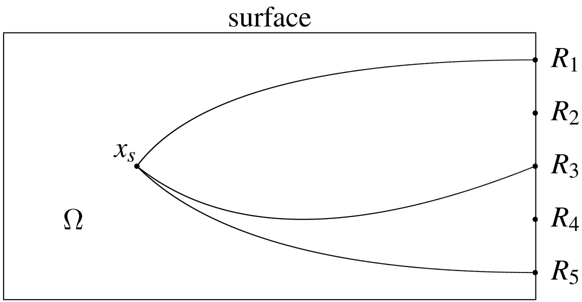
Numerical analysis of an inverse problem for the eikonal equation

Klaus Deckelnick (Magdeburg)
Charles M. Elliott (Warwick)
Vanessa Styles (Sussex)
Num. Math. to appear

Warwick, May 2011

Travel-time tomography



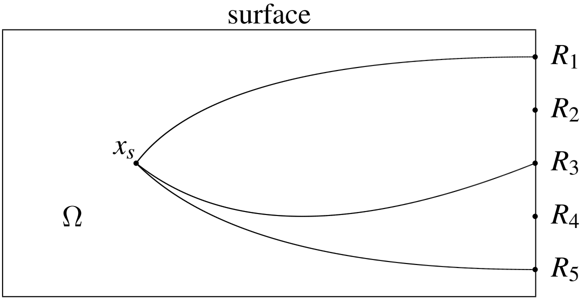
given measured first arrival times T_i^* at the receivers R_i

find $v^* : \bar{\Omega} \to \mathbb{R}_{>0}$ velocity distribution

Idea $\min_{v \in \mathcal{A}} \sum_i |T_i(v) - T_i^*|^2$, where

$T_i(v)$ is the first arrival time at R_i *given* the velocity v

Travel-time tomography



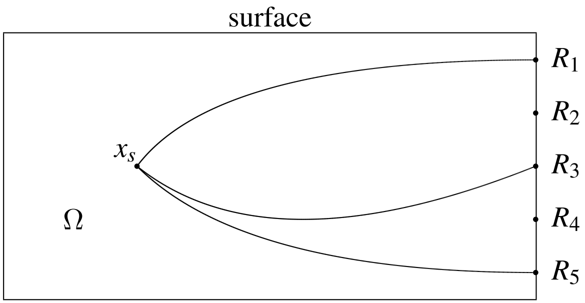
given measured first arrival times T_i^* at the receivers R_i

find $v^* : \bar{\Omega} \to \mathbb{R}_{>0}$ velocity distribution

Idea $\min_{v \in \mathcal{A}} \sum_i |T_i(v) - T_i^*|^2$, where

$T_i(v)$ is the first arrival time at R_i *given* the velocity v

Travel-time tomography



given measured first arrival times T_i^* at the receivers R_i

find $v^* : \bar{\Omega} \to \mathbb{R}_{>0}$ velocity distribution

Idea $\min_{v \in \mathcal{A}} \sum_i |T_i(v) - T_i^*|^2$, where

$T_i(v)$ is the first arrival time at R_i *given* the velocity v

Mathematical problem

- $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded domain,
- $x_s \in \Omega$ source point,
- $T^* : \partial\Omega \to \mathbb{R}_{>0}$ measurements,
- velocity (slowness) model:

  $K = \{a : \bar{\Omega} \to \mathbb{R} \,|\, a(x) = \sum_{i=1}^{L} a_i \phi_i(x), a_{\min} \leq a_i \leq a_{\max}\}$

$$(\mathbf{P}) \qquad \min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \int_{\partial\Omega} |T_a(x) - T^*(x)|^2 dS$$

where

$$T_a(x) = \inf\{\int_0^1 a(\gamma(r))|\gamma'(r)|dr \mid \gamma \in W^{1,\infty}([0,1], \bar{\Omega}), \gamma(0) = x_s, \gamma(1) = x\}.$$

Mathematical problem

- $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded domain,
- $x_s \in \Omega$ source point,
- $T^* : \partial\Omega \to \mathbb{R}_{>0}$ measurements,
- velocity (slowness) model:

  $K = \{a : \bar{\Omega} \to \mathbb{R} \,|\, a(x) = \sum_{i=1}^{L} a_i \phi_i(x), a_{\min} \leq a_i \leq a_{\max}\}$

$$(\mathbf{P}) \qquad \min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \int_{\partial\Omega} |T_a(x) - T^*(x)|^2 dS$$

where

$$T_a(x) = \inf\{\int_0^1 a(\gamma(r))|\gamma'(r)|dr \mid \gamma \in W^{1,\infty}([0,1], \bar{\Omega}), \gamma(0) = x_s, \gamma(1) = x\}.$$

Mathematical problem

- $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded domain,
- $x_s \in \Omega$ source point,
- $T^* : \partial\Omega \to \mathbb{R}_{>0}$ measurements,
- velocity (slowness) model:

  $K = \{a : \bar{\Omega} \to \mathbb{R} \,|\, a(x) = \sum_{i=1}^{L} a_i \phi_i(x), a_{\min} \le a_i \le a_{\max}\}$

$$(\mathbf{P}) \qquad \min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \int_{\partial\Omega} |T_a(x) - T^*(x)|^2 dS$$

where

$$T_a(x) = \inf\{\int_0^1 a(\gamma(r))|\gamma'(r)|dr \mid \gamma \in W^{1,\infty}([0,1], \bar{\Omega}), \gamma(0) = x_s, \gamma(1) = x\}.$$

Mathematical problem

- $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded domain,
- $x_s \in \Omega$ source point,
- $T^* : \partial\Omega \to \mathbb{R}_{>0}$ measurements,
- velocity (slowness) model:

  $K = \{a : \bar{\Omega} \to \mathbb{R} \,|\, a(x) = \sum_{i=1}^{L} a_i \phi_i(x), a_{\min} \le a_i \le a_{\max}\}$

$$(\mathbf{P}) \qquad \min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \int_{\partial\Omega} |T_a(x) - T^*(x)|^2 dS$$

where

$$T_a(x) = \inf\{ \int_0^1 a(\gamma(r))|\gamma'(r)|dr \,|\, \gamma \in W^{1,\infty}([0,1], \bar{\Omega}), \gamma(0) = x_s, \gamma(1) = x\}.$$

Mathematical problem

- $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ bounded domain,
- $x_s \in \Omega$ source point,
- $T^* : \partial\Omega \to \mathbb{R}_{>0}$ measurements,
- velocity (slowness) model:

  $K = \{a : \bar{\Omega} \to \mathbb{R} \,|\, a(x) = \sum_{i=1}^{L} a_i \phi_i(x), a_{\min} \leq a_i \leq a_{\max}\}$

$$(\mathbf{P}) \qquad \min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \int_{\partial\Omega} |T_a(x) - T^*(x)|^2 dS$$

where

$$T_a(x) = \inf\{\int_0^1 a(\gamma(r))|\gamma'(r)|dr \mid \gamma \in W^{1,\infty}([0, 1], \bar{\Omega}), \gamma(0) = x_s, \gamma(1) = x\}.$$

Evaluation of T_a

- Ray tracing (solution of BVPs for ODEs)

- Eikonal equation (Sei & Symes '94)

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$ continuous, $x_s \in \Omega$

find $T : \bar{\Omega} \to \mathbb{R}$, such that

$$\begin{aligned} |\nabla T(x)| &= a(x), \quad x \in \Omega \setminus \{x_s\} &(1)\\ \nabla T(x) \cdot \nu(x) &\geq 0, \quad x \in \partial\Omega &(2)\\ T(x_s) &= 0 &(3) \end{aligned}$$

$\rightsquigarrow$ PDE constrained optimisation problem

Evaluation of T_a

- Ray tracing (solution of BVPs for ODEs)

- Eikonal equation (Sei & Symes '94)

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$ continuous, $x_s \in \Omega$

find $T : \bar{\Omega} \to \mathbb{R}$, such that

$$
\begin{aligned}
|\nabla T(x)| &= a(x), \quad x \in \Omega \setminus \{x_s\} && (1)\\
\nabla T(x) \cdot \nu(x) &\geq 0, \quad x \in \partial\Omega && (2)\\
T(x_s) &= 0 && (3)
\end{aligned}
$$

$\rightsquigarrow$ PDE constrained optimisation problem

Evaluation of T_a

- Ray tracing (solution of BVPs for ODEs)

- Eikonal equation (Sei & Symes '94)

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$ continuous, $x_s \in \Omega$

find $T : \bar{\Omega} \to \mathbb{R}$, such that

$$
\begin{aligned}
|\nabla T(x)| &= a(x), \quad x \in \Omega \setminus \{x_s\} &(1)\\
\nabla T(x) \cdot \nu(x) &\geq 0, \quad x \in \partial\Omega &(2)\\
T(x_s) &= 0 &(3)
\end{aligned}
$$

$\rightsquigarrow$ PDE constrained optimisation problem

Evaluation of T_a

- Ray tracing (solution of BVPs for ODEs)

- Eikonal equation (Sei & Symes ’94)

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$ continuous, $x_s \in \Omega$

find $T : \bar{\Omega} \to \mathbb{R}$, such that

$$
\begin{aligned}
|\nabla T(x)| &= a(x), \quad x \in \Omega \setminus \{x_s\} && (1)\\
\nabla T(x) \cdot \nu(x) &\geq 0, \quad x \in \partial\Omega && (2)\\
T(x_s) &= 0 && (3)
\end{aligned}
$$

$\rightsquigarrow$ PDE constrained optimisation problem

Eikonal equation with Soner boundary condition

Definition A function $T \in C^0(\bar{\Omega})$ is called a viscosity solution of (1), (2) if for each $\zeta \in C^\infty(\mathbb{R}^n)$: if $T - \zeta$ has a local maximum (minimum) at a point $x \in \Omega \setminus \{x_s\}$ ($\bar{\Omega} \setminus \{x_s\}$), then

$$|\nabla\zeta(x)| \leq a(x) \quad (|\nabla\zeta(x)| \geq a(x)).$$

Theorem (Soner '86, Capuzzo–Dolcetta & Lions '90)

Problem (1), (2), (3) has a unique viscosity solution $T \in C^{0,1}(\bar{\Omega})$, which is given by

$$T(x) = \inf\{\int_0^1 a(\gamma(r))|\gamma'(r)|dr \mid \gamma \in W^{1,\infty}([0,1],\bar{\Omega}), \gamma(0) = x_s, \gamma(1) = x\}.$$

Eikonal equation with Soner boundary condition

Definition A function $T \in C^0(\bar{\Omega})$ is called a viscosity solution of (1), (2) if for each $\zeta \in C^\infty(\mathbb{R}^n)$: if $T - \zeta$ has a local maximum (minimum) at a point $x \in \Omega \setminus \{x_s\}$ ($\bar{\Omega} \setminus \{x_s\}$), then

$$|\nabla \zeta(x)| \leq a(x) \quad (|\nabla \zeta(x)| \geq a(x)).$$

Theorem (Soner '86, Capuzzo–Dolcetta & Lions '90)

Problem (1), (2), (3) has a unique viscosity solution $T \in C^{0,1}(\bar{\Omega})$, which is given by

$$T(x) = \inf\{\int_0^1 a(\gamma(r))|\gamma'(r)|dr \mid \gamma \in W^{1,\infty}([0,1], \bar{\Omega}), \\ \gamma(0) = x_s, \gamma(1) = x\}.$$

Formal idea for uniqueness

Suppose that $T_1, T_2 \in C^0(\bar{\Omega}) \cap C^1(\bar{\Omega} \setminus \{x_s\})$ are two solutions of (1),(2),(3).

Fix $0 < \theta << 1$ and choose $x_0 \in \bar{\Omega}$ such that

$$(1-\theta)T_1(x_0) - T_2(x_0) = \max_{x \in \bar{\Omega}} \{(1-\theta)T_1(x) - T_2(x)\}.$$

Case 1 $x_0 \in \Omega \setminus \{x_s\}$:

$$\Rightarrow \qquad (1-\theta)\nabla T_1(x_0) - \nabla T_2(x_0) = 0$$

$$\Rightarrow \qquad (1-\theta)\underbrace{|\nabla T_1(x_0)|}_{=a(x_0)} = \underbrace{|\nabla T_2(x_0)|}_{=a(x_0)} \quad \lightning$$

Formal idea for uniqueness

Suppose that $T_1, T_2 \in C^0(\bar{\Omega}) \cap C^1(\bar{\Omega} \setminus \{x_s\})$ are two solutions of (1),(2),(3).

Fix $0 < \theta << 1$ and choose $x_0 \in \bar{\Omega}$ such that

$$(1-\theta)T_1(x_0) - T_2(x_0) = \max_{x \in \bar{\Omega}} \{(1-\theta)T_1(x) - T_2(x)\}.$$

Case 1 $x_0 \in \Omega \setminus \{x_s\}$:

$$\Rightarrow \quad (1-\theta)\nabla T_1(x_0) - \nabla T_2(x_0) = 0$$

$$\Rightarrow \quad (1-\theta)\underbrace{|\nabla T_1(x_0)|}_{=a(x_0)} = \underbrace{|\nabla T_2(x_0)|}_{=a(x_0)} \quad ↯$$

Formal idea for uniqueness

Suppose that $T_1, T_2 \in C^0(\bar{\Omega}) \cap C^1(\bar{\Omega} \setminus \{x_s\})$ are two solutions of (1),(2),(3).

Fix $0 < \theta << 1$ and choose $x_0 \in \bar{\Omega}$ such that

$$(1-\theta)T_1(x_0) - T_2(x_0) = \max_{x \in \bar{\Omega}} \{(1-\theta)T_1(x) - T_2(x)\}.$$

Case 1 $x_0 \in \Omega \setminus \{x_s\}$:

$$\Rightarrow \quad (1-\theta)\nabla T_1(x_0) - \nabla T_2(x_0) = 0$$

$$\Rightarrow \quad (1-\theta)\underbrace{|\nabla T_1(x_0)|}_{=a(x_0)} = \underbrace{|\nabla T_2(x_0)|}_{=a(x_0)} \quad ↯$$

Formal idea for uniqueness

Suppose that $T_1, T_2 \in C^0(\bar{\Omega}) \cap C^1(\bar{\Omega} \setminus \{x_s\})$ are two solutions of (1),(2),(3).

Fix $0 < \theta << 1$ and choose $x_0 \in \bar{\Omega}$ such that

$$(1-\theta)T_1(x_0) - T_2(x_0) = \max_{x \in \bar{\Omega}} \{(1-\theta)T_1(x) - T_2(x)\}.$$

Case 1 $x_0 \in \Omega \setminus \{x_s\}$:

$$\Rightarrow \quad (1-\theta)\nabla T_1(x_0) - \nabla T_2(x_0) = 0$$

$$\Rightarrow \quad (1-\theta) \underbrace{|\nabla T_1(x_0)|}_{=a(x_0)} = \underbrace{|\nabla T_2(x_0)|}_{=a(x_0)} \quad \lightning$$

Case 2 $x_0 \in \partial\Omega$:

Let $v := -\frac{\nabla T_2(x_0)}{|\nabla T_2(x_0)|}$. Then

$$v \cdot \nu(x_0) \leq 0, \quad \frac{\partial}{\partial v}\big((1-\theta)T_1 - T_2\big)(x_0) \leq 0$$

$$\Rightarrow |\nabla T_2(x_0)| \leq (1-\theta)\nabla T_1(x_0) \cdot v \leq (1-\theta)|\nabla T_1(x_0)| \quad \lightning$$

In conclusion $x_0 = x_s$

$$\Rightarrow \quad \max_{x \in \bar{\Omega}} \{(1-\theta)T_1(x) - T_2(x)\} = 0$$

$$\theta \searrow 0 : \quad T_1(x) \leq T_2(x), x \in \bar{\Omega}.$$

Rigorous argument: Doubling of variables

$$\Phi(x,y) = (1-\theta)T_1(x) - T_2(y) - \frac{1}{\epsilon}|x - y - \epsilon\nu(x_0)|^2 - \frac{1}{\rho}|y - x_0|^2$$

Case 2 $x_0 \in \partial\Omega$:

Let $v := -\frac{\nabla T_2(x_0)}{|\nabla T_2(x_0)|}$. Then

$$v \cdot \nu(x_0) \leq 0, \quad \frac{\partial}{\partial v}\big((1-\theta)T_1 - T_2\big)(x_0) \leq 0$$

$$\Rightarrow |\nabla T_2(x_0)| \leq (1-\theta)\nabla T_1(x_0) \cdot v \leq (1-\theta)|\nabla T_1(x_0)| \quad ↯$$

In conclusion $x_0 = x_s$

$$\begin{aligned} \Rightarrow \quad & \max_{x\in\bar{\Omega}}\{(1-\theta)T_1(x) - T_2(x)\} = 0 \\ \theta \searrow 0: \quad & T_1(x) \leq T_2(x), x \in \bar{\Omega}. \end{aligned}$$

Rigorous argument: Doubling of variables

$$\Phi(x,y) = (1-\theta)T_1(x) - T_2(y) - \frac{1}{\epsilon}|x - y - \epsilon\nu(x_0)|^2 - \frac{1}{\rho}|y - x_0|^2$$

Case 2 $x_0 \in \partial\Omega$:

Let $v := -\frac{\nabla T_2(x_0)}{|\nabla T_2(x_0)|}$. Then

$$v \cdot \nu(x_0) \leq 0, \quad \frac{\partial}{\partial v}\big((1-\theta)T_1 - T_2\big)(x_0) \leq 0$$

$$\Rightarrow |\nabla T_2(x_0)| \leq (1-\theta)\nabla T_1(x_0) \cdot v \leq (1-\theta)|\nabla T_1(x_0)| \quad ↯$$

In conclusion $x_0 = x_s$

$$\Rightarrow \quad \max_{x \in \bar{\Omega}}\{(1-\theta)T_1(x) - T_2(x)\} = 0$$

$$\theta \searrow 0: \quad T_1(x) \leq T_2(x), x \in \bar{\Omega}.$$

Rigorous argument: Doubling of variables

$$\Phi(x,y) = (1-\theta)T_1(x) - T_2(y) - \frac{1}{\epsilon}|x - y - \epsilon\nu(x_0)|^2 - \frac{1}{\rho}|y - x_0|^2$$

Case 2 $x_0 \in \partial\Omega$:

Let $v := -\frac{\nabla T_2(x_0)}{|\nabla T_2(x_0)|}$. Then

$$v \cdot \nu(x_0) \leq 0, \quad \frac{\partial}{\partial v}\big((1-\theta)T_1 - T_2\big)(x_0) \leq 0$$

$$\Rightarrow |\nabla T_2(x_0)| \leq (1-\theta)\nabla T_1(x_0) \cdot v \leq (1-\theta)|\nabla T_1(x_0)| \quad ↯$$

In conclusion $x_0 = x_s$

$$\Rightarrow \qquad \max_{x \in \bar{\Omega}}\{(1-\theta)T_1(x) - T_2(x)\} = 0$$

$$\theta \searrow 0: \qquad T_1(x) \leq T_2(x), x \in \bar{\Omega}.$$

Rigorous argument: Doubling of variables

$$\Phi(x,y) = (1-\theta)T_1(x) - T_2(y) - \frac{1}{\epsilon}|x - y - \epsilon\nu(x_0)|^2 - \frac{1}{\rho}|y - x_0|^2$$

Discretisation of the eikonal equation

Abgrall '03, Leung & Qian '06

- $\bar{\Omega}_h = \{x_\alpha\}$: finite difference grid on $\bar{\Omega} \subset \mathbb{R}^2, x_s \in \Omega_h$
- $\mathcal{N}_\alpha = \{x_\beta \in \bar{\Omega}_h \,|\, x_\beta$ is a neighbour of $x_\alpha\}$.

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$

find $T_h : \bar{\Omega}_h \to \mathbb{R}$ such that $T_h(x_s) = 0$ and

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big[\Big(\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}}\Big)^+\Big]^2 = a(x_\alpha)^2, \quad x_\alpha \in \bar{\Omega}_h \setminus \{x_s\}.$$

Numerical solution: Fast Marching Method

Discretisation of the eikonal equation

Abgrall '03, Leung & Qian '06

- $\bar{\Omega}_h = \{x_\alpha\}$: finite difference grid on $\bar{\Omega} \subset \mathbb{R}^2, x_s \in \Omega_h$
- $\mathcal{N}_\alpha = \{x_\beta \in \bar{\Omega}_h \,|\, x_\beta$ is a neighbour of $x_\alpha\}$.

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$

find $T_h : \bar{\Omega}_h \to \mathbb{R}$ such that $T_h(x_s) = 0$ and

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big[\Big(\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}}\Big)^+\Big]^2 = a(x_\alpha)^2, \quad x_\alpha \in \bar{\Omega}_h \setminus \{x_s\}.$$

Numerical solution: Fast Marching Method

Discretisation of the eikonal equation

Abgrall '03, Leung & Qian '06

- $\bar{\Omega}_h = \{x_\alpha\}$: finite difference grid on $\bar{\Omega} \subset \mathbb{R}^2, x_s \in \Omega_h$
- $\mathcal{N}_\alpha = \{x_\beta \in \bar{\Omega}_h \,|\, x_\beta$ is a neighbour of $x_\alpha\}$.

given $a : \bar{\Omega} \to \mathbb{R}_{>0}$

find $T_h : \bar{\Omega}_h \to \mathbb{R}$ such that $T_h(x_s) = 0$ and

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big[\Big(\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}}\Big)^+\Big]^2 = a(x_\alpha)^2, \quad x_\alpha \in \bar{\Omega}_h \setminus \{x_s\}.$$

Numerical solution: Fast Marching Method

Fast marching

A discrete solution can be found efficiently, without iteration, using the fast marching procedure, Sethian.

The idea behind this method is that the unique solution T_α at a grid point x_α, only depends on neighbouring values T_β such that $0 \leq T_\beta < T_\alpha$ so that the solution can be obtained in increasing order of magnitude of the grid values T_α. Solving the equation then becomes an issue of sorting the grid values.

Fast marching

First tag x_{α_0} as *known* and tag as *trial* all points that are one grid point away from this *known* point. Finally tag as *far* all remaining points. Now cycle through the following *Fast Marching Procedure*:

- [Step 1] Compute a trial value of $\tilde{T}_\alpha$ for every $x_\alpha \in$ *trial* according to discrete equation assuming that it is smaller than or equal to its *trial* neighbours.
- [Step 2] Set x_μ to be any *trial* point such that the trial values satisfy $\tilde{T}_\mu \leq \tilde{T}_\alpha$ for all $x_\alpha \in$ *trial*.
- [Step 3] Set $T_\mu = \tilde{T}_\mu$ for all such x_μ and add x_μ to *known* and remove from *trial*.
- [Step 4] Tag all neighbours of *known* as *trial* if they are not *known*.
- [Step 5] If *trial* $= \{\emptyset\}$ STOP.
- [Step 6] Return to Step 1.

Lemma

The Fast Marching Procedure terminates in K cycles where K is the number of distinct positive values taken by the solution T_h .

Observe that the unique solution of the equation

$$\sum_{x_\beta \in \mathcal{N}_{m,\alpha}} \left[\left(\frac{r - U_\beta}{h_{\alpha\beta}} \right)^+ \right]^2 = a(x_\alpha)^2$$

defining the trial values may be found by solving a quadratic equation and taking the largest root.

Error bound

Theorem

Let $a : \bar{\Omega} \to \mathbb{R}_{>0}$ be Lipschitz continuous, $T : \bar{\Omega} \to \mathbb{R}$ the viscosity solution of (1), (2), (3) and $T_h : \bar{\Omega}_h \to \mathbb{R}$ the corresponding discrete solution. Then

$$\max_{x_\alpha \in \bar{\Omega}_h} |T(x_\alpha) - T_h(x_\alpha)| \leq C\sqrt{h}.$$

The constant C depends on Ω, $\min_{\bar{\Omega}} a$ and the Lipschitz constant of a.

Idea of proof

Choose $x_\beta \in \bar{\Omega}_h$ with

$$(1-\mu\sqrt{h})T(x_\beta) - T_h(x_\beta) = \max_{x_\alpha \in \bar{\Omega}_h} \{(1-\mu\sqrt{h})T(x_\alpha) - T_h(x_\alpha)\}$$

Case 1 $|x_s - x_\beta| > \sqrt{h}$

Use the fact that T is a viscosity solution and the properties of the scheme in order to exclude this case.

Case 2 $|x_s - x_\beta| \leq \sqrt{h}$

$$T(x_\beta) - T_h(x_\beta) \leq |T(x_\beta) - T(x_s)| + |T_h(x_s) - T_h(x_\beta)| \leq C\sqrt{h}$$

Idea of proof

Choose $x_\beta \in \bar{\Omega}_h$ with

$$(1-\mu\sqrt{h})T(x_\beta) - T_h(x_\beta) = \max_{x_\alpha \in \bar{\Omega}_h} \{(1-\mu\sqrt{h})T(x_\alpha) - T_h(x_\alpha)\}$$

Case 1 $|x_s - x_\beta| > \sqrt{h}$

Use the fact that T is a viscosity solution and the properties of the scheme in order to exclude this case.

Case 2 $|x_s - x_\beta| \le \sqrt{h}$

$$T(x_\beta) - T_h(x_\beta) \le |T(x_\beta) - T(x_s)| + |T_h(x_s) - T_h(x_\beta)| \le C\sqrt{h}$$

Idea of proof

Choose $x_\beta \in \bar{\Omega}_h$ with

$$(1-\mu\sqrt{h})T(x_\beta) - T_h(x_\beta) = \max_{x_\alpha \in \bar{\Omega}_h} \{(1-\mu\sqrt{h})T(x_\alpha) - T_h(x_\alpha)\}$$

Case 1 $|x_s - x_\beta| > \sqrt{h}$

Use the fact that T is a viscosity solution and the properties of the scheme in order to exclude this case.

Case 2 $|x_s - x_\beta| \leq \sqrt{h}$

$$T(x_\beta) - T_h(x_\beta) \leq |T(x_\beta) - T(x_s)| + |T_h(x_s) - T_h(x_\beta)| \leq C\sqrt{h}$$

The discrete functional

$$(\mathbf{P_h}) \qquad \min_{a\in K} \mathcal{J}_h(a) = \frac{1}{2}\sum_{x_\alpha\in\partial\Omega_h} h_\alpha |T_h(x_\alpha) - T^*(x_\alpha)|^2 + \frac{\delta_h}{2}\int_\Omega |\nabla a|^2,$$

where $\lim_{h\to 0}\delta_h = 0$.

Theorem

(i) $(\mathbf{P_h})$ has a solution $\bar{a}_h \in K$. There exists a sequence $h \to 0$ such that $\bar{a}_h \to \bar{a}$ and $\bar{a}$ is a solution of $(\mathbf{P})$.

(ii) If $\displaystyle\lim_{h\to 0}\frac{\delta_h}{\sqrt{h}} = 0$, then $\displaystyle\int_\Omega |\nabla\bar{a}|^2 \le \int_\Omega |\nabla\tilde{a}|^2$ for all solutions $\tilde{a}$ of $(\mathbf{P})$.

The discrete functional

$$(\mathbf{P_h}) \qquad \min_{a\in K} \mathcal{J}_h(a) = \frac{1}{2}\sum_{x_\alpha \in \partial\Omega_h} h_\alpha |T_h(x_\alpha) - T^*(x_\alpha)|^2 + \frac{\delta_h}{2}\int_\Omega |\nabla a|^2,$$

where $\lim_{h\to 0} \delta_h = 0$.

Theorem

(i) $(\mathbf{P_h})$ has a solution $\bar{a}_h \in K$. There exists a sequence $h \to 0$ such that $\bar{a}_h \to \bar{a}$ and $\bar{a}$ is a solution of $(\mathbf{P})$.

(ii) If $\lim_{h\to 0} \frac{\delta_h}{\sqrt{h}} = 0$, then $\int_\Omega |\nabla \bar{a}|^2 \le \int_\Omega |\nabla \tilde{a}|^2$ for all solutions $\tilde{a}$ of $(\mathbf{P})$.

The discrete functional

$$(\mathbf{P_h}) \qquad \min_{a\in K} \mathcal{J}_h(a) = \frac{1}{2} \sum_{x_\alpha \in \partial\Omega_h} h_\alpha |T_h(x_\alpha) - T^*(x_\alpha)|^2 + \frac{\delta_h}{2} \int_\Omega |\nabla a|^2,$$

where $\lim_{h\to 0} \delta_h = 0$.

Theorem

(i) $(\mathbf{P_h})$ has a solution $\bar{a}_h \in K$. There exists a sequence $h \to 0$ such that $\bar{a}_h \to \bar{a}$ and $\bar{a}$ is a solution of $(\mathbf{P})$.

(ii) If $\displaystyle\lim_{h\to 0} \frac{\delta_h}{\sqrt{h}} = 0$, then $\displaystyle\int_\Omega |\nabla \bar{a}|^2 \leq \int_\Omega |\nabla \tilde{a}|^2$ for all solutions $\tilde{a}$ of $(\mathbf{P})$.

Adjoint equation

given $a = \sum_{i=1}^{L} a_i \phi_i$ with corresponding $T = T_a$

solve $\quad \nabla \cdot (p \nabla T_a) = 0 \text{ in } \Omega \setminus \{x_s\}; \quad p \dfrac{\partial T_a}{\partial \nu} = T_a - T^* \text{ on } \partial\Omega$

$$
\begin{aligned}
\frac{\partial \mathcal{J}}{\partial a_m}(a) &= \frac{1}{2} \frac{\partial}{\partial a_m} \int_{\partial\Omega} |T_a - T^*|^2 = \int_{\partial\Omega} \underbrace{(T_a - T^*)}_{= p \frac{\partial T_a}{\partial \nu}} \frac{\partial T_a}{\partial a_m} \\
&= \int_\Omega \underbrace{\nabla \cdot (p \nabla T_a)}_{=0} \frac{\partial T_a}{\partial a_m} + \int_\Omega p \quad \underbrace{\nabla T_a \cdot \nabla \frac{\partial T_a}{\partial a_m}}_{= \frac{1}{2} \frac{\partial}{\partial a_m} |\nabla T_a|^2 = \frac{1}{2} \frac{\partial}{\partial a_m} a^2} \\
&= \int_\Omega p a \phi_m
\end{aligned}
$$

Adjoint equation

given $a = \sum_{i=1}^{L} a_i \phi_i$ with corresponding $T = T_a$

solve $\quad \nabla \cdot (p \nabla T_a) = 0 \ \text{ in } \Omega \setminus \{x_s\}; \quad p \dfrac{\partial T_a}{\partial \nu} = T_a - T^* \ \text{ on } \partial\Omega$

$$
\begin{aligned}
\frac{\partial \mathcal{J}}{\partial a_m}(a) &= \frac{1}{2} \frac{\partial}{\partial a_m} \int_{\partial\Omega} |T_a - T^*|^2 = \int_{\partial\Omega} \underbrace{(T_a - T^*)}_{= p \frac{\partial T_a}{\partial \nu}} \frac{\partial T_a}{\partial a_m} \\
&= \int_\Omega \underbrace{\nabla \cdot (p \nabla T_a)}_{=0} \frac{\partial T_a}{\partial a_m} + \int_\Omega p \quad \underbrace{\nabla T_a \cdot \nabla \frac{\partial T_a}{\partial a_m}}_{= \frac{1}{2} \frac{\partial}{\partial a_m} |\nabla T_a|^2 = \frac{1}{2} \frac{\partial}{\partial a_m} a^2} \\
&= \int_\Omega p a \phi_m
\end{aligned}
$$

Adjoint equation

given $a = \sum_{i=1}^{L} a_i \phi_i$ with corresponding $T = T_a$

solve $\quad \nabla \cdot (p \nabla T_a) = 0 \text{ in } \Omega \setminus \{x_s\}; \quad p \frac{\partial T_a}{\partial \nu} = T_a - T^* \text{ on } \partial\Omega$

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial a_m}(a) &= \frac{1}{2} \frac{\partial}{\partial a_m} \int_{\partial\Omega} |T_a - T^*|^2 = \int_{\partial\Omega} \underbrace{(T_a - T^*)}_{= p \frac{\partial T_a}{\partial \nu}} \frac{\partial T_a}{\partial a_m} \\ &= \int_{\Omega} \underbrace{\nabla \cdot (p \nabla T_a)}_{=0} \frac{\partial T_a}{\partial a_m} + \int_{\Omega} p \quad \underbrace{\nabla T_a \cdot \nabla \frac{\partial T_a}{\partial a_m}}_{= \frac{1}{2} \frac{\partial}{\partial a_m} |\nabla T_a|^2 = \frac{1}{2} \frac{\partial}{\partial a_m} a^2} \\ &= \int_{\Omega} p a \phi_m \end{aligned}$$

Adjoint equation

given $a = \sum_{i=1}^{L} a_i \phi_i$ with corresponding $T = T_a$

solve $\quad \nabla \cdot (p \nabla T_a) = 0 \;\text{ in } \Omega \setminus \{x_s\}; \quad p \dfrac{\partial T_a}{\partial \nu} = T_a - T^* \;\text{ on } \partial\Omega$

$$
\begin{aligned}
\frac{\partial \mathcal{J}}{\partial a_m}(a) &= \frac{1}{2} \frac{\partial}{\partial a_m} \int_{\partial\Omega} |T_a - T^*|^2 = \int_{\partial\Omega} \underbrace{(T_a - T^*)}_{= p \frac{\partial T_a}{\partial \nu}} \frac{\partial T_a}{\partial a_m} \\
&= \int_{\Omega} \underbrace{\nabla \cdot (p \nabla T_a)}_{=0} \frac{\partial T_a}{\partial a_m} + \int_{\Omega} p \quad \underbrace{\nabla T_a \cdot \nabla \frac{\partial T_a}{\partial a_m}}_{= \frac{1}{2} \frac{\partial}{\partial a_m} |\nabla T_a|^2 = \frac{1}{2} \frac{\partial}{\partial a_m} a^2} \\
&= \int_{\Omega} p a \phi_m
\end{aligned}
$$

Adjoint equation

given $a = \sum_{i=1}^{L} a_i \phi_i$ with corresponding $T = T_a$

solve $\nabla \cdot (p \nabla T_a) = 0$ in $\Omega \setminus \{x_s\}$; $\quad p \dfrac{\partial T_a}{\partial \nu} = T_a - T^*$ on $\partial\Omega$

$$
\begin{aligned}
\frac{\partial \mathcal{J}}{\partial a_m}(a) &= \frac{1}{2} \frac{\partial}{\partial a_m} \int_{\partial\Omega} |T_a - T^*|^2 = \int_{\partial\Omega} \underbrace{(T_a - T^*)}_{= p \frac{\partial T_a}{\partial \nu}} \frac{\partial T_a}{\partial a_m} \\
&= \int_{\Omega} \underbrace{\nabla \cdot (p \nabla T_a)}_{=0} \frac{\partial T_a}{\partial a_m} + \int_{\Omega} p \underbrace{\nabla T_a \cdot \nabla \frac{\partial T_a}{\partial a_m}}_{= \frac{1}{2} \frac{\partial}{\partial a_m} |\nabla T_a|^2 = \frac{1}{2} \frac{\partial}{\partial a_m} a^2} \\
&= \int_{\Omega} p a \phi_m
\end{aligned}
$$

Adjoint equation

given $a = \sum_{i=1}^{L} a_i \phi_i$ with corresponding $T = T_a$

solve $\nabla \cdot (p \nabla T_a) = 0$ in $\Omega \setminus \{x_s\}$; $p \dfrac{\partial T_a}{\partial \nu} = T_a - T^*$ on $\partial\Omega$

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial a_m}(a) &= \frac{1}{2} \frac{\partial}{\partial a_m} \int_{\partial\Omega} |T_a - T^*|^2 = \int_{\partial\Omega} \underbrace{(T_a - T^*)}_{= p \frac{\partial T_a}{\partial \nu}} \frac{\partial T_a}{\partial a_m} \\
&= \int_\Omega \underbrace{\nabla \cdot (p \nabla T_a)}_{=0} \frac{\partial T_a}{\partial a_m} + \int_\Omega p \underbrace{\nabla T_a \cdot \nabla \frac{\partial T_a}{\partial a_m}}_{= \frac{1}{2} \frac{\partial}{\partial a_m} |\nabla T_a|^2 = \frac{1}{2} \frac{\partial}{\partial a_m} a^2} \\
&= \int_\Omega p a \phi_m
\end{aligned}$$

Discrete adjoint equation

given $a \in K$ with corresponding discrete state $T_h : \bar{\Omega}_h \to \mathbb{R}$

find $P_h : \bar{\Omega}_h \setminus \{x_s\} \to \mathbb{R}$ such that

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big(\big[\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}}\big]^+ \frac{P_h(x_\alpha)}{h_{\alpha\beta}} - \big[\frac{T_h(x_\beta) - T_h(x_\alpha)}{h_{\alpha\beta}}\big]^+ \frac{P_h(x_\beta)}{h_{\alpha\beta}} \Big)$$

$$= \begin{cases} 0 & , x_\alpha \in \Omega_h \setminus \{x_s\} \\ \frac{h_\alpha}{h^2}\big(T_h(x_\alpha) - T^*(x_\alpha)\big) & , x_\alpha \in \partial\Omega_h . \end{cases}$$

Lemma The above problem has a unique solution.

Remark P_h can be efficiently calculated by using an ordering of the grid points with respect to the size of T_h.

Discrete adjoint equation

given $a \in K$ with corresponding discrete state $T_h : \bar{\Omega}_h \to \mathbb{R}$

find $P_h : \bar{\Omega}_h \setminus \{x_s\} \to \mathbb{R}$ such that

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big(\big[\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}}\big]^+ \frac{P_h(x_\alpha)}{h_{\alpha\beta}} - \big[\frac{T_h(x_\beta) - T_h(x_\alpha)}{h_{\alpha\beta}}\big]^+ \frac{P_h(x_\beta)}{h_{\alpha\beta}} \Big)$$

$$= \begin{cases} 0 & , x_\alpha \in \Omega_h \setminus \{x_s\} \\ \frac{h_\alpha}{h^2} \big(T_h(x_\alpha) - T^*(x_\alpha)\big) & , x_\alpha \in \partial\Omega_h. \end{cases}$$

Lemma The above problem has a unique solution.

Remark P_h can be efficiently calculated by using an ordering of the grid points with respect to the size of T_h.

Discrete adjoint equation

given $a \in K$ with corresponding discrete state $T_h : \bar{\Omega}_h \to \mathbb{R}$

find $P_h : \bar{\Omega}_h \setminus \{x_s\} \to \mathbb{R}$ such that

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big(\big[\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}}\big]^+ \frac{P_h(x_\alpha)}{h_{\alpha\beta}} - \big[\frac{T_h(x_\beta) - T_h(x_\alpha)}{h_{\alpha\beta}}\big]^+ \frac{P_h(x_\beta)}{h_{\alpha\beta}} \Big)$$

$$= \begin{cases} 0 & , x_\alpha \in \Omega_h \setminus \{x_s\} \\ \frac{h_\alpha}{h^2}\big(T_h(x_\alpha) - T^*(x_\alpha)\big) & , x_\alpha \in \partial\Omega_h . \end{cases}$$

Lemma The above problem has a unique solution.

Remark P_h can be efficiently calculated by using an ordering of the grid points with respect to the size of T_h.

Discrete adjoint equation

given $a \in K$ with corresponding discrete state $T_h : \bar{\Omega}_h \to \mathbb{R}$

find $P_h : \bar{\Omega}_h \setminus \{x_s\} \to \mathbb{R}$ such that

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \Big(\big[\frac{T_h(x_\alpha) - T_h(x_\beta)}{h_{\alpha\beta}} \big]^+ \frac{P_h(x_\alpha)}{h_{\alpha\beta}} - \big[\frac{T_h(x_\beta) - T_h(x_\alpha)}{h_{\alpha\beta}} \big]^+ \frac{P_h(x_\beta)}{h_{\alpha\beta}} \Big)$$

$$= \begin{cases} 0 & , x_\alpha \in \Omega_h \setminus \{x_s\} \\ \frac{h_\alpha}{h^2} \big(T_h(x_\alpha) - T^*(x_\alpha) \big) & , x_\alpha \in \partial\Omega_h. \end{cases}$$

Lemma The above problem has a unique solution.

Remark P_h can be efficiently calculated by using an ordering of the grid points with respect to the size of T_h.

Derivative of $\mathcal{J}_h$

Lemma

Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$. Then, for $m = 1, \ldots, L$

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = h^2 \sum_{x_\alpha \in \bar{\Omega}_h \setminus \{x_s\}} P_h(x_\alpha) a(x_\alpha) \phi_m(x_\alpha) + \delta_h \int_\Omega \nabla a \cdot \nabla \phi_m$$

In practice

- $\min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \sum_{j=1}^{p} \int_{\partial\Omega} |T_a^j(x) - T^{j,*}(x)|^2 dS$

- Minimisation of $\mathcal{J}_h$ by a projected gradient method

Derivative of $\mathcal{J}_h$

Lemma

Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$. Then, for $m = 1, \dots, L$

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = h^2 \sum_{x_\alpha \in \bar{\Omega}_h \setminus \{x_s\}} P_h(x_\alpha) a(x_\alpha) \phi_m(x_\alpha) + \delta_h \int_\Omega \nabla a \cdot \nabla \phi_m$$

In practice

- $\min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \sum_{j=1}^{p} \int_{\partial\Omega} |T_a^j(x) - T^{j,*}(x)|^2 dS$

- Minimisation of $\mathcal{J}_h$ by a projected gradient method

Derivative of $\mathcal{J}_h$

Lemma

Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$. Then, for $m = 1, \ldots, L$

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = h^2 \sum_{x_\alpha \in \bar{\Omega}_h \setminus \{x_s\}} P_h(x_\alpha) a(x_\alpha) \phi_m(x_\alpha) + \delta_h \int_\Omega \nabla a \cdot \nabla \phi_m$$

In practice

- $\min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \sum_{j=1}^{p} \int_{\partial\Omega} |T_a^j(x) - T^{j,*}(x)|^2 dS$

- Minimisation of $\mathcal{J}_h$ by a projected gradient method

Test example

$\Omega = [-1, 1] \times [0, 2]$

$h = 0.02, \delta_h = h, L = 121$

$T^*(x_\alpha) = T_h(x_\alpha) + \Lambda n(x_\alpha), x_\alpha \in \bar{\Omega}_h$, where

- T_h is the discrete solution for a given $a : \bar{\Omega} \to \mathbb{R}_{>0}$;

- $n(x_\alpha) \in [-1, 1]$ is random noise

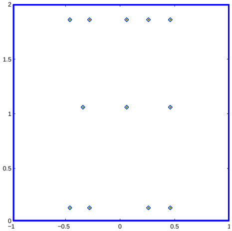


Figure: The distribution of 12 source points in Ω

Test example

$\Omega = [-1, 1] \times [0, 2]$

$h = 0.02, \delta_h = h, L = 121$

$T^*(x_\alpha) = T_h(x_\alpha) + \Lambda n(x_\alpha), x_\alpha \in \bar{\Omega}_h$, where

- T_h is the discrete solution for a given $a : \bar{\Omega} \to \mathbb{R}_{>0}$;

- $n(x_\alpha) \in [-1, 1]$ is random noise

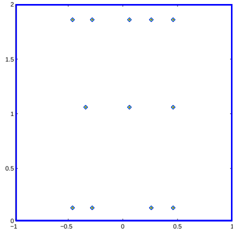


Figure: The distribution of 12 source points in Ω

Test example

$\Omega = [-1, 1] \times [0, 2]$

$h = 0.02, \delta_h = h, L = 121$

$T^*(x_\alpha) = T_h(x_\alpha) + \Lambda n(x_\alpha), x_\alpha \in \bar{\Omega}_h$, where

- T_h is the discrete solution for a given $a : \bar{\Omega} \to \mathbb{R}_{>0}$;

- $n(x_\alpha) \in [-1, 1]$ is random noise

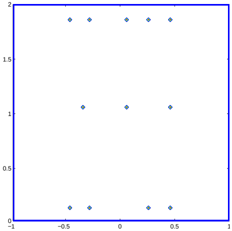


Figure: The distribution of 12 source points in Ω

Behaviour of $\|a - a_h\|$

h	$\|a_{h_f} - a_h\|_0$	eoc
0.04	$5.69 \cdot 10^{-3}$	-
0.03	$5.04 \cdot 10^{-3}$	0.665
0.025	$4.29 \cdot 10^{-3}$	0.560
0.02	$3.48 \cdot 10^{-3}$	0.938
0.016	$2.91 \cdot 10^{-3}$	0.981

Effect of noise

Λ	$\mathcal{J}_h(a_h)$	$\|a - a_h\|_0$
0	$2.22 \cdot 10^{-6}$	$1.30 \cdot 10^{-3}$
0.01	$1.57 \cdot 10^{-3}$	$2.45 \cdot 10^{-3}$
0.05	$3.92 \cdot 10^{-2}$	$1.11 \cdot 10^{-2}$
0.1	$1.57 \cdot 10^{-1}$	$2.24 \cdot 10^{-2}$

Behaviour of $\|a - a_h\|$

h	$\|a_{h_f} - a_h\|_0$	eoc
0.04	$5.69 \cdot 10^{-3}$	-
0.03	$5.04 \cdot 10^{-3}$	0.665
0.025	$4.29 \cdot 10^{-3}$	0.560
0.02	$3.48 \cdot 10^{-3}$	0.938
0.016	$2.91 \cdot 10^{-3}$	0.981

Effect of noise

Λ	$\mathcal{J}_h(a_h)$	$\|a - a_h\|_0$
0	$2.22 \cdot 10^{-6}$	$1.30 \cdot 10^{-3}$
0.01	$1.57 \cdot 10^{-3}$	$2.45 \cdot 10^{-3}$
0.05	$3.92 \cdot 10^{-2}$	$1.11 \cdot 10^{-2}$
0.1	$1.57 \cdot 10^{-1}$	$2.24 \cdot 10^{-2}$

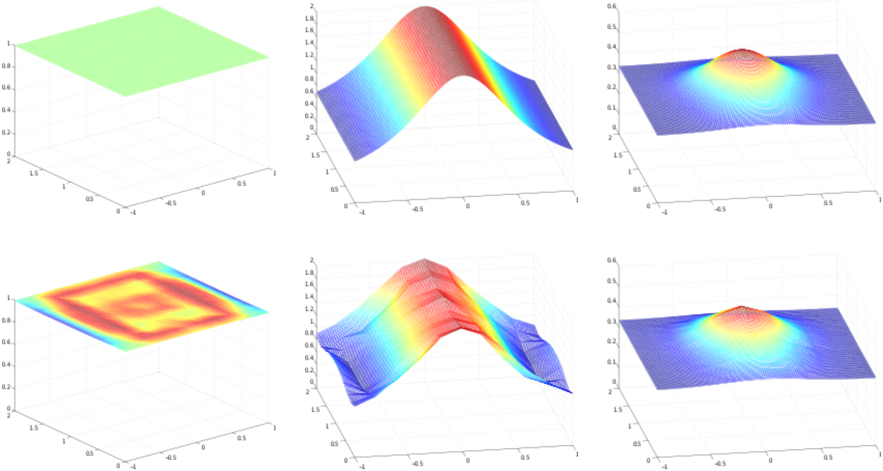


Figure: $a(x)$ (upper plots), $a_h(x)$ with $L = 121$, $\delta_h = h$ and 12 source points (lower plots)

Concluding remarks

- Contribution
  - Analysis of model and numerical analysis of scheme
  - Efficient solution of discrete adjoint equation
- Issues
  - Observations may not be the first arrival time
  - Velocity model: the slowness may be discontinuous across interfaces
    Another model

$$K = \{a : \bar{\Omega} \to \mathbb{R} | a(x) = (a_1 - a_0)\phi(x) + a_0\phi(x)\}$$

$$\mathcal{J}_\phi(a) = \mathcal{J}(a) + \sigma\Big(\int_\Omega [\frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{\epsilon}W(\phi)]dx\Big)$$