Nonlinear Stochastic Markov Processes and Modeling Uncertainty in Populations

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Summary:

- Consider an alternative approach to the use of nonlinear stochastic Markov processes in modeling uncertainty in populations.
- alternate formulations $\equiv$ probabilistic structures on family of deterministic dynamical systems, yield pointwise equivalent population densities–lead to fast efficient calculations in inverse problems.
- Here present class of stochastic formulations for which an alternate representation is readily found.

Summary of Previous Findings

We compared the *probabilistic rate distribution (PRD)* model approach to incorporating the class rate uncertainty into a structured population model with the *stochastic rate model (SRM)* formulation. The earlier discussions indicate that these two stochastic and probabilistic formulations are *conceptually* quite different. One entails imposing a probabilistic structure on the set of possible transition rates permissible in the entire population while the other involves formulating transition as a stochastic diffusion process. *However, the analysis in [Shrimp2] reveals that in some cases the structure distribution (the probability density function of $X(t)$) obtained from the stochastic rate model is* **exactly** *the same as that obtained from* *the PRD model.* For example, if we consider the two models

$$
\begin{aligned}
&\text{stochastic formulation:}\\
&dX(t) = b_0(X(t)+c_0)dt + \sqrt{2t}\sigma_0(X(t)+c_0)dW(t)\\
&\text{probabilistic formulation:}\\
&\tfrac{dx(t;b)}{dt} = (b-\sigma_0^2 t)(x(t;b)+c_0),\\
&\qquad\qquad\qquad\qquad b \in \mathbb{R} \text{ with } B \sim \mathcal{N}(b_0,\sigma_0^2),
\end{aligned}
\tag{1}
$$

and assume their initial structure distributions are the same, then we obtain at each time t the same structure distribution from these two distinct formulations. Here b_0, σ_0 and c_0 are positive constants (for application purposes), and B is a normal random variable with b a realization of B. Moreover, by using the same analysis as in

[Shrimp2] we can show that if we compare

stochastic formulation:

$$dX(t) = (b_0 + \sigma_0^2 t)(X(t) + c_0)dt + \sqrt{2t}\sigma_0(X(t) + c_0)dW(t) \tag{2}$$

probabilistic formulation:

$$\frac{dx(t;b)}{dt} = b(x(t;b) + c_0), \quad b \in \mathbb{R} \text{ with } B \sim \mathcal{N}(b_0, \sigma_0^2),$$

with the same initial structure distributions, then we can also obtain at each time t the same structure distribution for these two formulations. In addition, we see that both the stochastic rate models and the probabilistic rate models in (1) and (2) reduce to the same deterministic growth model $\dot{x} = b_0(x + c_0)$ when there is no uncertainty or variability in rate (i.e., $\sigma_0 = 0$) even though both models in (2) *do not* satisfy the mean rate dynamics

$$\frac{d\mathrm{E}(X(t))}{dt} = b_0(\mathrm{E}(X(t)) + c_0) \tag{3}$$

while both models in (1) *do*. This last observation was critical in the early efforts of [Shrimp2, Shrimp3] which were derived under the additional constraint that (3) must hold. This was motivated by available shrimp data of longitudinal measurements of *average* shrimp weight (in gms), i.e., an observation of $\bar{x}(t) = \mathrm{E}(X(t))$. In this earlier work it was found that an affine growth law $\frac{d\bar{x}(t)}{dt} = g(\bar{x}(t)) = b_0(\bar{x}(t) + c_0)$ yielded a good fit to this data for early shrimp growth. This led to a search for equivalent mathematical representations which also satisfied this extra condition.

More specifically, one can prove that the formulations in (1) generate stochastic processes $X(t)$ which both satisfy the mean rate dynamics

(3) and yield processes

$$X(t) = -c_0 + (X_0 + c_0)Y(t)$$

where

$$Y_{PRD}(t) = \exp(Bt - \tfrac{1}{2}\sigma_0^2 t^2), \ \text{where } B \sim \mathcal{N}(b_0, \sigma_0^2). \tag{4}$$

$$Y_{SRM}(t) = \exp\left((b_0 t - \tfrac{1}{2}\sigma_0^2 t^2) + \sigma_0 \int_0^t \sqrt{2\tau}\, dW(\tau) \right). \tag{5}$$

Moreover it can be shown that for each time t, both $Y_{PRD}(t)$ and $Y_{SRM}(t)$ are log normally distributed with identical means and variances. Thus under the additional reasonable assumption (trivially true for non-random initial data) that the random variables X_0 and each of $Y_{PRD}(t)$ and $Y_{SRM}(t)$ are independent we find that each of the stochastic processes derived from (1) possess at each time t the same distribution. That is, at each time t each of the processes $X(t)$ have the *same probability density*.

Finally, the two stochastic processes are *NOT* the same. This can be seen immediately from (4) and (5), but also from a direct calculation of the covariances for Y_{PRD} and Y_{SRM}.

In establishing the above results and to discuss the corresponding covariances, the following relationship between normal distribution and log-normal distribution [CasBerg, page 109] is heavily used.

Lemma 1. *If* $\ln Z \sim \mathcal{N}(\mu, \sigma^2)$, *then* Z *is log-normally distributed, where its probability density function* $f_Z(z)$ *is defined by*

$$f_Z(z) = \frac{1}{z\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln z - \mu)^2}{2\sigma^2}\right),$$

and its mean and variance are given as follows

$$E(Z) = \exp(\mu + \tfrac{1}{2}\sigma^2), \;\; Var(Z) = [\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2).$$

In our subsequent arguments we shall also need the following basic result on the process generated by Ito integrals of Wiener processes that can be found in [Klebner, Sec 4.3, Thm 4.11].

Lemma 2. *For a non-random function $f \in L_2(0,T)$, the Ito integrals $Q(t) = \int_0^t f(s)dW(s)$ for $0 < t \leq T$ yield a Gaussian stochastic process with pointwise distributions $\mathcal{N}\left(0, \int_0^t f^2(s)ds\right)$. Moreover, $Cov(Q(t), Q(t+\xi)) = \int_0^t f^2(s)ds$ for all $\xi \geq 0$.*

We can use these lemmas to find the covariance function of the stochastic processes $Y_{PRD}(t)$ in the probabilistic formulation and $Y_{SRM}(t)$ in the stochastic formulation.

Probabilistic formulation: In this case we have

$$Y_{PRD}(t) = \exp(Bt - \tfrac{1}{2}\sigma_0^2 t^2), \text{ where } B \sim \mathcal{N}(b_0, \sigma_0^2).$$

By Lemma 1, we find immediately

$$\mathrm{E}(Y_{PRD}(t)) = \exp(b_0 t). \tag{6}$$

Then using Lemma 1 and (6) we find the covariance function for the process $\{Y(t)\} = \{Y_{PRD}(t)\}$ given by

$$\begin{aligned}
\mathrm{Cov}(Y(t), Y(s)) &= \mathrm{E}(Y(t)Y(s)) - \mathrm{E}(Y(t))\mathrm{E}(Y(s)) \\
&= \mathrm{E}\left\{\exp\left(B(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2)\right)\right\} - \exp(b_0(t+s)) \\
&= \exp\left(b_0(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2) + \tfrac{1}{2}\sigma_0^2(t+s)^2\right) \\
&- \exp(b_0(t+s)) \\
&= \exp\left(b_0(t+s) + st\sigma_0^2\right) - \exp(b_0(t+s)) \\
&= \exp(b_0(t+s))\left[\exp\left(st\sigma_0^2\right) - 1\right].
\end{aligned}$$

Stochastic formulation: We found

$$Y_{SRM}(t) = \exp\left((b_0 t - \tfrac{1}{2}\sigma_0^2 t^2) + \sigma_0 \int_0^t \sqrt{2\tau} dW(\tau) \right).$$

Let $Q(t) = \sigma_0 \int_0^t \sqrt{2\tau} dW(\tau)$. Then by Lemma 2, we have that $\{Q(t)\}$ is a Gaussian process with zero mean and covariance function given by

$$\mathrm{Cov}(Q(t), Q(s)) = \sigma_0^2 \min\{t^2, s^2\}. \tag{7}$$

Using Lemma 1 and (7) we find that

$$\mathrm{E}(Y_{SRM}(t)) = \exp(b_0 t). \tag{8}$$

Note that for any fixed s and t, both $Q(t)$ and $Q(s)$ are Gaussian distributions with zero mean. Hence, $Q(t) + Q(s)$ is also a Gaussian distribution with zero mean and variance defined by

$$\begin{aligned} \operatorname{Var}(Q(t)+Q(s)) &= \operatorname{Var}(Q(t))+\operatorname{Var}(Q(s))+2\operatorname{Cov}(Q(t),Q(s)) \\ &= \sigma_0^2\left(t^2+s^2+2\min\{t^2,s^2\}\right). \end{aligned} \tag{9}$$

Now we use Lemma 1, along with equations (8) and (9) to find the covariance function of $\{Y(t)\} = \{Y_{SRM}(t)\}$.

$$
\begin{aligned}
&\mathrm{Cov}(Y(t), Y(s)) \\
&\quad = \mathrm{E}(Y(t)Y(s)) - \mathrm{E}(Y(t))\mathrm{E}(Y(s)) \\
&\quad = \mathrm{E}\left\{\exp(b_0(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2) + Q(t) + Q(s))\right\} - \exp(b_0(t+s)) \\
&\quad = \exp\left(b_0(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2) + \tfrac{1}{2}\sigma_0^2\left(t^2+s^2+2\min\{t^2,s^2\}\right)\right) \\
&\quad - \exp(b_0(t+s)) \\
&\quad = \exp\left(b_0(t+s) + \sigma_0^2\min\{t^2,s^2\}\right) - \exp(b_0(t+s)) \\
&\quad = \exp(b_0(t+s))\left[\exp\left(\sigma_0^2\min\{t^2,s^2\}\right) - 1\right].
\end{aligned}
$$

In summary, while the two formulations of (1) generally lead to different processes, one can argue that they are *equivalent* in the sense that *they possess the same probability density at any time t*. We refer to this as *pointwise equivalence in density*. This density must satisfy the corresponding Fokker-Planck or Forward Kolmogorov equation for the stochastic formulation in (1). Thus if one wishes to obtain a numerical solution of such a Fokker-Planck equation, one possibility is to consider the numerical solution of the equivalent but more readily solved CRDSS formulation of (1). For the particular systems of (1) and (2), this approach was demonstrated to be a computationally advantageous strategy in [BaDavHu].

Natural research question: Are there general classes of Fokker-Planck systems that can be converted to an equivalent (in the distributional sense described above) CRDSS system and hence efficiently solved numerically for the desired probability density function? A positive answer to this question is given in *Banks-Hu, Jan 2011.*

Equivalence between Probabilistic and Stochastic Formulations

In this section, we turn to several cases for which one can establish the desired equivalence between the probabilistic and stochastic formulations given above. The probabilistic formulations we consider here involve a finite-dimensional parameter family of structure rates of change; that is, all the subsystems have the same functional form $g(x, t; b_0, b_1, \ldots, b_{n-1}) = g(x, t; \bar{b})$ for the structure rates of change but the values of parameters $\bar{b} = (b_0, b_1, \ldots, b_{n-1})$ vary across the system.

Case I

In the first case we derive conditions under which the probabilistic and stochastic formulations generate stochastic processes with the same distributions (normal in the case the initial condition is a fixed constant) at each time t. The probabilistic formulation considered has the following form

$$\frac{dx(t;\bar{b})}{dt} = \alpha(t)x(t;\bar{b}) + \gamma(t) + \bar{b}\cdot\bar{\varrho}(t), \tag{10}$$

where $\bar{b} = (b_0, b_1, \ldots, b_{n-1}) \in \mathbb{R}^n$, α, γ and $\bar{\varrho} = (\varrho_0, \varrho_1, \ldots, \varrho_{n-1})$ are non-random functions of t, $B_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$, $j = 0, 1, 2, \ldots, n-1$, and are mutually independent, with the $\bar{b}$ chosen as realizations of $\bar{B} = (B_0, B_1, \ldots, B_{n-1})$. Hence, the dynamics of an individual with initial condition x_0 in a subsystem with its rates of change having parameter values $\bar{b}$ is described by the deterministic model (10) with initial condition $x(0) = x_0$.

We assume that all the subsystems have the same probability density function for initial condition X_0, and let

$X(t) = x(t; X_0, B_0, B_1, \ldots, B_{n-1}) = x(t; X_0, \bar{B})$ and

$$Y(t) = \int_0^t \gamma(s) \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds + \bar{B} \cdot \int_0^t \bar{\varrho}(s) \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds.$$

Then we have that

$$X(t) \quad = \quad X_0 \exp\left(\int_0^t \alpha(s) ds\right) + Y(t). \tag{11}$$

Note that $B_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$, and B_j, $j = 0, 1, 2, \ldots, n-1$, are mutually independent. Hence, we find that for any fixed t, $Y(t)$ is normally distributed with mean defined by

$$\int_0^t (\gamma(s) + \bar{\mu} \cdot \bar{\varrho}(s)) \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds, \tag{12}$$

where $\bar{\mu} = (\mu_0, \mu_1, \ldots, \mu_{n-1})$, and variance defined by

$$\sum_{j=0}^{n-1} \sigma_j^2 \left[\int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right]^2 . \tag{13}$$

Hence, if all the individuals in the entire system have the same fixed initial condition x_0, then $X(t)$ is also normally distributed for any fixed time t, i.e., $X(t)$ is a Gaussian process. Based on this piece of information, the stochastic model is chosen to have the form

$$dX(t) = [\alpha(t)X(t) + \xi(t)]dt + \eta(t)dW(t), \quad X(0) = X_0, \tag{14}$$

where α, ξ and η are non-random functions of t.

Can argue that if functions ξ, η and ϱ_j, and constants μ_j, σ_j and n

satisfy the following two equalities

$$\int_0^t \xi(s) \exp\left(\int_s^t \alpha(\tau)d\tau\right) ds = \int_0^t [\gamma(s) + \bar{\mu}\cdot\bar{\varrho}(s)] \exp\left(\int_s^t \alpha(\tau)d\tau\right) ds \tag{15}$$

and

$$\int_0^t \left[\eta(s) \exp\left(\int_s^t \alpha(\tau)d\tau\right)\right]^2 ds = \sum_{j=0}^{n-1} \sigma_j^2 \left[\int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau)d\tau\right) ds\right]^2, \tag{16}$$

then the probabilistic formulation (10) and the stochastic formulation (14) yield stochastic processes that are pointwise equivalent in density.

Probabilistic Formulation to Stochastic Formulation

Here we assume that probabilistic formulation (10) is known, and we want to determine its corresponding stochastic formulation. In other words, we need to determine functions ξ and η in terms of functions ϱ_j, and constants μ_j, σ_j and n. By (15), it is obvious that if function ξ is chosen to be

$$\xi(t) = \gamma(t) + \sum_{j=0}^{n-1} \mu_j \varrho_j(t) = \gamma(t) + \bar{\mu} \cdot \bar{\varrho}(t),$$

then (15) holds. Can ague that the function η such that (16) is given by

$$\eta(t) = \left[2 \sum_{j=0}^{n-1} \sigma_j^2 \varrho_j(t) \int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right]^{\frac{1}{2}}.$$

Stochastic Formulation to Probabilistic Formulation

Next we assume that stochastic formulation (14) is known, and we wish to determine its corresponding probabilistic formulation. In other words, we need to determine function ρ_j, and constants μ_j, σ_j and n in terms of functions ξ and η . By (15) and (16) we know that we have numerous different choices for the probabilistic formulation. Here we choose one of the simple formulations. Let $n = 2$ and $\mu_1 = 0$. Then by (15) we have

$$\int_0^t [\gamma(s) + \mu_0 \varrho_0(s)] \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds = \int_0^t \xi(s) \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds. \tag{17}$$

It is obvious that if we set

$$\gamma(t) + \mu_0 \varrho_0(t) = \xi(t), \tag{18}$$

then (17) holds. But we see that we still have different choices for probabilistic formulation. One simple case is just to choose $\gamma \equiv 0$, $\varrho_0(t) = \xi(t)$, and $\mu_0 = 1$. Then by (16) we have

$$\sigma_1^2 \left[\int_0^t \varrho_1(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right]^2$$
$$= \int_0^t \eta^2(s) \exp\left(2 \int_s^t \alpha(\tau) d\tau \right) ds - \sigma_0^2 \left[\int_0^t \xi(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right]^2, \tag{19}$$

which implies that we need to choose σ_0 sufficiently small such that its right-hand side is greater than 0. Now by (19) we have

$$\sigma_1 \int_0^t \varrho_1(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds$$
$$= \left[\int_0^t \eta^2(s) \exp\left(2 \int_s^t \alpha(\tau) d\tau \right) ds - \sigma_0^2 \left(\int_0^t \xi(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right)^2 \right]^{\frac{1}{2}}$$

Differentiating both sides of the above equation with respect to t we obtain that

$$
\begin{aligned}
\sigma_1 \varrho_1(t) &= \tfrac{d}{dt}[\textstyle\int_0^t \eta^2(s) \exp\left(2 \int_s^t \alpha(\tau) d\tau\right) ds \\
&- \sigma_0^2 \left(\textstyle\int_0^t \xi(s) \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds\right)^2]^{\frac{1}{2}} \\
& \quad -\alpha(t)[\textstyle\int_0^t \eta^2(s) \exp\left(2 \int_s^t \alpha(\tau) d\tau\right) ds \\
&- \sigma_0^2 \left(\textstyle\int_0^t \xi(s) \exp\left(\int_s^t \alpha(\tau) d\tau\right) ds\right)^2]^{\frac{1}{2}}.
\end{aligned}
$$

Hence, we can just assign any positive value for σ_1, and then use the above equality to determine function ϱ_1.

Remarks and Examples

Other cases discussed in [Banks-Hu]. Based on discussions there, we see that we can find the corresponding probabilistic formulation for the following two types of stochastic differential equations

$$dX(t) = [\alpha(t)X(t) + \xi(t)]dt + \eta(t)dW(t),$$

and

$$dX(t) = \xi(t)(X(t) + c)dt + \eta(t)(X(t) + c)dW(t),$$

where ξ, η, and α are all deterministic function of t, and c is a given constant. **Hence, if a nonlinear stochastic differential equation can be reduced to one of the above forms by some invertible transformation, then one can find its corresponding probabilistic formulation.**

First we will consider some special cases of nonlinear stochastic differential equations that can be reduced to linear stochastic differential equations after some transformation. First consider the stochastic differential equation

$$dX(t) = g(X(t), t)dt + \sigma(X(t), t)dW(t) \tag{20}$$

where g and σ are non-random functions of x and t. Under certain conditions on g and σ can show [Gard] that (20) can be reduced to a linear SDE of the form

$$dh(X(t), t) = \bar{g}(t)dt + \bar{\sigma}(t)dW(t),$$

where $\bar{g}(t)$ can be readily computed.

In addition, it was shown in [Gard] that the autonomous stochastic differential equation

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t),$$

can be reduced to the linear stochastic differential equation

$$dh(X) = (\lambda_0 + \lambda_1 h(X))dt + (\nu_0 + \nu_1 h(X))dW(t)$$

if and only if

$$\psi'(x) = 0 \text{ or } \left(\frac{(\sigma\psi')'}{\psi'}\right)'(x) = 0, \tag{21}$$

where λ_0, λ_1, ν_0 and ν_1 are some constants, and $\psi(x) = \frac{g(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x)$.

If the latter part of (21) is satisfied, then we see that $\frac{(\sigma\psi')'}{\psi'}$ is some constant. Let $\nu_1 = -\frac{(\sigma\psi')'}{\psi'}$. If $\nu_1 \neq 0$, then we can choose

$$h(x) = c\exp\left(\nu_1 \int_a^x \frac{1}{\sigma(\tau)} d\tau\right),$$

where c is some constant. If $\nu_1 = 0$, then we can choose

$$h(x) = \nu_0 \int_a^x \frac{1}{\sigma(\tau)} d\tau + c.$$

Examples to illustrate this transformation method to find the corresponding equivalent formulations.

Example 1: Use transformation method to find equivalent probabilistic formulation for nonlinear stochastic differential equation:

$$dX(t) = \left[1 - \frac{1}{2}\exp(-2X(t))\right] dt + \exp(-X(t))dW(t).$$

Find:

$$\frac{dx(t;b)}{dt} = 1 + b\left[\frac{\exp(2t)}{\sqrt{2[\exp(2t)-1]}} - \sqrt{\frac{\exp(2t)-1}{2}}\right]\exp(-x(t;b)),$$

where $b \in \mathbb{R};\ B \sim \mathcal{N}(0,1)$,

yields process that is pointwise equivalent in density.

Example 2: We consider the deterministic logistic equation

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{\kappa}\right), \; x(0) = x_0, \tag{22}$$

where b is some constant representing the intrinsic growth rate, and κ is a given constant representing the carrying capacity.

We find the probabilistic formulation

$$\frac{dx(t;b)}{dt} = bx(t;b)\left(1 - \frac{x(t;b)}{\kappa}\right), \; b \in \mathbb{R}; \; B \sim \mathcal{N}(\mu_0, \sigma_0^2) \tag{23}$$

and the stochastic formulation

$$\begin{aligned} dX(t) = X(t)\left[(\mu_0 - \sigma_0^2 t)\left(1 - \tfrac{X(t)}{\kappa}\right) + 2t\sigma_0^2\left(1 - \tfrac{X(t)}{\kappa}\right)^2\right] dt \\ -\sqrt{2t}\sigma_0 X(t)\left(1 - \tfrac{X(t)}{\kappa}\right) dW(t) \end{aligned} \tag{24}$$

are pointwise equivalent in density. Figures 1 and 2 depict the probability density function $p(x,t)$ at different times t for the probabilistic formulation (23) and the stochastic formulation (24) with $\kappa = 100$, $x_0 = 10$, $\mu_0 = 1$ and $\sigma_0 = 0.1$, where $p(x,t)$ is obtained by simulating 10^5 sample paths for each formulation.

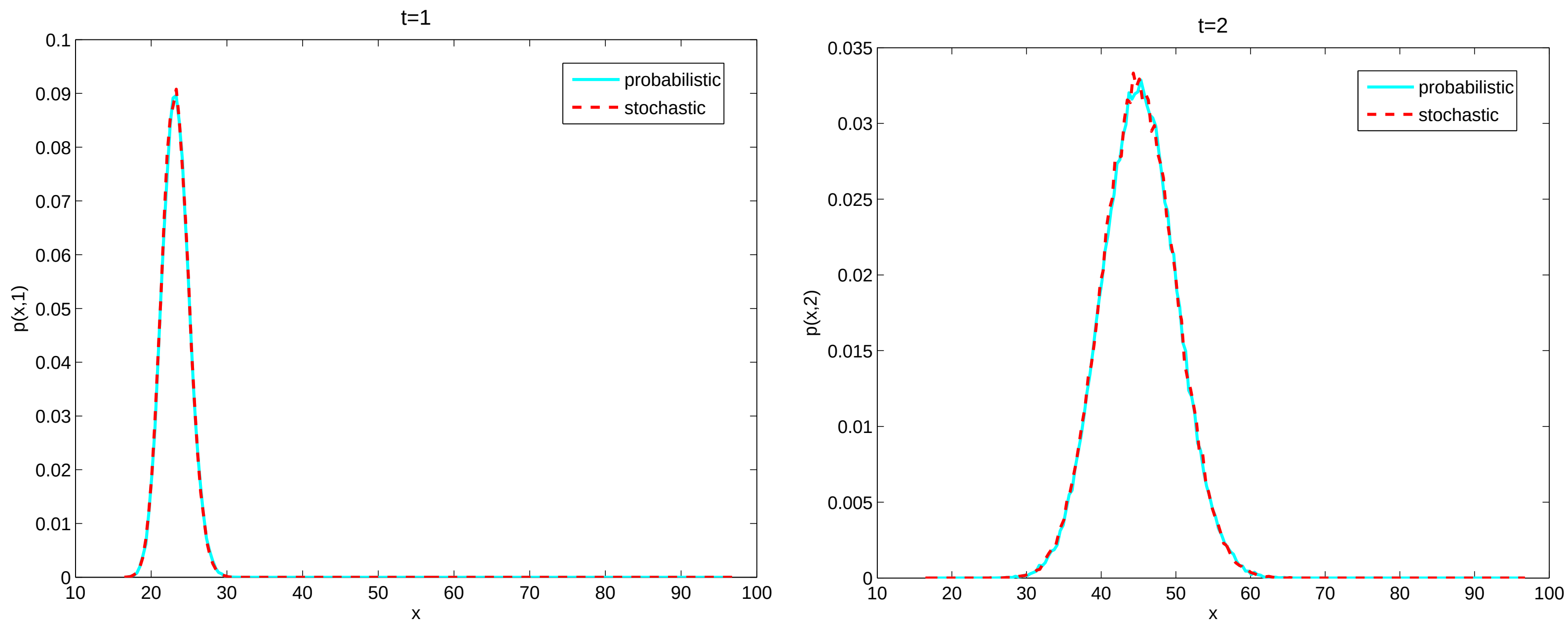

Figure 1: Probability density function $p(x, t)$ are obtained by simulating 10^5 sample paths for probabilistic formulation (23) and stochastic formulation (24) at $t = 1$ and 2 where $\Delta t = 0.004$ is used in (**??**), and $T = 4$.

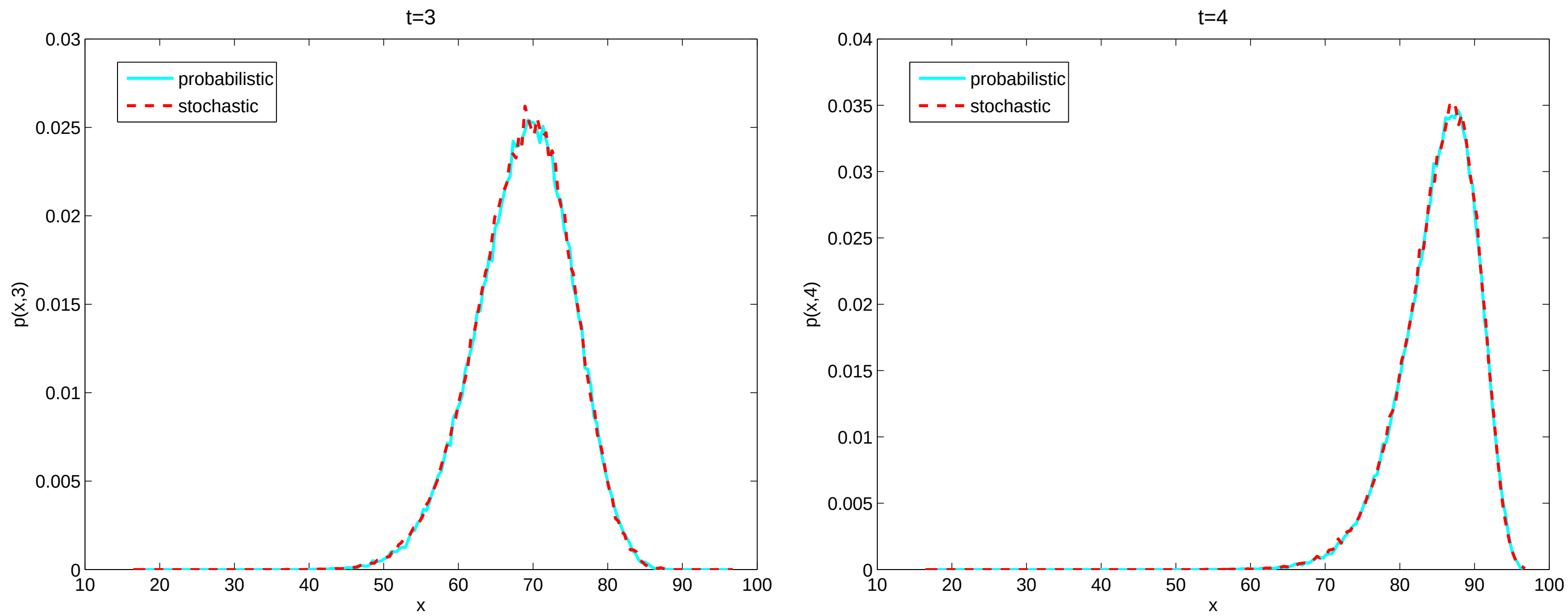

Figure 2: Probability density function $p(x, t)$ are obtained by simulating 10^5 sample paths for probabilistic formulation (23) and stochastic formulation (24) at $t = 3$ and 4, where $\Delta t = 0.004$ is used in (**??**), and $T = 4$.

Concluding Remarks

- Derived several classes of examples for which we can establish pointwise equivalence in density for the corresponding probabilistic and stochastic formulations.
- Well documented: difficulties arise in numerically solving the F-P when the drift g dominates the diffusion σ^2.
- Results here lead to alternative methods that can be fast and efficient in numerically solving the Fokker-Planck by employing its pointwise equivalent in density probabilistic formulation.
- Have shown efficacy in inverse problems calculations; current efforts on control problems.

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