\Delta\boldsymbol{V} + \boldsymbol{f}\big]\times\boldsymbol{\nabla}\Theta + \big[(\sigma\mathrm{Re})^{-1}\boldsymbol{\zeta}\Delta\Theta + h\big]\Big\}\,.$$

$$\partial_t\mathbf{B} - \mathrm{curl}\,(\mathbf{U}\times\mathbf{B}) = \mathbf{D} \qquad \mathrm{div}\,\mathbf{U} \neq 0$$

$$\mathbf{D} = -\boldsymbol{\nabla}\Big(\mathsf{q}\mathsf{Q}'(\mathsf{q})\,\mathrm{div}\,\mathbf{U}\Big)\times\boldsymbol{\nabla}\Theta\,.$$

$$\frac{d}{dt}\int_{\mathsf{S}(\mathbf{U})}\mathbf{B}\cdot d\mathbf{S} = \int_{\mathsf{S}(\mathbf{U})}\mathbf{D}\cdot d\mathbf{S}\,.$$



ECMWF : http://www.met.rdg.ac.uk/Data/CurrentWeather : "Analyzed data" every 6 hrs of the N. Hemisphere showing contours of θ on a level set $q = 2$ in the tropopause. High values of $\nabla\theta$ lie at sharp contour interfaces. Thanks to Brian Hoskins, Paul Berrisford & Nicholas Klingaman.

Euler singularity literature

1. Morf, Orszag & Frisch U 1980; Chorin A J 1982; Brachet, Meiron, Orszag, Nickel, Morf & Frisch 1983; Siggia 1984; Kida 1985; Ashurst & Meiron 1987; Pumir & Kerr 1987; Pumir & Siggia 1990; Grauer & Sideris 1991; Bell & Marcus 1992; Brachet, Meneguzzi, Vincent, Politano & Sulem 1992; Kerr 1993, 2005; Boratav & Pelz 1993, 1995; Pelz 1997; Pelz & Gulak 1997; Grauer, Marliani & Germaschewski 1998; Pelz 2001; Cichowlas & Brachet 2005; Gulak & Pelz 2005; Pelz & Ohkitani 2005; Hou & Li 2006, 2008.

   See the articles in : Proc. of "Euler Equations 250 years on" Aussois conference : Physica D, (2008) **vol 237**.

2. Controversy concerning the development of a singularity :
   (a) **Kerr 1993 & Bustamante & Kerr 2008; Orlandi & Carnevale 2007; Grafke, Homann, Dreher & Grauer 2008 suggest singular behaviour.**
   (b) **Hou & Li R 2006, 2008 suggest double exponential growth.**

3. The Kerr and Hou/Li calculations agree until the last phase.

A diagnostic for Euler codes : by Ch. Doering, DDH & JDG

arXiv:1002.2961v1

We propose the following test on the accuracy of the codes.

$$\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p\,, \qquad\qquad \frac{D}{Dt} = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\,,$$

Now treat θ simply as a passive tracer concentration (with initial data)

$$\frac{D\theta}{Dt} = 0\,.$$

Define

$$q = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\theta \qquad \Rightarrow \qquad \frac{Dq}{Dt} = 0\,.$$

$$\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} q \times \boldsymbol{\nabla}\theta \qquad \Rightarrow \qquad \mathrm{div}\,\boldsymbol{\mathcal{B}} = 0$$

The vector field $\mathcal{B}$ contains information on ω, $\nabla\omega$, $\nabla\theta$ and $\nabla^2\theta$.

$$\partial_t \boldsymbol{\mathcal{B}} = \mathrm{curl}\,(\boldsymbol{u} \times \boldsymbol{\mathcal{B}}) \qquad \text{or} \qquad \frac{D\boldsymbol{\mathcal{B}}}{Dt} = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla}\boldsymbol{u}\,.$$

1. Choose initial data for $\boldsymbol{u}$ and θ, thereby fixing initial data for q and $\boldsymbol{\mathcal{B}}$.
2. Use one's code to evolve $\boldsymbol{u}$ & simultaneously solve $D\theta/Dt = 0$ for θ and $D\boldsymbol{\mathcal{B}}/Dt = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla}\boldsymbol{u}$ for $\boldsymbol{\mathcal{B}}$.
3. Construct $q_1(\cdot\,,\,t) = \boldsymbol{\omega}(\cdot\,,\,t) \cdot \boldsymbol{\nabla}\theta(\cdot\,,\,t)$ from data and then :
   (a) compare the solution for $\boldsymbol{\mathcal{B}}(\cdot,\,t)$ obtained from solving $D\boldsymbol{\mathcal{B}}/Dt = \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\nabla}\boldsymbol{u}$ with $\boldsymbol{\mathcal{B}}_1(\cdot\,,\,t) = \boldsymbol{\nabla}q_1(\cdot\,,\,t) \times \boldsymbol{\nabla}\theta(\cdot\,,\,t)$
   (b) and, furthermore, compare these with $\boldsymbol{\mathcal{B}}_2(\cdot\,,\,t) = \boldsymbol{\nabla}q(\cdot\,,\,t) \times \boldsymbol{\nabla}\theta(\cdot\,,\,t)$ where $q(\cdot\,,\,t)$ is the evolved solution of $Dq/Dt = 0$.
4. For fixed initial data for $\boldsymbol{u}$ this procedure may be implemented for a variety of "markers" $\theta_n(\cdot\,,\,t)$ evolving from distinct initial data $\theta_n(\cdot\,,\,0)$ to diagnose the numerical accuracy in different regions of the flow.

Because $\boldsymbol{\mathcal{B}}$ contains $\boldsymbol{\nabla}\omega$, comparison of $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{B}}_1$ and $\boldsymbol{\mathcal{B}}_2$ tests the accuracy of the computation of some of the small scale flow-structures.

The behaviour of $\|\ln \boldsymbol{\mathcal{B}}\|_\infty$

$$\tfrac{1}{2}\frac{D|\boldsymbol{\mathcal{B}}|^2}{Dt} = \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\nabla u}\cdot\boldsymbol{\mathcal{B}} \qquad \text{or as} \qquad \frac{D\ln|\boldsymbol{\mathcal{B}}|}{Dt} = \hat{\boldsymbol{\mathcal{B}}}\cdot\boldsymbol{\nabla u}\cdot\hat{\boldsymbol{\mathcal{B}}}\,.$$

With $\mathcal{B} = |\boldsymbol{\mathcal{B}}|$, the L^p-norm of $\ln\mathcal{B}$ gives

$$\frac{d}{dt}\|\ln\mathcal{B}\|_p \leq \|\boldsymbol{\nabla u}\|_p\,.$$

For $p=\infty$

$$\|\ln\mathcal{B}(\cdot\,,\,t)\|_\infty = \|\ln\mathcal{B}(\cdot\,,\,0)\|_\infty \int_0^t \|\boldsymbol{\nabla u}(\cdot\,,\,\tau)\|_\infty\,d\tau$$

The BKM Theorem $\Rightarrow \int_0^t \|\boldsymbol{\nabla u}(\cdot\,,\,\tau)\|_\infty\,d\tau$ is controlled by $\int_0^t \|\boldsymbol{\omega}(\cdot\,,\,\tau)\|_\infty\,d\tau$.

Lemma : *If $\int_0^t \|\boldsymbol{\omega}(\cdot\,,\,\tau)\|_\infty\,d\tau$ is bounded at time t then :*

(i) No singularity in $\boldsymbol{\nabla}q$ and $\boldsymbol{\nabla}\theta$ can occur without the simultaneous alignment or anti-alignment of the vectors $\boldsymbol{\nabla}q$ and $\boldsymbol{\nabla}\theta$.

(ii) No alignment or anti-alignment of the vectors $\boldsymbol{\nabla}q$ nor $\boldsymbol{\nabla}\theta$ can occur if they both remain finite.

Compressible Euler

Consider the dimensionless form of the Euler equations in their simplest form

$$\rho\left(\frac{D\boldsymbol{u}}{Dt}\right) = -\boldsymbol{\nabla} p\,, \qquad\qquad \partial_t\rho + \mathrm{div}\,(\rho\,\boldsymbol{u}) = 0\,.$$

Define

$$q := \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\rho$$

where

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\omega}\,\mathrm{div}\,\boldsymbol{u} - \boldsymbol{\nabla}\left(\rho^{-1}\right)\times\boldsymbol{\nabla} p$$

Now formally manipulate :

$$\begin{aligned}\frac{D}{Dt}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\rho) &= \left(\frac{D\boldsymbol{\omega}}{Dt}\right)\cdot\boldsymbol{\nabla}\rho + \boldsymbol{\omega}\cdot\frac{D}{Dt}(\boldsymbol{\nabla}\rho)\\ &= \left(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)\cdot\boldsymbol{\nabla}\rho + \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\left(\frac{D\rho}{Dt}\right)\end{aligned}$$

Thus q satisfies

$$\frac{Dq}{Dt} + \left\{\boldsymbol{\nabla}\left(\rho^{-1}\right) \times \boldsymbol{\nabla} p + \boldsymbol{\omega}\,\mathrm{div}\,\boldsymbol{u}\right\} \cdot \boldsymbol{\nabla}\rho + \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\left(\rho\,\mathrm{div}\,\boldsymbol{u}\right) = 0\,.$$

$$\partial_t q + \mathrm{div}\,\{q\boldsymbol{\mathcal{U}}\} = 0\,, \qquad q \neq 0\,, \qquad \partial_t\rho + \boldsymbol{\mathcal{U}} \cdot \boldsymbol{\nabla}\rho = 0$$

where

$$\boldsymbol{\mathcal{U}} = \boldsymbol{u} + q^{-1}\rho\,\boldsymbol{\omega}\,\mathrm{div}\,\boldsymbol{u}\,, \qquad \mathrm{div}\boldsymbol{\mathcal{U}} \neq 0\,.$$

$$\partial_t\boldsymbol{\mathcal{B}} - \mathrm{curl}\,(\boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{B}}) = \boldsymbol{\mathcal{D}}$$

and where the divergence-less vector $\boldsymbol{\mathcal{D}}$ is given by

$$\boldsymbol{\mathcal{D}} = -\nabla\left(q\,\mathrm{div}\boldsymbol{\mathcal{U}}\right) \times \nabla\theta\,.$$

In the Boussinesq approximation, $\boldsymbol{\mathcal{D}} = 0$.

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