On implicitly constituted incompressible fluids

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Governing equations

Equations in a bounded domain $\Omega \subset \mathcal{R}^3$ for $t \in [0, T]$:

$$\boxed{\operatorname{div} \mathbf{v} = 0}$$

$$\boxed{\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \mathbf{f}}$$

$$\boxed{\mathbf{S} = \mathbf{S}^T}$$

- $\mathbf{v}$ is the velocity of the fluid
- p is the mean normal stress (pressure) $p := -\frac{1}{3} \operatorname{tr} \mathbf{T}$
- $\mathbf{f}$ external body forces ($\equiv \mathbf{0}$)
- $\mathbf{S}$ is the constitutively determined (deviatoric) part of the Cauchy stress

The Cauchy stress: $\boxed{\mathbf{T} = -p\mathbf{I} + \mathbf{S}}$ $\qquad \mathbf{T} = \left(\mathbf{T} - \frac{1}{3}(\operatorname{tr} \mathbf{T})\mathbf{I}\right) + \frac{1}{3}(\operatorname{tr} \mathbf{T})\mathbf{I}$

Point-wisely given constitutive equations

• $\mathbf{D}(\mathbf{v})$ - the symmetric part of the velocity gradient: $2\mathbf{D}(\mathbf{v}) := \nabla\mathbf{v} + (\nabla\mathbf{v})^T$.

• Consider merely **point-wise** relations between $\mathbf{D}$ and $\mathbf{S}$ (or $\mathbf{D}$ and $\mathbf{T}$)
NO integral, differential (rate-type) or stochastic constitutive relations:

$$\boxed{\mathbf{G}(\mathbf{S},\mathbf{D}) = \mathbf{0}} \quad \text{or} \quad \tilde{\mathbf{G}}(\mathbf{T},\mathbf{D}) = \mathbf{0}$$

- robust class of fluids
- justification to adhoc models
- easy incorporation of constraints (as $\operatorname{div}\mathbf{v} = 0$)
- new class of explicit models $\mathbf{D} = \mathbf{H}(\mathbf{S})$ vrs $\mathbf{S} = \tilde{\mathbf{H}}(\mathbf{D})$

Extensions:

$$\boxed{\mathbf{G}(\mathbf{S},\mathbf{D},p,x,t,\text{temperature, density, concentration, etc.}) = \mathbf{0}}$$

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Navier-Stokes fluids

$$\mathbf{T} = -p\mathbf{I} + 2\mu^*\mathbf{D} \iff \mathbf{S} = 2\mu^*\mathbf{D} \iff \mathbf{D} = \frac{1}{2\mu^*}\mathbf{S}$$

$$\begin{aligned} 0 \le \boxed{\xi = \mathbf{S}\cdot\mathbf{D}} \\ &= 2\mu^*|\mathbf{D}|^2 = (2\mu^*)^{-1}|\mathbf{S}|^2 \\ &= \mu^*|\mathbf{D}|^2 + \frac{1}{2}(2\mu^*)^{-1}|\mathbf{S}|^2 \end{aligned}$$

$$\mu^* = 1/2 \qquad \Longrightarrow \quad \mathbf{S} = \mathbf{D}$$

$$\xi = \mathbf{S}\cdot\mathbf{D} = \tfrac{\mathbf{S}\cdot\mathbf{D}}{2} + \tfrac{\mathbf{S}\cdot\mathbf{D}}{2} = \tfrac{1}{2}|\mathbf{D}|^2 + \tfrac{1}{2}|\mathbf{S}|^2$$

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$$\mathbf{T} = -p\mathbf{I} + 2\mu^*|\mathbf{D}|^{r-2}\mathbf{D} \iff \mathbf{S} = 2\mu^*|\mathbf{D}|^{r-2}\mathbf{D} \iff \mathbf{D} = [2\mu^*]^{-\frac{1}{r-1}}|\mathbf{S}|^{\frac{2-r}{r-1}}\mathbf{S}$$

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Also, for all $\mathbf{D}$, $\mathbf{E} \in \mathcal{R}^{3\times 3}$

$$(\tilde{\mathbf{S}}(\mathbf{D}) - \tilde{\mathbf{S}}(\mathbf{E}))\cdot(\mathbf{D}-\mathbf{E}) \geq 0\,, \qquad \text{where} \quad \tilde{\mathbf{S}}(\mathbf{B}) := 2\mu^*|\mathbf{B}|^{r-2}\mathbf{B}$$

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$$\mathbf{S} = (1+|\mathbf{D}|^2)^{r-2}\mathbf{D} \qquad \mathbf{D} = (1+|\mathbf{S}|^2)^{\frac{2-r}{r-1}}\mathbf{S}$$
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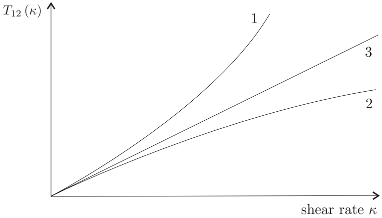
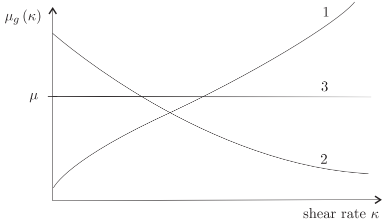
Fluids with shear-rate dependent viscosities

Continuous Explicit Standard Power-Law models ($\mathbf{S} := \mathbf{S}(\mathbf{D})$)

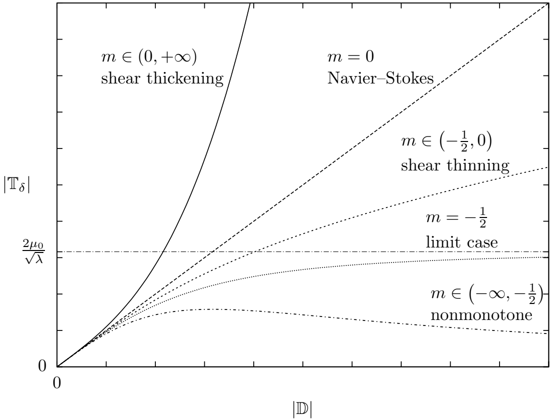
$\boxed{\mu(|\mathbf{D}|^2)}$ $\quad \mathbf{v} = (u(y), 0, 0) \implies |\mathbf{D}(\mathbf{v})|^2 = 1/2|u'|^2 := \kappa$ shear rate

- $\mu(|\mathbf{D}|^2) = 2\mu^* |\mathbf{D}|^{r-2} \qquad 1 < r < \infty$
- $\mu(|\mathbf{D}|^2) = 2\mu_0^* + \mu_1^* |\mathbf{D}|^{r-2} \qquad 1 < r < \infty$
- $\mu(|\mathbf{D}|^2) = 2\mu_0^* (\epsilon + |\mathbf{D}|^2)^{r-2} \qquad 1 \in \mathcal{R}$
- power-law like fluids $\implies$ r-coercivity, $(r-1)$-growth and strict monotonicity
- fluids with shear-rate dependent viscosity





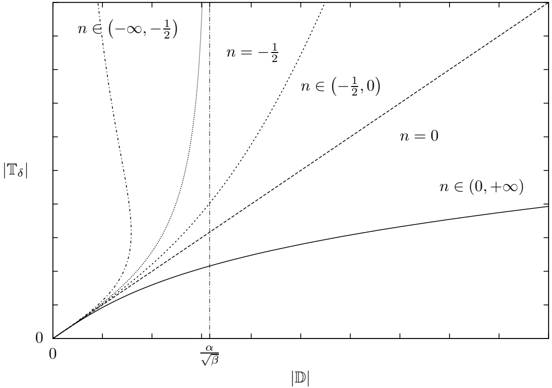
Classical power-law model for various power-law index



$$\mathbf{S} = (1 + |\mathbf{D}|^2)^m \mathbf{D}$$

Stress power-law model for various power-law index

Continuous Explicit Stress Power-law models ($\mathbf{D} := \mathbf{D}(\mathbf{S})$)



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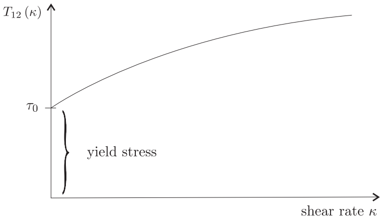
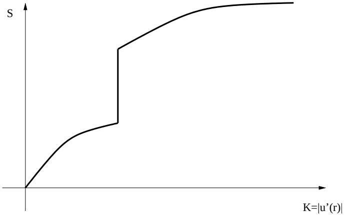
Power-law like fluids with activation criteria/discontinuous stresses





- threshold value for the stress to start flow
- Bingham fluid
- Herschel-Bingham fluid

- drastic changes of the properties when certain criterion is met
- formation and dissolution of blood
- chemical reactions/time scale

Power-law like fluids with activation criteria/II

$$|\mathbf{S}| > \tau^* \quad \text{if and only if} \quad \mathbf{S} = \tau^* \frac{\mathbf{D}}{|\mathbf{D}|} + 2\mu_i(|\mathbf{D}|^2)\mathbf{D}$$
$$|\mathbf{S}| \le \tau^* \quad \text{if and only if} \quad \mathbf{D} = \mathbf{0}$$

is equivalent to

$$\boxed{2\mu_i(|\mathbf{D}|^2)\left(\tau^* + (|\mathbf{S}| - \tau^*)^+\right)\mathbf{D} = \left(|\mathbf{S}| - \tau^*\right)^+\mathbf{S}}$$

Similarly:

$$\begin{aligned} \mathbf{S} &= \mu_\alpha(|\mathbf{D}|^2)\mathbf{D} && \text{if} \quad |\mathbf{D}| < d^* \\ \mathbf{S} &= \mu_\beta(|\mathbf{D}|^2)\mathbf{D} && \text{if} \quad |\mathbf{D}| > d^* \\ \mathbf{S} &= \mu^*\mathbf{D} && \text{if} \quad |\mathbf{D}| = d^*, \end{aligned}$$

μ^* takes any value between $\mu^*_\alpha := \lim_{s\to d^*_-} \mu_\alpha(s)$ and $\mu^*_\beta := \lim_{s\to d^*_+} \mu_\beta(s)$

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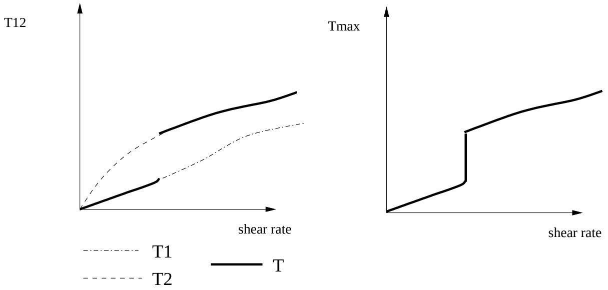
Discontinuous response described by a maximal monotone graph



Perfect plasticity

Ugly "discontinuous" explicit models such as

- Perfect plasticity

$$|\mathbf{D}| = 0 \implies |\mathbf{S}| \leq 1$$
$$|\mathbf{D}| > 0 \implies \mathbf{S} := \frac{\mathbf{D}}{|\mathbf{D}|}$$

can be described by a nice continuous implicit formula

$$||\mathbf{D}|\mathbf{S} - \mathbf{D}| + (|\mathbf{S}| - 1)_+ = 0$$

Implicit theories/I - KR Rajagopal since 2003

Implicit constitutive theory: ability to capture responses of larger set of materials

$$\boxed{\mathbf{G}(\mathbf{T},\mathbf{D}) = \mathbf{0}}$$

Isotropy of the material implies

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{TD}+\mathbf{DT})\\ + \alpha_6(\mathbf{T}^2\mathbf{D}+\mathbf{DT}^2) + \alpha_7(\mathbf{TD}^2+\mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2+\mathbf{D}^2\mathbf{T}^2) = \mathbf{0}\end{aligned}$$

α_i being a functions of

$$\operatorname{tr}\mathbf{T},\ \operatorname{tr}\mathbf{D},\ \operatorname{tr}\mathbf{T}^2,\ \operatorname{tr}\mathbf{D}^2,\ \operatorname{tr}\mathbf{T}^3,\ \operatorname{tr}\mathbf{D}^3,\ \operatorname{tr}(\mathbf{TD}),\ \operatorname{tr}(\mathbf{T}^2\mathbf{D}),\ \operatorname{tr}(\mathbf{D}^2\mathbf{T}),\ \operatorname{tr}(\mathbf{D}^2\mathbf{T}^2)$$

For incompressible fluids

$$\mathbf{T} = \tfrac{\operatorname{tr}\mathbf{T}}{3}\mathbf{I} + \mu(\operatorname{tr}\mathbf{T},\operatorname{tr}\mathbf{D}^2)\mathbf{D}$$

Implicit theories/II

Implicit constitutive theory: ability to include constraints in an easy way

If

$$\boxed{\mathbf{T} = \mathbf{G}_1(\mathbf{D})}$$

isotropy of the material implies

$$\mathbf{T} = \beta_0\mathbf{I} + \beta_1\mathbf{D} + \beta_2\mathbf{D}^2 \qquad \beta_i = \beta_i(\operatorname{tr}\mathbf{D}^2, \operatorname{tr}\mathbf{D}^3)$$

If

$$\boxed{\mathbf{D} = \mathbf{G}_2(\mathbf{T})}$$

isotropy of the material leads to

$$\begin{aligned}\mathbf{D} &= \gamma_0\mathbf{I} + \gamma_1\mathbf{T} + \gamma_2\mathbf{T}^2 \qquad \gamma_i = \gamma_i(\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{T}^2, \operatorname{tr}\mathbf{T}^3)\\ &= \gamma_1\mathbf{S} + \gamma_2(\mathbf{T}^2 - \tfrac{\operatorname{tr}\mathbf{T}^2}{3}\mathbf{I})\end{aligned}$$

Implicit theories/II

Implicit constitutive theory: ability to include constraints in an easy way

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$$\mathbf{T} = \beta_0 \mathbf{I} + \beta_1 \mathbf{D} + \beta_2 \mathbf{D}^2 \qquad \beta_i = \beta_i(\operatorname{tr} \mathbf{D}^2, \operatorname{tr} \mathbf{D}^3)$$

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Implicit formulation - maximal monotone ψ-graph setting

$$(\mathbf{S},\mathbf{D}) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathbf{G}(\mathbf{D},\mathbf{S}) = \mathbf{0}$$

Assumptions ($\mathcal{A}$ is a ψ-maximal monotone graph):

- **(A1)** $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$
- **(A2)** **Monotone graph:** For any $(\mathbf{S}_1,\mathbf{D}_1), (\mathbf{S}_2,\mathbf{D}_2) \in \mathcal{A}$

$$(\mathbf{S}_1 - \mathbf{S}_2) : (\mathbf{D}_1 - \mathbf{D}_2) \geq 0$$

  No strict monotonicity is needed!

- **(A3)** **Maximal graph:** If for some $(\mathbf{S},\mathbf{D})$ there holds

$$(\mathbf{S} - \tilde{\mathbf{S}}) : (\mathbf{D} - \tilde{\mathbf{D}}) \geq 0 \qquad \forall\, (\tilde{\mathbf{S}},\tilde{\mathbf{D}}) \in \mathcal{A}$$

  then

$$(\mathbf{S},\mathbf{D}) \in \mathcal{A}$$

- **(A4)** **ψ-graph:** There are $\alpha \in (0,1]$ and $g > 0$ so that for any $(\mathbf{S},\mathbf{D}) \in \mathcal{A}$

$$\mathbf{S} : \mathbf{D} \geq \alpha(\psi(\mathbf{D}) + \psi^*(\mathbf{S})) - g$$

What is ψ? An excursion to Orlicz spaces

Assume that $\psi : \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}$ is an N - function (if it depends only on the modulus then Young function), i.e.,

- ψ is convex and continuous
- $\psi(\mathbf{D}) = \psi(-\mathbf{D})$
- $$\lim_{|\mathbf{D}|\to 0_+} \frac{\psi(\mathbf{D})}{|\mathbf{D}|} = 0, \qquad \lim_{|\mathbf{D}|\to \infty} \frac{\psi(\mathbf{D})}{|\mathbf{D}|} = \infty$$

We define the conjugate function $\psi*$:

$$\boxed{\psi^*(\mathbf{S}) := \max_{\mathbf{D}} (\mathbf{S}\cdot\mathbf{D} - \psi(\mathbf{D}))}$$

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$$\boxed{\psi^*(\mathbf{S}) := \max_{\mathbf{D}} \left(\mathbf{S}\cdot\mathbf{D} - \psi(\mathbf{D})\right)}$$

What is ψ? An excursion to Orlicz spaces/2

- Young inequality:

$$\boxed{\mathbf{S} : \mathbf{D} \le \psi(\mathbf{D}) + \psi^*(\mathbf{S})}$$

- Orlicz spaces: The Orlicz space $L^\psi(\Omega)^{d\times d}$ is the set of all measurable function $\mathbf{D} : \Omega \to \mathbb{R}^{3\times 3}_{sym}$ such that

$$\lim_{\lambda\to\infty} \int_\Omega \psi(\lambda^{-1}\mathbf{D})\, dx = 0$$

with the norm

$$\|\mathbf{D}\|_{L^\psi} := \inf\{\lambda;\ \int_\Omega \psi(\lambda^{-1}\mathbf{D})\, dx \le 1\}$$

- Hölder inequality

$$\int_\Omega ab\, dx \le 2\|a\|_{L^\psi(\Omega)}\|b\|_{L^{\psi^*}(\Omega)}$$

- Δ_2-condition

$$\psi(2\mathbf{D}) \le C_1\psi(\mathbf{D}) + C_2$$

Maximization of entropy production

In order to specify the constitutive relations, the principle of maximal entropy production (laziness, economy) is used (**KR Rajagopal, A Srinivasa**):

$$\boxed{\mathbf{S} \cdot \mathbf{D} = \xi \geq 0} \qquad (*)$$

Let us assume that $\xi := \xi(\mathbf{D}) \geq 0$ and for some fixed $\mathbf{S}$ we would like to maximize ξ with the constraint (*).

- $\xi := 2\nu_0 |\mathbf{D}|^2$

$$\mathbf{S} = 2\nu_0 \mathbf{D}$$

- $\xi = \nu(|\mathbf{D}|)|\mathbf{D}|^2$

$$\mathbf{S} = \nu(|\mathbf{D}|)\mathbf{D}$$

Maximization of entropy production - dual view

Let us assume that $\xi := \xi(\mathbf{S}) \geq 0$ and for some fixed $\mathbf{D}$ we would like to maximize ξ with the constraint $\xi = \mathbf{S} \cdot \mathbf{D}$.

- $\xi := \frac{2}{\nu_0}|\mathbf{S}|^2$

$$\mathbf{D} = 2\nu_0\mathbf{S}$$

- $\xi = \nu^*(|\mathbf{S}|)|\mathbf{S}|^2$

$$\mathbf{D} = \nu^*(|\mathbf{S}|)\mathbf{S}$$

Maximization of entropy production

Let us assume that $\xi := \xi(\mathbf{D}, \mathbf{S}) \geq 0$ and
(i) for some fixed $\mathbf{S}$ we would like to maximize ξ with the constraint $\xi = \mathbf{S} \cdot \mathbf{D}$
or
(ii) for some fixed $\mathbf{D}$ we would like to maximize ξ with the constraint $\xi = \mathbf{S} \cdot \mathbf{D}$

- $\xi := \frac{|\mathbf{D}|^2 + |\mathbf{S}|^2}{2}$

$$\mathbf{S} = \mathbf{D}$$

- $\xi = \frac{|\mathbf{D}|^r}{r} + \frac{|\mathbf{S}|^{r'}}{r'}$

$$\mathbf{S} = |\mathbf{D}|^{r-1}\mathbf{D}$$

Optimality of ψ and ψ^*

Let us assume that $\xi := \xi_1(\mathbf{D}) + \xi_2(\mathbf{S}) \geq 0$ - not necessarily conjugate and
(i) for some fixed $\mathbf{S}$ we would like to maximize ξ with the constraint $\xi = \mathbf{S} \cdot \mathbf{D}$
or
(ii) for some fixed $\mathbf{D}$ we would like to maximize ξ with the constraint $\xi = \mathbf{S} \cdot \mathbf{D}$
It is the same as maximize ξ_1 with the constraint

$$\mathbf{S} \cdot \mathbf{D} - \xi_2(\mathbf{S}) = \xi_1(\mathbf{D})$$

Hence, for $\mathbf{D}$ - the point where maximum is reached - we interchange the role of $\mathbf{S}$ and $\mathbf{D}$, so at this point

$$\max_{\mathbf{S}} (\mathbf{S} \cdot \mathbf{D} - \xi_2(\mathbf{S})) = \xi_1(\mathbf{D})$$

But it implies

$$\xi_2^*(\mathbf{D}) := \max_{\mathbf{S}} (\mathbf{S} \cdot \mathbf{D} - \xi_2(\mathbf{S})) = \xi_1(\mathbf{D})$$

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Optimality of ψ and ψ^* - more general models

- Non-polynomial growth

$$\mathbf{S} \sim (1+|\mathbf{D}|^2)^{\frac{r-2}{2}} \ln(1+|\mathbf{D}|)\mathbf{D} \implies \psi(\mathbf{D}) \sim |\mathbf{D}|^r \ln(1+|\mathbf{D}|)$$

- Anisotropic case - different growth

$$\mathbf{S}_{ij} \sim |\mathbf{D}|^{r_{ij}-2}\mathbf{D} \implies \psi(\mathbf{D}) \sim \sum |\mathbf{D}|^{r_{ij}}$$

- Different upper and lower growth in principle - ψ has different polynomial upper and lower growth, for $\psi(\mathbf{D}) := \psi(|\mathbf{D}|)$: for certain $1 < r \leq q < \infty$ there are positive constants c_1, c_2, c_3 and c_4 so that

$$\boxed{c_1 s^r - c_2 \leq \psi(s) \leq c_3 s^q + c_4}$$

$\implies$

$$\boxed{c_1^* s^{q'} - c_2^* \leq \psi^*(s) \leq c_3^* s^{r'} + c_4^*}$$

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What is the goal?

- Goal = existence result for as general constitutive relationships as possible
- Using large data apriori estimates (Ω bounded and nice, nice b.c.)
  - Steady case
$$\int_\Omega \psi(\mathbf{D}) + \psi^*(\mathbf{S})\, dx \le C$$
  - Unsteady case
$$\sup_t \|\mathbf{v}\|_2^2 + \int_0^T \int_\Omega \psi(\mathbf{D}) + \psi^*(\mathbf{S})\, dx\, dt \le C$$
- If the function spaces that are under control are "slightly better" than just to guarantee that all terms in weak formulation are meaningful, does there exist a weak solution?

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The goal more precisely

- If the function spaces generated by the apriori large data energy estimates are compactly embedded into L^2, does there exist a weak solution?

Results - power-law like fluid - Explicit

Compact embedding is available if $r > \frac{6}{5}$
Lebesgue and Sobolev spaces

- $r = 2$ **Lerray** (1934)
- $r \geq \frac{11}{5}$ for unsteady, $r \geq \frac{9}{5}$ steady; **Ladyzhenskaya, JL Lions** 60's
- $r \geq \frac{9}{5}$ unsteady; **Bellout, Bloom, Nečas**, Málek, **Růžička** 90's
- $r \geq \frac{8}{5}$ unsteady; **Frehse**, Málek, **Steinhauer** (2000)
- $r > \frac{6}{5}$ steady; **Frehse**, Málek, **Steinhauer** (2003) **Diening**, Málek, **Steinhauer** (2008)
- $r > \frac{6}{5}$ unsteady; **Diening, Růžička, Wolf** (2009)

Results - power-law like fluid - implicit (discontinuous)

Lebesgue and Sobolev spaces

- $r \geq \frac{11}{5}$ - strictly monotone operators - **Gwiazda, Málek, Ṡwierczewska** (2007)
- $r > \frac{9}{5}$ - Herschel-Bulkley model - **Málek, Růžička, Shelukhin**(2005)
- $r > \frac{6}{5}$ steady - strictly monotone graph - **Bulíček, Gwiazda, Málek, Ṡwierczewska** (2009)
- $r > \frac{6}{5}$ unsteady; **Bulíček, Gwiazda**, Málek, Ṡwierczewska-Gwiazda (2010)

Answers - implicit

Orlicz and Orlicz-Sobolev spaces

- subcritical - **Gwiazda, Ṡwierczevska-Gwiazda** et al (2009)
  - strict monotone operators
  - the energy equality holds
  - the term $(\mathbf{v} \otimes \mathbf{v}) \cdot \mathbf{D}(\mathbf{v}) \in L^1$ - the solution is an admissible test function in the weak formulation
- supercritical - **Bulíček, Gwiazda**, Málek, **Ṡwierczewska-Gwiazda** (2010)
  - **(A1)-(A4)** - maximal monotone ψ-graph
  - $\psi(\mathbf{D}) = \psi(|\mathbf{D}|)$;

Methods

- subcritical case
  - **energy equality - v** is an addmissible test function
  - **Minty's method**

  difficulties if ψ does not satisfy Δ_2 condition
- supercritical case
  - **generalized Minty's method**
  - **Lipschitz approximation in Orlicz-Sobolev spaces**

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Generalized Minty's method - Convergence lemma

Assume that

- $\mathcal{A}$ is a maximal monotone ψ-graph satisfying **(A1)–(A4)**
- $\{\mathbf{S}^n\}_{n=1}^{\infty}$ and $\{\mathbf{D}^n\}_{n=1}^{\infty}$ satisfy for some $Q' \subset Q$

$$\begin{aligned} (\mathbf{S}^n, \mathbf{D}^n) &\in \mathcal{A} && \text{for a.a. } (t,x) \in Q', \\ \mathbf{D}^n &\rightharpoonup \mathbf{D} && \text{weakly in } L^{\psi}(Q'), \\ \mathbf{S}^n &\rightharpoonup \mathbf{S} && \text{weakly in } L^{\psi^*}(Q'), \end{aligned}$$

$$\limsup_{n\to\infty} \int_{Q'} \mathbf{S}^n \cdot \mathbf{D}^n \, dx\, dt \leq \int_{Q'} \mathbf{S} \cdot \mathbf{D} \, dx\, dt.$$

Then for almost all $(t,x) \in Q'$ we have

$$(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$$

Lemma - *Local* version

Application of Convergence lemma to Stokes-like problems

To find $(\mathbf{v}, p, \mathbf{S})$ such that

$$\begin{array}{llll} \mathbf{v} \in W_0^{1,r}(\Omega) & p \in L^{r'}(\Omega) & \mathbf{D} \in L^{\psi}(\Omega) & \mathbf{S} \in L^{\psi^*}(\Omega) \\ \operatorname{div} \mathbf{v} = 0 & -\operatorname{div} \mathbf{S} = -\nabla p + \mathbf{f} \text{ in } \mathcal{D}'(\Omega), & & \\ (\mathbf{S}(x), \mathbf{D}(\mathbf{v}(x))) \in \mathcal{A} \text{ for a.a. } x \in \Omega & & & \end{array}$$

Theorem. For any $\mathbf{f} \in \left(W_0^{1,r}\right)^*$ and Ω there is a weak solution to the Problem $\mathcal{P}$.

Proof is based on the following steps:

- Take any selection ($\forall \mathbf{D}$ take one $\mathbf{S}^* := \mathbf{S}^*_{\mathbf{D}}$ so that $(\mathbf{S}^*, \mathbf{D}) \in \mathcal{A}$) and its η-regularizations leads to η-approximations $\mathcal{P}_\eta$
- Galerkin N-approximations of $\mathcal{P}_\eta$ give finitedimensional problems $\mathcal{P}_{N,\eta}$
- For fix $N \in \mathcal{N}$: letting $\eta \to 0$ we obtain $\mathcal{P}_N$
- Uniform estimates follow from **(A4)**; Letting $N \to \infty$ and applying Convergence lemma one concludes $(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$ a.e. in Ω

Extension to subcritical problems

If

- compactness in L^2 is available (which follows if $r > \frac{9}{5}$ for steady problems and $r > \frac{11}{5}$ for unsteady problems)
- $\mathbf{v}$ is an admissible test function in the weak formulation of the problem

Then the results holds for

- $\operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \mathbf{f}$ in Ω
- $\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \mathbf{f}$ in $(0, T) \times \Omega$ and $\mathbf{v}(0, \cdot) = \mathbf{v}_0 \in L^2_{\mathrm{div}}$

Supercritical problems: $\mathbf{v}^n - \mathbf{v}$ is not admissible test function $\implies$ need for its appropriate truncation

Lipschitz approximations of Sobolev function/1

Calderon, Ziemer, Acerbi and Fusco, ...

Theorem. (**Diening**, Málek, **Steinhauer** '08, **Frehse**, Málek, **Steinhauer** '03)
Let $1 < q < \infty$ and $\Omega \in \mathcal{C}^{0,1}$. Let

$$\mathbf{u}^n \in W_0^{1,q}(\Omega)^d \quad \text{and } \mathbf{u}^n \rightharpoonup \mathbf{0} \text{ weakly in } W_0^{1,q}(\Omega)^d.$$

Set

$$K := \sup_n \|\mathbf{u}^n\|_{1,q} < \infty,$$
$$\gamma_n := \|\mathbf{u}^n\|_q \to 0 \qquad (n \to \infty).$$

Let $\theta_n > 0$ be such that (e.g. $\theta_n := \sqrt{\gamma_n}$)

$$\theta_n \to 0 \qquad \text{and} \qquad \frac{\gamma_n}{\theta_n} \to 0 \qquad (n \to \infty).$$

Let $\mu_j := 2^{2^j}$.

Lipschitz approximations of Sobolev function/2

Then there exists a sequence $\lambda_{n,j} > 0$ with

$$\mu_j \leq \lambda_{n,j} \leq \mu_{j+1},$$

and a sequence $\mathbf{u}^{n,j} \in W_0^{1,\infty}(\Omega)^d$ such that for all $j, n \in \mathbb{N}$

$$\begin{aligned} \|\mathbf{u}^{n,j}\|_\infty &\leq \theta_n \to 0 \qquad (n \to \infty), \\ \|\nabla \mathbf{u}^{n,j}\|_\infty &\leq c\,\lambda_{n,j} \leq c\,\mu_{j+1} \end{aligned}$$

and

$$\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\} \subset \Omega \cap \big(\{M\mathbf{u}^n > \theta_n\} \cup \{M(\nabla \mathbf{u}^n) > 2\,\lambda_{n,j}\}\big),$$

and for all $j \in \mathbb{N}$ and $n \to \infty$

$$\begin{aligned} \mathbf{u}^{n,j} &\to \mathbf{0} \quad \text{strongly in } L^s(\Omega)^d \text{ for all } s \in [1,\infty], \\ \mathbf{u}^{n,j} &\rightharpoonup \mathbf{0} \quad \text{weakly in } W_0^{1,s}(\Omega)^d \text{ for all } s \in [1,\infty), \\ \nabla \mathbf{u}^{n,j} &\overset{*}{\rightharpoonup} \mathbf{0} \quad \text{weakly-}* \text{ in } L^\infty(\Omega)^{d\times d}. \end{aligned}$$

Lipschitz approximations of Sobolev function/3

Furthermore, for all $n, j \in \mathbb{N}$

$$|\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}|_d \leq \frac{c\|\mathbf{u}^n\|_{1,q}^q}{\lambda_{n,j}^q} + c\left(\frac{\gamma^n}{\theta^n}\right)^q$$

and

$$\|\nabla \mathbf{u}^{n,j}\,\chi_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}}\|_q \leq c\,\|\lambda_{n,j}\chi_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}}\|_q \leq c\,\frac{\gamma_n}{\theta_n}\,\mu_{j+1} + c\,\epsilon_j,$$

where $\epsilon_j := K\,2^{-j/q}$ vanishes as $j \to \infty$. The constant c depends on Ω.

• based on the continuity of the Hardy-Littlewood maximal function in L^p - In Orlicz space setting it requires Δ_2-condition and *log*-continuity w.r.t. x or (t, x)

• Goal is to avoid using continuity of Hardy-Littelwood maximal function; apply weak $(1, 1)$-estimates

Lipschitz approximations of "Orlicz-Sobolev" functions

Lemma

$\{\mathbf{u}^n\}_{n=1}^\infty$ tends strongly to $\mathbf{0}$ in L^1 and $\{\mathbf{S}^n\}_{n=1}^\infty$ such that

$$\int_\Omega \psi^*(|\mathbf{S}^n|) + \psi(|\nabla \mathbf{u}^n|)\, dx \le C^* \quad (C^* > 1).$$

Then for arbitrary $\lambda^* \in \mathbb{R}_+$ and $k \in \mathbb{N}$ there exists $\lambda^{\max} < \infty$ and there exists sequence of $\{\lambda_n^k\}_{n=1}^\infty$ and the sequence $\mathbf{u}_k^n$ (going to zero) and open sets $E_n^k := \{\mathbf{u}_k^n \neq \mathbf{u}^n\}$ such that $\lambda_n^k \in [\lambda^*, \lambda^{\max}]$ and for any sequence α_k^n

$$\mathbf{u}_k^n \in W^{1,p}, \quad \|\mathbf{D}(\mathbf{u}_k^n)\|_\infty \le C\lambda_n^k,$$

$$|\Omega \cap E_n^k| \le C\frac{C^*}{\psi(\lambda_n^k)},$$

$$\int_{\Omega\cap E_n^k} |\mathbf{S}^n \cdot \mathbf{D}(\mathbf{u}_k^n)|\ dx \le CC^* \left(\frac{\alpha_n^k}{k} + \frac{\alpha_n^k \psi(\lambda_n^k/\alpha_n^k)}{\psi(\lambda_n^k)}\right)$$

Application of Lipschitz approximations to steady flows

- We have suitable approximations $(\mathbf{v}^n, \mathbf{S}^n)$ and their weak limits $(\mathbf{v}, \overline{\mathbf{S}})$, we need to show that $(\overline{\mathbf{S}}, \mathbf{D}(\mathbf{v})) \in \mathcal{A}$
- Test the approximative n- problem by Lipschitz approximation of $\mathbf{v}^n - \mathbf{v}$, i.e., $\mathbf{u}_k^n := (\mathbf{v}^n - \mathbf{v})_k$
- One gets (here $\mathbf{S}$ is such that $(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$

$$\lim_{n\to\infty} \int_{\mathbf{u}_k^n = \mathbf{u}^n} (\mathbf{S}^n - \mathbf{S}) : \mathbf{D}(\mathbf{u}_k^n) \le CC^* \left(\frac{\alpha_n^k}{k} + \frac{\alpha_n^k \psi(\lambda_n^k/\alpha_n^k)}{\psi(\lambda_n^k)} \right)$$

- Hölder inequality gives

$$\lim_{n\to\infty} \int_\Omega |(\mathbf{S}^n - \mathbf{S}) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v})|^\varepsilon \le \int_{\mathbf{u}^n = \mathbf{u}_k^n} + \int_{\mathbf{u}^n \ne \mathbf{u}_k^n} \le \text{ small terms} \to 0$$

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Application of Generalized Minty's method/Convergence lemma

- point-wise convergence of $(\mathbf{S}^n - \mathbf{S}) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v})$ to 0; if the strictly monotone property available the proof is finished
- if only monotone property is available: apply bf biting lemma; Since $(\mathbf{S}^n - \mathbf{S}) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v})$ is bounded in L^1 there is sequence of non-increasing sets $A_{k+1} \subset A_k$, $\lim_{k\to\infty} |A_k| = 0$ such that

$$(\mathbf{S}^n - \mathbf{S}) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v}) \text{ converges to 0 weakly in } L^1(\Omega \setminus A_k)$$

- nonnegativity & point-wise & weak implies strong in $L^1(\Omega \setminus A_k)$
- strong & weak implies for any bounded φ

$$\lim_{n\to\infty} \int_{\Omega\setminus A_k} \mathbf{S}^n \cdot \mathbf{D}(\mathbf{v}^n)\varphi = \int_{\Omega\setminus A_k} \overline{\mathbf{S}} \cdot \mathbf{D}(\mathbf{v})\varphi$$

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- apply Convergence lemma

Concluding Remarks

- Extension of (homogeneous, incompressible) fluids of power-law type to fully implicit constitutive theory characterized by maximal monotone ψ-graphs (not necessarilly of power-law type)
  - Thermodynamically consistent
  - Ability to capture shear thinning/thickening, activation criteria, pressure thickening
- "Complete" large data existence theory both for steady and unsteady flows
- From explicit to implicit, from $(\mathbf{v}, p)$ formulation to $(\mathbf{v}, p, \mathbf{S})$ setting
- Extension 1: from $\psi(|\mathbf{D}|)$ to $\psi(\mathbf{D})$
- Extension 2: unsteady flows, *full thermodynamic setting*

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Newtonian and Fourier fluids

Newtonian homogeneous incompressible fluid

$$\mathbf{S} = 2\nu(e)\mathbf{D}(\mathbf{v}) \quad \text{or } \mathbf{T} = -p\mathbf{I} + 2\nu(e)\mathbf{D}(\mathbf{v})$$

Fourier fluid

$$\mathbf{q} = -\kappa(e)\nabla e$$

"Equivalent" formulation of the balance of energy/1

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\
(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v})
\end{aligned}
$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\
e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v})
\end{aligned}
$$

Helmholtz decomposition $\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \nabla g^{\mathbf{v}}$

Leray's projector $\mathbb{P} : \mathbf{u} \mapsto \mathbf{u}_{\mathrm{div}}$

"Equivalent" formulation of the balance of energy/2

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \mathbb{P} \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \mathbb{P} \operatorname{div} \mathbf{S} &= \mathbf{0} \\ e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \end{aligned}$$

Advantages/Disadvantages

- $+$ pressure is not included into the 2nd formulation
- $+$ minimum principle for e if $\mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \geq 0$
- $-$ $\mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \in L^1$ while $\mathbf{S}\mathbf{v} \in L^q$ with $q > 1$, 2nd form is **derived** form of BE

Newtonian case - $\mathbf{S} = \nu(e)\mathbf{D}(\mathbf{v})$ and $\mathbf{q} = -\kappa(e)\nabla e$ - bounded ν, κ

Theorem 1. (**M. Bulíček, E. Feireisl**, J. Málek '06 and '08) Assume that

- $\nu^* \geq \nu(s) \geq \nu_* > 0$ and $\kappa^* \geq \kappa(s) \geq \kappa_* > 0$
- $\partial\Omega \in C^{1,1}$, $\mathbf{v}_0 \in L^2_{\mathbf{n},div}$, $e_0 \in L^1$, $e_0 \geq C^* > 0$ in Ω, $h \in L^1(0,T)$.

Then for all $T > 0$, $0 \leq \lambda < 1$ there is suitable weak solution $\{\mathbf{v}, p, e\}$

- $\mathbf{v} \in C(0,T; L^2_{weak}) \cap L^2(0,T; W^{1,2}_{\mathbf{n},div})$
- $\operatorname{tr}\mathbf{v} \in L^2(0,T; L^2(\partial\Omega))$
- $p \in L^{\frac{5}{3}}(0,T; L^{\frac{5}{3}})$ $\qquad \int_\Omega p(t,x)dx = g(t)$
- $e \in L^\infty(0,T; L^1) \cap L^m(Q)$, $\nabla e \in L^n(Q)$ with $m \in \langle 1, \frac{5}{3})$, $n \in \langle 1, \frac{5}{4})$

$$(p + \frac{|\mathbf{v}|^2}{2})\mathbf{v} \in L^{\frac{10}{9}}(0,T; L^{\frac{10}{9}}) \qquad \mathbf{D}(\mathbf{v})\mathbf{v} \in L^{\frac{5}{4}}([0,T]; L^{\frac{5}{4}})$$

Weak vrs Suitable weak solutions

Governing equations

$$\operatorname{div} \mathbf{v} = 0 \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = -\nabla p + \operatorname{div}\Big(\nu(\dots)\mathbf{D}(\mathbf{v})\Big)$$

$$\left(e + \frac{|\mathbf{v}|^2}{2}\right)_{,t} + \operatorname{div}\left(\left(e + p + \frac{|\mathbf{v}|^2}{2}\right)\mathbf{v}\right) - \operatorname{div}(\kappa(\dots)\nabla e) = \operatorname{div}\Big(\nu(\dots)\mathbf{D}(\mathbf{v})\,\mathbf{v}\Big)$$

Formulation of the second law of thermodynamics

$$e_{,t} + \operatorname{div}(e\mathbf{v}) - \operatorname{div}(\kappa(e)\nabla e) \geq \nu(e)|\mathbf{D}(\mathbf{v})|^2$$

equivalent to

$$(\tfrac{1}{2}|\mathbf{v}|^2)_{,t} + \nu|\mathbf{D}(\mathbf{v})|^2 \leq \operatorname{div}(\nu(e)\mathbf{D}(\mathbf{v})\mathbf{v} - (p + \tfrac{1}{2}|\mathbf{v}|^2)\mathbf{v})$$

Energy estimates and their consequences

$$\operatorname{div} \mathbf{v} = 0 \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = -\nabla p + \operatorname{div}\Big(\nu(\dots)\mathbf{D}(\mathbf{v})\Big)$$

$$\left(e + \frac{|\mathbf{v}|^2}{2}\right)_{,t} + \operatorname{div}\left(\left(e + p + \frac{|\mathbf{v}|^2}{2}\right)\mathbf{v}\right) - \operatorname{div}(\kappa(\dots)\nabla e) = \operatorname{div}\Big(\nu(\dots)\mathbf{D}(\mathbf{v})\,\mathbf{v}\Big)$$

$$e_{,t} + \operatorname{div}(e\mathbf{v}) - \operatorname{div}(\kappa(\dots)\nabla e)(\geq) = \nu(\dots)|\mathbf{D}(\mathbf{v})|^2$$

- $\int_\Omega \left(e + \frac{|\mathbf{v}|^2}{2}\right)(t,x)dx \leq \int_\Omega \left(e_0 + \frac{|\mathbf{v}_0|^2}{2}\right)dx \quad \Longrightarrow \quad \boxed{e \in L^\infty(L^1) \quad \mathbf{v} \in L^\infty(L^2)}$
- $\int_0^T \nu(\dots)|\mathbf{D}(\mathbf{v})|^2\,dx \leq C \quad \Longrightarrow \quad \boxed{\nabla\mathbf{v} \in L^2(L^2)}$
- $\nu(\dots)|\mathbf{D}(\mathbf{v})|^2 \geq 0, \quad \Longrightarrow \quad \boxed{e > C^* \text{ a.e. } , e \in L^m(L^m)\,,\ \nabla(e)^{(1-s)/2} \in L^2(L^2)}$

Estimates for the pressure

Equation for the pressure: $\boxed{p = (-\Delta)^{-1} \operatorname{div} \operatorname{div}(\mathbf{v} \otimes \mathbf{v} - \nu(\dots)\mathbf{D}(\mathbf{v}))}$

- $\mathbf{v} \in L^\infty(L^2)$ and $\nabla \mathbf{v} \in L^2(L^2) \implies \boxed{\mathbf{v} \in L^{10/3}(L^{10/3}) \text{ and } p \in L^{5/3}(L^{5/3})}$

No-slip BCs for NSEs: L^p maximal regularity for the evolutionary Stokes system (Solonnikov '77, Giga, Giga, Sohr '85)

$$\mathbf{f} \in L^p(L^q) \implies \mathbf{v}_{,t}, \nabla^{(2)}\mathbf{v}, \nabla p \in L^p(L^q)$$

No-slip BC for generalized NSEs with $\nu(e)$ does not hold.

Navier's slip: $\mathbf{v} \cdot \mathbf{n} = 0$ solutions of homogeneous Neumann problem for Laplace equations are admissible

$$(p, \operatorname{div} \varphi) = -(p, -\Delta h) \qquad\qquad -\Delta h = |p|^\alpha p$$

Integrable pressure exists for domains with Lipschitz boundary, etc.

Further consequences of energy estimates

$$\operatorname{div} \mathbf{v} = 0 \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = -\nabla p + \operatorname{div}\Big(\nu(\dots)\mathbf{D}(\mathbf{v})\Big)$$

$$\left(e + \frac{|\mathbf{v}|^2}{2}\right)_{,t} + \operatorname{div}\left(\left(e + p + \frac{|\mathbf{v}|^2}{2}\right)\mathbf{v}\right) - \operatorname{div}(\kappa(\dots)\nabla e) = \operatorname{div}\Big(\nu(\dots)\mathbf{D}(\mathbf{v})\,\mathbf{v}\Big)$$

$$e_{,t} + \operatorname{div}(e\mathbf{v}) - \operatorname{div}(\kappa(\dots)\nabla e)(\geq) = \nu(\dots)|\mathbf{D}(\mathbf{v})|^2$$

- $\mathbf{v}_{,t} \in \left(L^{5/2}(W^{1,5/2})\right)^* = L^{-5/3}(W^{1,-5/3})$
- $e_{,t} \in L^1(W^{-1,q'})$ with $q > 10$
- Aubin-Lions lemma and its generalization: $\mathbf{v}$ and e precompact in $L^m(L^m)$ for $m \in [1, \frac{5}{3})$
- Trace theorem and Aubin-Lions lemma: pre-compactness of $\mathbf{v}$ on $\partial\Omega$

Two steps in the proof of existence

- Stability of the system w.r.t. weakly converging sequences
- Constructions of approximations (several levels), derivation of uniform estimates, weak limits - candidates for the solutions, taking limits in nonlinearities

Newtonian and Fourier fluids - unbounded ν and κ

$\alpha,\ \beta > 0$:

$$\boxed{\nu(e) \sim (1 + e^{\alpha}) \qquad \kappa(e) \sim (1 + e^{\beta})}$$

- NSF: ν decreases with increasing e $\qquad \nu(e) = \nu_0 \exp(\frac{a}{b+e})$
- TKE: ν increases with increasing k $\qquad \nu(k, \ell) = \nu_0 + \nu_1 \ell \sqrt{k}$

Conjecture - Bulíček, Lewandowski, Málek

$$\nu(e) := \nu_0 e^{\alpha} \quad \text{and} \quad \kappa(e) := \mu_0 e^{\alpha}. \tag{1}$$

Conjecture. Let $\alpha \in \mathbb{R}$, ν and μ are of the form (1). Then there exists a $\delta > 0$ and $C^* > 0$ such that for any suitable weak solution $(\mathbf{v}, p, e)$ to NSF with unbounded material coefficients the following implication holds:
If

$$\int_{-1}^{0} \int_{B_1(0)} \nu(e) |\mathbf{D}(\mathbf{v})|^2 \, dx \, dt \leq \delta$$

then

$$|\mathbf{v}(t,x)| \leq C^* \quad \text{in } (-\frac{1}{2}, 0) \times B_{\frac{1}{2}}(0).$$

For $\alpha \equiv 0$: NSEs - Conjecture holds (CKN '82, Vasseur '07). Statement:

If Conjecture holds for $\alpha \geq \frac{1}{6}$ then the corresponding suitable weak solution has bounded velocity

Scaling property of NSF

$(\mathbf{v}, p, e)$ solve NSF on some neighborhood of $(0,0)$: $(-\ell_0^A, 0) \times B_{\ell_0}(0)$ with some $A > 0$ and $\ell_0 > 0$. Then

$$\mathbf{v}_\ell(t,x) := \ell^B \mathbf{v}(\ell^A t, x) \quad p_\ell(t,x) \quad := \ell^{2B} p(\ell^A t, x) \quad e_\ell(t,x) := \ell^{2B} e(\ell^A t, \ell x)$$

with

$$A := \frac{2-2\alpha}{1-2\alpha}, \qquad B := \frac{1}{1-2\alpha} \qquad \alpha \neq \frac{1}{2}$$

solves NSF in $(-1,0) \times B_1(0)$. Conjecture applied on $(\mathbf{v}_\ell, e_\ell)$ leads to

$$\begin{aligned}
\delta &\geq \int_{-1}^{0} \int_{B_1(0)} \nu(k_\ell) |\mathbf{D}(\mathbf{v}_\ell)|^2 \, dx \, dt \\
&= \int_{-1}^{1} \int_{B_1(0)} \ell^{2B\alpha+2B+2} (k(\ell^A t, \ell x))^\alpha |\mathbf{D}(\mathbf{v}(\ell^A t, \ell x))|^2 \, dx \, dt \\
&= \ell^{\frac{6\alpha-1}{1-2\alpha}} \int_{-\ell^A}^{0} \int_{B_\ell(0)} k^\alpha |\mathbf{D}(\mathbf{v})|^2 \, dx \, dt.
\end{aligned}$$

Existence result - unbounded ν and κ

Theorem 2. (**M. Bulíček, R. Lewandowski**, J. Málek, 2010) Assume that ν, κ fulfil the growth condition

$$\boxed{\beta \geq 0 \qquad\qquad 0 \leq \alpha < \frac{2\beta}{5} + \frac{2}{3}}$$

Then for any set of data there exists (suitable) weak solution $(\mathbf{v}, p, e)$ to the system in consideration, completed by Navier's slip boundary conditions, such that

$$\mathbf{v} \in C(0,T; L^2_{weak}) \cap L^2(0,T; W^{1,2}_{\mathbf{n},div}) \qquad \mathrm{tr}\,\mathbf{v} \in L^{8/5}(0,T; L^{8/5}(\partial\Omega))$$

$$p \in L^q(0,T; L^q) \qquad q < \min\{5/3, 2 - 2\alpha/(\alpha+\beta+5/3)\}$$

$$e \in L^\infty(0,T; L^1), \qquad e \geq 0 \qquad \text{and} \qquad (1+e)^s - 1 \in L^2(0,T; W^{1,2}) \qquad s < \frac{\beta+1}{2}$$

$$\mathbf{v}_{,t} \in L^{q'}(0,T; W^{-1,q'}_{\mathbf{n}}) \qquad e_{,t} \in \mathcal{M}(0,T; W^{-1,10/9}) \qquad E_{,t} \in L^1(0,T; W^{-1,10/9})$$

$$\lim_{t\to 0+} (\|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 + \|e(t) - e_0\|_1) = 0$$

Maximal L^2-regularity for Stokes-Fourier system

Navier-Stokes sytem

$$\operatorname{div} \mathbf{v} = 0 \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div}(2\nu_0 \mathbf{D}(\mathbf{v})) + \nabla p = \mathcal{F}$$

Maximal L^q-regularity for the evolutionary (linear) Stokes system

$$\operatorname{div} \mathbf{v} = 0 \qquad \mathbf{v}_{,t} - \operatorname{div}(2\nu_0 \mathbf{D}(\mathbf{v})) + \nabla p = \mathcal{F}$$

$$\mathcal{F} \in L^r(0, T; L^r(\Omega)^d) \implies \mathbf{v}_{,t}, \nabla p, \nabla^2 \mathbf{v} \in L^r(0, T; L^r(\Omega))$$

Q: Maximal L^q-regularity for the evolutionary (non-linear) Stokes-Fourier

$$\boxed{\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} - \operatorname{div}(\nu(e)\mathbf{D}(\mathbf{v})) + \nabla p &= \mathcal{F} \\ e_{,t} - \operatorname{div}(\kappa(e)\nabla e) &= \nu(e)|\mathbf{D}(\mathbf{v})|^2 \end{aligned}}$$

Simplifications: periodic problem, $\kappa(e) = 1$, $\nu_0 \le v(e) \le \nu_1$, $r = 2$

L^2-maximal regularity like result for Stokes-Fourier Eqs

Theorem 3. (**M. Bulíček, P. Kaplický**, J. Málek *Applicable Analysis 2010*)
Let $d \geq 2$. Let

$$\mathcal{F} \in L^2(0,T;L^2(\Omega)^d) \quad \sqrt{e_0} \in W^{1,2}(\Omega) \quad e_0 \geq e_{min}$$

$$\mathbf{v}_0 \in W^{1,2}_{\mathrm{div}}(\Omega)^d := \{\mathbf{u} \in W^{1,2}(\Omega)^d;\ \mathrm{div}\,\mathbf{u} = 0,\ \int_\Omega \mathbf{u} = \mathbf{0}\}$$

Assume that $\nu \in \mathcal{C}^{0,1}(\mathbb{R}_+)$ fulfills ($\varepsilon > 0$)

$$-\frac{2}{15(s - e_{min} + \varepsilon)} \leq \frac{\nu'(s)}{\nu(s)} \leq \frac{1}{40(s - e_{min} + \varepsilon)} \qquad \text{for all } s \in (e_{min}, \infty)$$

Then there exists a triple $(\mathbf{v}, e, p)$ that solves (SF) such that

$$\mathbf{v} \in L^\infty(0,T;W^{1,2}_{\mathrm{div}}(\Omega)^d) \cap L^2(0,T;W^{2,2}(\Omega)^d) \cap W^{1,2}(0,T;L^2(\Omega)^d)$$

$$e \in L^{\frac{d+2}{d}}(0,T;W^{2,\frac{d+2}{d}}(\Omega)) \cap W^{1,\frac{d+2}{d}}(0,T;L^{\frac{d+2}{d}}(\Omega))$$

$$\sqrt{e} \in L^\infty(0,T;W^{1,2}(\Omega)) \qquad p \in L^2(0,T;W^{1,2}(\Omega))$$

Example

$$-\frac{2}{15(s-e_{min}+\varepsilon)} \leq \frac{\nu'(s)}{\nu(s)} \leq \frac{1}{40(s-e_{min}+\varepsilon)} \qquad \text{for all } s \in (e_{min}, \infty)$$

If $\boxed{\nu(e) = \nu_0 \exp(\frac{a}{b+e})}$ that the above conditions holds if $e_{min} > 2a - b$