On Nonlinear Optimal Control Problems with an L^1 Norm

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Workshop on Inverse Problems and Optimal Control for PDEs

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Overview

1. Introduction and Problem Setting
2. 1st- and 2nd-Order Optimality Conditions
3. Finite Element Error Estimates and Examples
4. Extension: Directional Sparsity
(joint with Georg Stadler, ICES, Texas)

Problem Setting for this Talk

Control problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)} \\ \text{such that} \quad & u_a \le u \le u_b \qquad (u_a < 0 < u_b) \\ \text{and} \quad & y \text{ solves the PDE} \end{aligned}$$

Semilinear partial differential equation

$$\begin{aligned} -\Delta y + a(\cdot, y) &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma \end{aligned}$$

Why Consider $\|u\|_{L^1(\Omega)}$?

- The L^1-norm
$$\|u\|_{L^1(\Omega)} = \int_\Omega |u(x)| \, \mathrm{d}x$$
is often a natural measure of the true control cost.
- It also has the effect of promoting sparse controls.

[Vossen, Maurer (2006); Stadler (2009); Clason, Kunisch (2011)]

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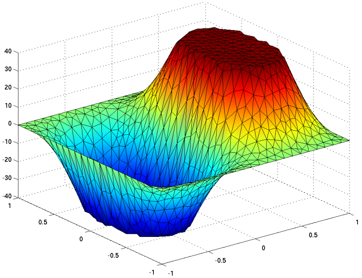
- Applications in control:
  - actuator placement
  - on/off control structure desired
  - true measure of control cost
- Other applications using the 1-norm:
  - compressed sensing
  - TV-based image restoration

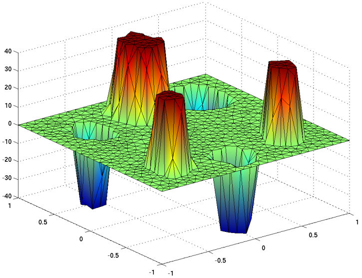
[Vossen, Maurer (2006); Stadler (2009); Clason, Kunisch (2011)]

A First Glance at Sparsity

$\mu = 0$



$\mu > 0$



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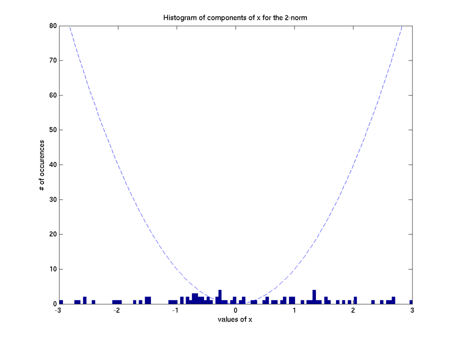
Smooth minimization problem

minimize $\frac{1}{2}\|x\|_2^2$ s.t. $Ax = b$

Histogram (solution components' sizes)



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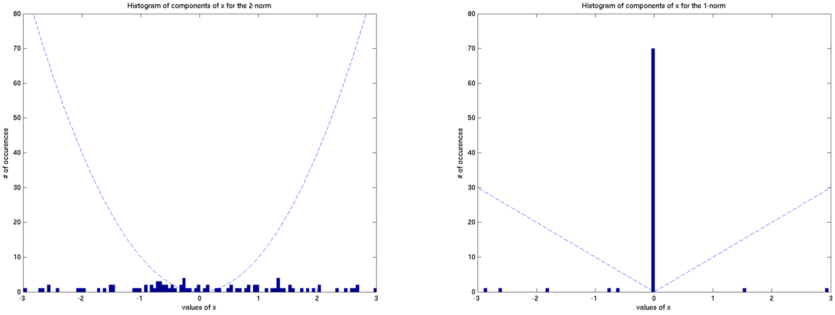

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$$x + A^\top p = 0$$
$$Ax - b = 0$$

Convex minimization problem

$$\text{minimize} \quad \|x\|_1 \quad \text{s.t.} \quad Ax = b$$

$$\lambda + A^\top p = 0, \quad \lambda \in \partial\|x\|_1$$
$$Ax - b = 0$$

$$\begin{aligned} \lambda_i &= -1 && \text{if } x_i < 0 \\ \lambda_i &= +1 && \text{if } x_i > 0 \\ \lambda_i &\in [-1, 1] && \text{if } x_i = 0 \end{aligned}$$

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$$x_i = \max\{0, x_i + c\,(\lambda_i - 1)\} + \min\{0, x_i + c\,(\lambda_i + 1)\}$$

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Basic Assumptions Concerning the PDE

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Assumptions

- $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, with $C^{1,1}$-boundary or convex, polygonal set
- a is Carathéodory-function, monotone, C^2 w.r.t. y

Properties

- For $u \in L^p(\Omega)$, $n/2 < p \le 2$ the solution $y = G(u) \in W^{2,p}(\Omega)$
- $G : L^p(\Omega) \to W^{2,p}(\Omega)$ is C^2, derivatives by linearization

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- $\xi^\top A(x)\,\xi \geq \underline{a}\,\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$, $\underline{a} > 0$

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- ▢ is differentiable w.r.t. $u \in L^2(\Omega)$

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Properties

- (orange box) is differentiable w.r.t. $u \in L^2(\Omega)$
- (blue box) is convex w.r.t. u

Problem Setting

Control problem

$$\text{Minimize} \quad \underbrace{\frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}}_{\text{f(u)}} + \underbrace{\mu\,\|u\|_{L^1(\Omega)}}_{\text{j(u)}}$$

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Sums of Differentiable and Convex Function

Definition of a generalized subdifferential

Let f be differentiable and j convex, $J = f + j$. The generalized subdifferential $\partial J(x)$ is defined as

$$\partial J(x) = \nabla f(x) + \partial j(x)$$

- This coincides with known generalized derivatives (e.g. Fréchet, Clarke) on this class of functions.
- This ensures the uniqueness, i.e. ∂J does not depend on the splitting of J into f and j.

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Necessary optimality condition of first order

$$0 \in \partial J(x) = \nabla f(x) + \partial j(x)$$

First-Order Necessary Condition

$$f(u) = \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}, \qquad j(u) = \|u\|_{L^1(\Omega)}$$

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$$\nabla f(\overline{u}) = \underbrace{G'(\overline{u})^\star(\overline{y} - y_d)}_{\text{adjoint state } \overline{p}} + \nu\,\overline{u}, \quad \text{where} \quad \overline{y} = G(\overline{u})$$

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First-order necessary optimality conditions

$$0 \in \nabla f(\overline{u}) + \mu\,\partial j(\overline{u})$$

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... with convex control constraints: $U_{\text{ad}} = \{u \in L^2(\Omega) : u_a \le u \le u_b\}$

$$0 \le \langle \nabla f(\overline{u}) + \mu\,\overline{\lambda},\ u - \overline{u}\rangle_{L^2(\Omega)} \quad \text{for all } u \in U_{\text{ad}}, \quad \overline{\lambda} \in \partial j(\overline{u})$$

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Theorem

Let $\overline{u}$ be a local min. with state $\overline{y} = G(\overline{u})$. Then there exist an adjoint state $\overline{p} = G'(\overline{u})^\star(\overline{y} - y_d)$ and a subgradient $\overline{\lambda} \in \partial j(\overline{u}) = \partial \|\overline{u}\|_{L^1(\Omega)}$ s.t.

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Subgradient of the L^1 norm

$$\overline{\lambda}(x) \begin{cases} = +1 & \text{where } \overline{u}(x) > 0 \\ \in [-1, 1] & \text{where } \overline{u}(x) = 0 \\ = -1 & \text{where } \overline{u}(x) < 0 \end{cases}$$

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Adjoint equation

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Corollary: projection formulas

$$\overline{u}(x) = \operatorname{proj}_{[u_a,u_b]}\Big(-\frac{1}{\nu}\big(\overline{p}(x) + \mu\,\overline{\lambda}(x)\big)\Big)$$

$$\overline{\lambda}(x) = \operatorname{proj}_{[-1,+1]}\Big(-\frac{1}{\mu}\overline{p}(x)\Big)$$

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It follows that $\overline{u}, \overline{\lambda} \in C^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$.
Moreover, $\overline{\lambda} \in \partial \|\overline{u}\|_{L^1(\Omega)}$ is unique.

Second-Order Optimality Conditions

Critical cone at stationary point $\overline{u}$ with associated $\overline{\lambda} \in \partial j(\overline{u})$

$$\mathcal{C}^+_{\overline{u}} := \left\{ v \in L^2(\Omega) : f'(\overline{u})\, v + \mu \, \langle \overline{\lambda},\, v \rangle = 0 \right\}$$

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$$\langle f''(\overline{u})\, v, \, v \rangle > 0 \quad \text{for all } v \in \mathcal{C}_{\overline{u}}^{+} \setminus \{0\} \quad \Rightarrow \quad \overline{u} \text{ is locally optimal}$$

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. . . with control constraints

$$\mathcal{C}_{\overline{u}} := \left\{ v \in L^2(\Omega) : \begin{array}{l} f'(\overline{u})\, v + \mu \, j'(\overline{u}; v) = 0 \\ \left.\begin{array}{l} v \geq 0 \text{ where } \overline{u} = u_a \\ v \leq 0 \text{ where } \overline{u} = u_b \end{array}\right\} \; v \in \mathcal{T}_{U_{ad}}(\overline{u}) \end{array} \right\}$$

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Second-Order Sufficient Conditions

Critical cone (closed, convex)

$$C_{\overline{u}} := \{ v \in \mathcal{T}_{U_{\text{ad}}}(\overline{u}) : f'(\overline{u})\, v + \mu\, j'(\overline{u}; v) = 0 \}$$

Theorem

Let $\overline{u} \in U_{\text{ad}}$ and $\overline{\lambda} \in \partial j(\overline{u})$ satisfy the first order necessary condition. Assume $\langle f''(\overline{u})\, v,\, v\rangle > 0$ holds for all $v \in C_{\overline{u}} \setminus \{0\}$. Then there exist $\delta > 0$, $\varepsilon > 0$ such that

$$J(\overline{u}) + \frac{\delta}{2} \| u - \overline{u} \|^2_{L^2(\Omega)} \le J(u) \quad \text{for all } u \in U_{\text{ad}} \cap B^{L^2}_{\varepsilon}(\overline{u}).$$

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Corollary

There exist $\tau > 0$, $\delta_2 > 0$ such that $\langle f''(\overline{u})\, v,\ v\rangle \ge \delta_2\, \|v\|^2_{L^2(\Omega)}$ for all

$$v \in C_{\overline{u}}^{\tau} = \{v \in \mathcal{T}_{U_{\text{ad}}}(\overline{u}) : f'(\overline{u})\, v + \mu\, j'(\overline{u}; v) \le \tau\, \|v\|_{L^2(\Omega)}\}$$

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Finite Element Approximation

- Regular triangulation $\{\mathcal{T}_h\}$ of Ω, $\Omega_h = \cup_{T\in\mathcal{T}_h} T$.
- Discrete space of (adjoint) states (piecewise linear):

$$Y_h = \{y_h \in C(\overline{\Omega}) : y_{h|T} \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ on } \overline{\Omega} \setminus \Omega_h\}$$

- Discrete PDE:

$$\int_{\Omega_h} \nabla z_h^\top A \nabla y_h + a(\cdot, y_h)\,\mathrm{d}x = \int_{\Omega_h} u\, z_h\,\mathrm{d}x \quad \text{for all } z_h \in Y_h$$

- Discrete space of controls (piecewise constant):

$$U_h = \{u_h \in L^2(\Omega_h) : u_{h|T} \equiv \mathrm{const} \text{ for all } T \in \mathcal{T}_h\}$$

Discrete Problem

Discrete optimization problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|G_h(u_h) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)} + \mu\,\|u_h\|_{L^1(\Omega)} \\ \text{such that} \quad & u_a \le u_h \le u_b \\ \text{and} \quad & u_h \in U_h \end{aligned}$$

Convergence of Minimizers

Theorem (approximation of global minima)

For every $h > 0$ let $\overline{u}_h$ be a global solution of the discrete problem. Then the sequence $\{\overline{u}_h\}_{h>0}$ is bounded in $L^\infty(\Omega)$ and there exist subsequences, denoted in the same way, converging to a point $\overline{u}$ in the weak* $L^\infty(\Omega)$ topology. Any of these limit points is a global solution of the continuous problem. Moreover, we have

$$\lim_{h\to 0} \left\{ \|\overline{u} - \overline{u}_h\|_{L^\infty(\Omega_h)} \right\} = 0 \quad \text{and} \quad \lim_{h\to 0} J_h(\overline{u}_h) = J(\overline{u}).$$

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Convergence of Minimizers

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Theorem (approximation of strict local minima)

Let $\overline{u}$ be a strict local minimum of the continuous problem, then there exists a sequence $\{\overline{u}_h\}_{h>0}$ of local minima of the discrete problems which converge towards $\overline{u}$.

Error Estimates

Theorem (piecewise constant discretization)

Let $\overline{u}$ be a solution of the continuous problem and $\{\overline{u}_h\}$ a sequence of solutions of the discrete problems converging towards $\overline{u}$. Moreover, assume that the second-order sufficient condition is satisfied.
Then there exists $C > 0$ such that

$$\|\overline{u} - \overline{u}_h\|_{L^\infty(\Omega_h)} + \|\overline{y} - \overline{y}_h\|_{L^\infty(\Omega_h)} + \|\overline{p} - \overline{p}_h\|_{L^\infty(\Omega_h)} + \|\overline{\lambda} - \overline{\lambda}_h\|_{L^\infty(\Omega_h)} \leq C\,h.$$

Error Estimates

Theorem (piecewise constant discretization)

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$$\|\overline{u}-\overline{u}_h\|_{L^\infty(\Omega_h)}+\|\overline{y}-\overline{y}_h\|_{L^\infty(\Omega_h)}+\|\overline{p}-\overline{p}_h\|_{L^\infty(\Omega_h)}+\|\overline{\lambda}-\overline{\lambda}_h\|_{L^\infty(\Omega_h)} \le C\,h.$$

Idea of the proof

Extend $\overline{u}_h$ to $\Omega \setminus \Omega_h$ by $\overline{u}$. We obtain by optimality

$$f'(\overline{u})(\overline{u}_h - \overline{u}) + \mu \int_\Omega \overline{\lambda}\ (\overline{u}_h - \overline{u}\ )\,\mathrm{d}x \ge 0$$

$$f_h'(\overline{u}_h)(u_h - \overline{u}_h) + \mu \int_\Omega \overline{\lambda}_h(u_h - \overline{u}_h)\,\mathrm{d}x \ge 0 \quad \text{for all } u_h \in U_h \cap U_{\mathrm{ad}}$$

Error Estimates

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Idea of the proof

$$\le \big[f'(\overline{u}_h) - f'(\overline{u})\big](\overline{u}_h - \overline{u}) \le \dots$$

Error Estimates

Theorem (piecewise constant discretization)

Let $\overline{u}$ be a solution of the continuous problem and $\{\overline{u}_h\}$ a sequence of solutions of the discrete problems converging towards $\overline{u}$. Moreover, assume that the second-order sufficient condition is satisfied.

Then there exists $C > 0$ such that

$$\|\overline{u}-\overline{u}_h\|_{L^\infty(\Omega_h)} + \|\overline{y}-\overline{y}_h\|_{L^\infty(\Omega_h)} + \|\overline{p}-\overline{p}_h\|_{L^\infty(\Omega_h)} + \|\overline{\lambda}-\overline{\lambda}_h\|_{L^\infty(\Omega_h)} \le C\, h.$$

Idea of the proof

$$\frac{\delta}{2}\|\overline{u}_h - \overline{u}\|^2_{L^2(\Omega)} \le \big[f'(\overline{u}_h) - f'(\overline{u})\big](\overline{u}_h - \overline{u}) \le \dots$$

since $\overline{u}_h - \overline{u} \in C^\tau_{\overline{u}}$ and SSC hold

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Error Estimates

Theorem (piecewise constant discretization)

Let $\overline{u}$ be a solution of the continuous problem and $\{\overline{u}_h\}$ a sequence of solutions of the discrete problems converging towards $\overline{u}$. Moreover, assume that the second-order sufficient condition is satisfied.
Then there exists $C > 0$ such that

$$\|\overline{u}-\overline{u}_h\|_{L^\infty(\Omega_h)}+\|\overline{y}-\overline{y}_h\|_{L^\infty(\Omega_h)}+\|\overline{p}-\overline{p}_h\|_{L^\infty(\Omega_h)}+\|\overline{\lambda}-\overline{\lambda}_h\|_{L^\infty(\Omega_h)} \le C\,h.$$

Theorem (variational discretization, Hinze (2005))

Let $\overline{u}$ be a solution of the continuous problem and $\{\overline{u}_h\}$ a sequence of solutions of the variational discretized probem, converging towards $\overline{u}$. Moreover, assume that the second-order sufficient condition is satisfied.
Then there is $C > 0$, such that

$$\|\overline{u}-\overline{u}_h\|_{L^2(\Omega_h)}+\|\overline{y}-\overline{y}_h\|_{L^2(\Omega_h)}+\|\overline{p}-\overline{p}_h\|_{L^2(\Omega_h)}+\|\overline{\lambda}-\overline{\lambda}_h\|_{L^2(\Omega_h)} \le C\,h^2.$$

Test Problem

Control problem

$$\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + 10^{-3}\,\|u\|^2_{L^2(\Omega)} + 3\cdot 10^{-2}\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b$$

- $y_d(x_1, x_2) = 2\,\sin(2\,\pi\,x_1)\,\sin(\pi\,x_2)\,\exp(x_1)$
- PDE:

$$\begin{aligned} -\Delta y + y^3 &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma \end{aligned}$$

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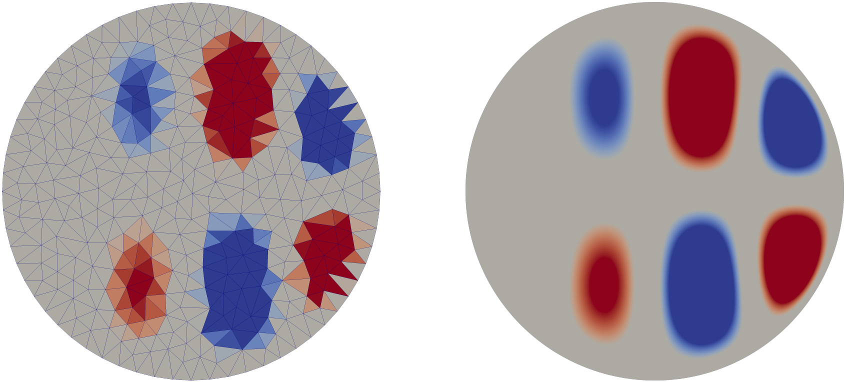
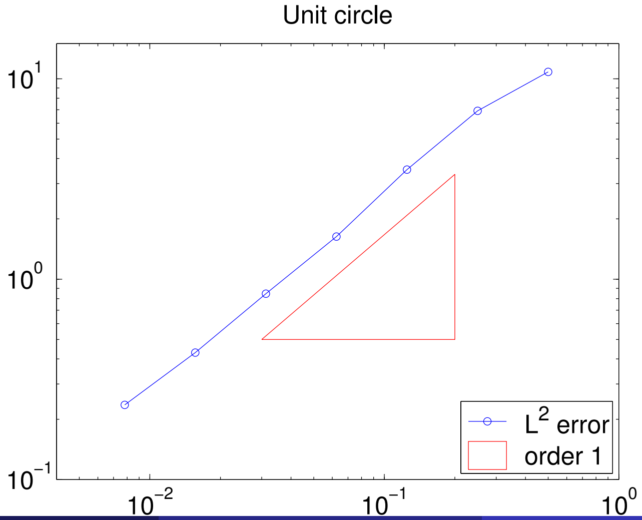
Solutions for $h = 2^{-3}$ and $h = 2^{-8}$



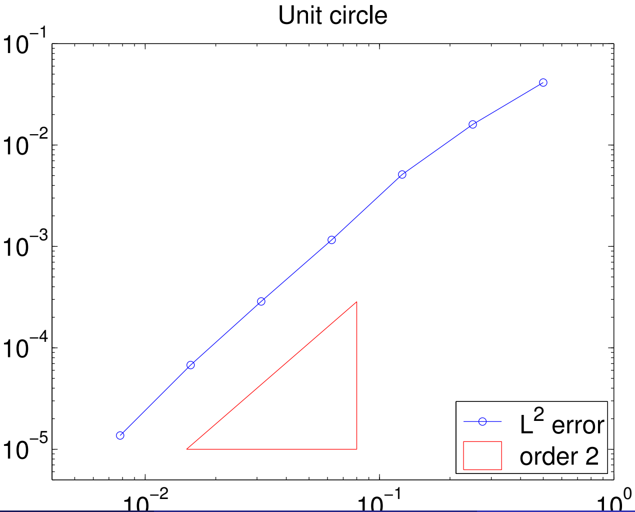
Convergence (Full Discretization)

Error in the control:



Convergence (Variational Discretization)

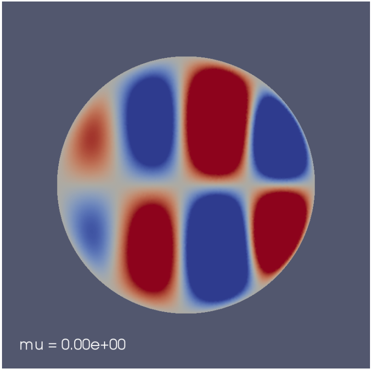
Error in the adjoint:



Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$



$\mu = 0.00$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu \, \|u\|_{L^1(\Omega)}$$

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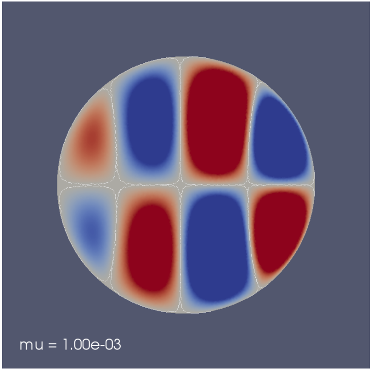
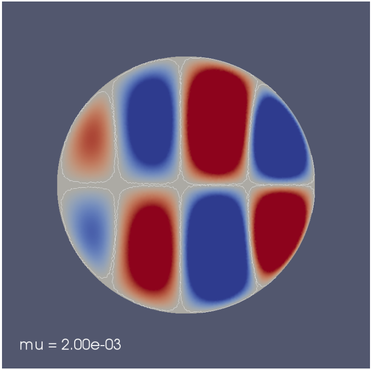


$\mu = 1.00\text{E–}03$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$

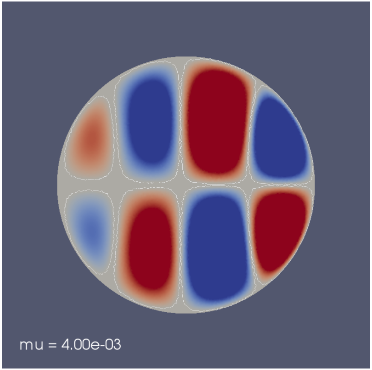


$\mu = 2.00\text{E–}03$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$

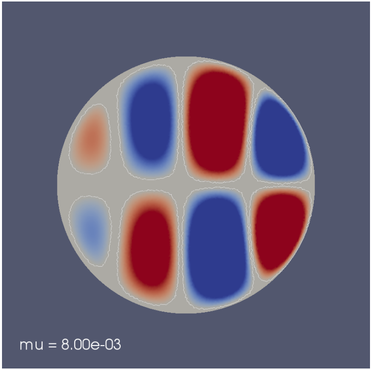


$\mu = 4.00\text{E–}03$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$

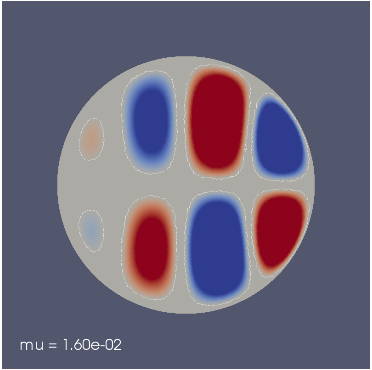


$\mu = 8.00\text{E–}03$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$

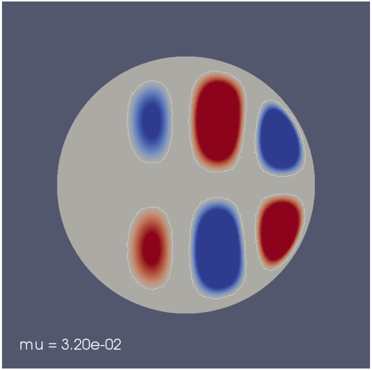


$\mu = 1.60\text{E–}02$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$

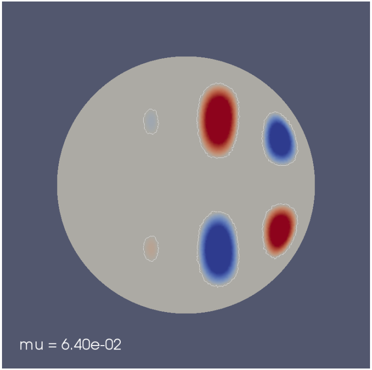


$\mu = 3.20\text{E–}02$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$

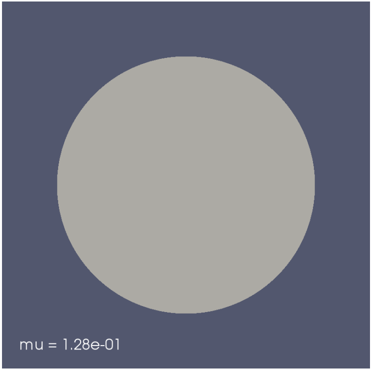


$\mu = 6.40\text{E}–02$

Influence of Parameter μ

$$\text{Minimize} \quad \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} + \mu\,\|u\|_{L^1(\Omega)}$$

$$\text{such that} \quad u_a \le u \le u_b \qquad (u_a < 0 < u_b)$$



$\mu = 1.28\text{E}{-}01$

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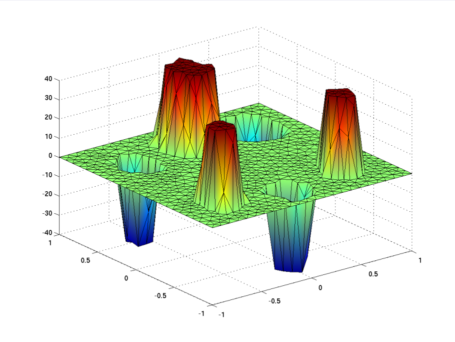
Overview

1. Introduction and Problem Setting
2. 1st- and 2nd-Order Optimality Conditions
3. Finite Element Error Estimates and Examples
4. Extension: Directional Sparsity
(joint with Georg Stadler, ICES, Texas)

Can we do Better Than Just Sparse?

Sparsity

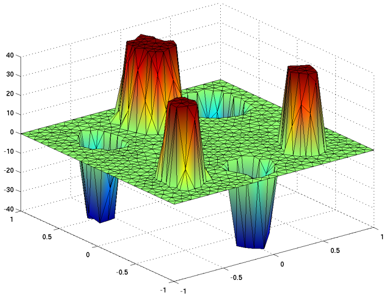


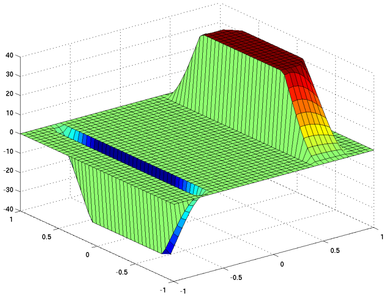
Objective function

$$\frac{1}{2}\|y - y_d\|_{L^2}^2 + \beta\,\|u\|_{L^1}$$

Can we do Better Than Just Sparse?

Sparsity vs. directional sparsity





Objective function

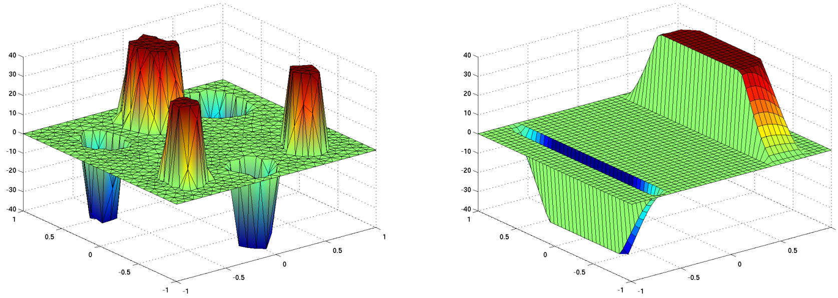
$$\frac{1}{2}\|y - y_d\|_{L^2}^2 + \beta \, \|u\|_{L^1}$$

Objective function

$$\frac{1}{2}\|y - y_d\|_{L^2}^2 + \beta \, \|u\|_{L^1(L^2)}$$

Can we do Better Than Just Sparse?

Sparsity vs. directional sparsity

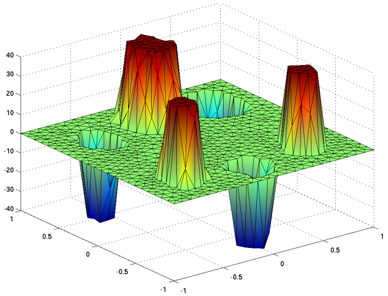


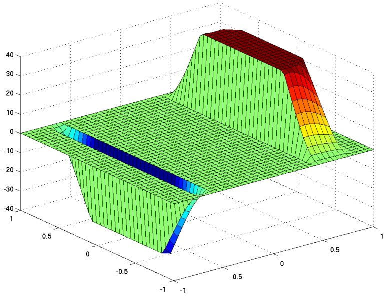
Properties

- no structural assumptions made

Can we do Better Than Just Sparse?

Sparsity vs. directional sparsity





Properties

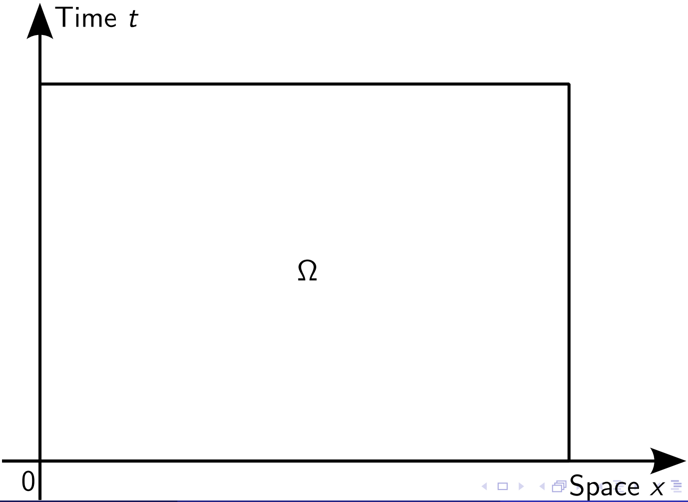
- no structural assumptions made

Properties

- exploits known or desired group sparsity structure

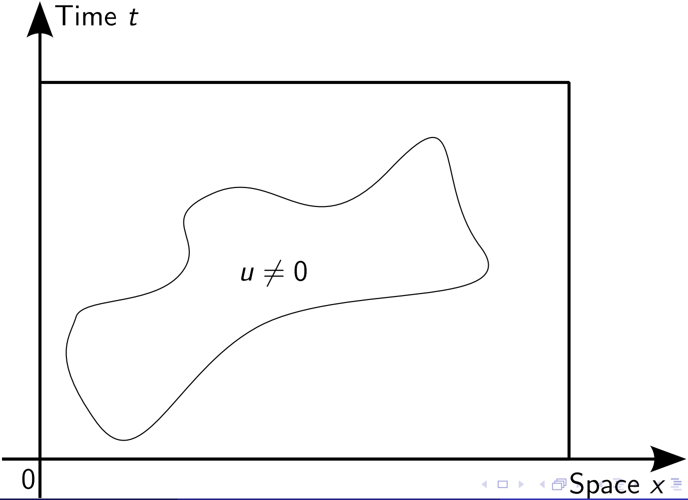
Directional Sparsity with Parabolic PDEs

Placement of actuators for a parabolic problem



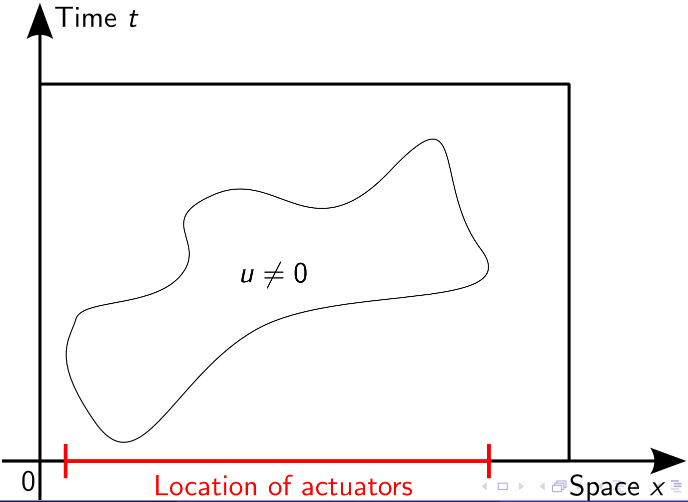
Directional Sparsity with Parabolic PDEs

With Sparsity functional



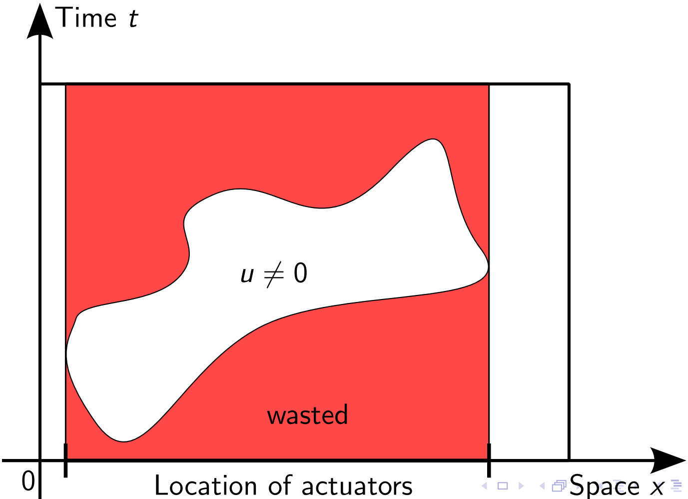
Directional Sparsity with Parabolic PDEs

With Sparsity functional



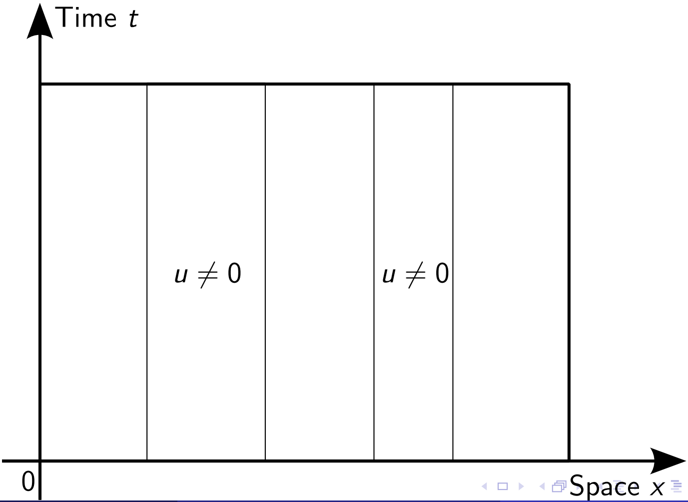
Directional Sparsity with Parabolic PDEs

With Sparsity functional



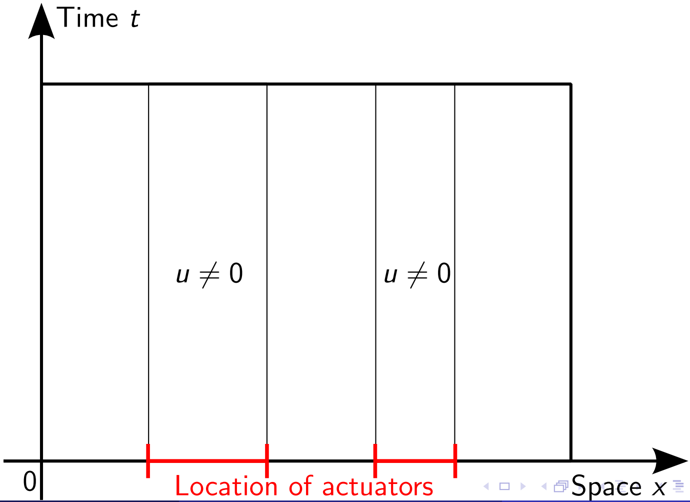
Directional Sparsity with Parabolic PDEs

With Directional Sparsity functional



Directional Sparsity with Parabolic PDEs

With Directional Sparsity functional

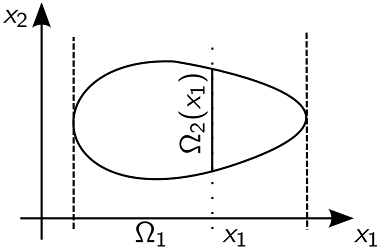


Directional Sparsity: Basic Definition

Problem formulation

$$\min \quad \frac{1}{2}\|\mathcal{S}u - y_d\|_H^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\,\|u\|_{L^1(L^2)}$$

$$\text{s.t.} \quad u_a \le u \le u_b \quad \text{a.e. in } \Omega$$

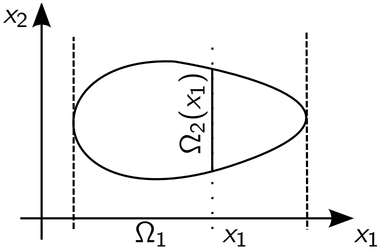


Directional Sparsity: Basic Definition

Problem formulation

$$\min \quad \frac{1}{2}\|\mathcal{S}u - y_d\|_H^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta \int_{\Omega_1} \left(\int_{\Omega_2(x_1)} u(x_1,x_2)^2 \, \mathrm{d}x_2 \right)^{1/2} \mathrm{d}x_1$$

$$\text{s.t.} \quad u_a \le u \le u_b \quad \text{a.e. in } \Omega$$



Related Approaches

Joint sparsity in image restoration

$$\Psi(u) = \sum_{\lambda \in \Lambda} \omega_\lambda \, |\vec{u}_\lambda|_q^p, \quad q = 2, p = 1$$

- $\Omega_1 \mathrel{\widehat{=}} \Lambda$
- $\Omega_2 = \{1, 2, \ldots, \# \text{ of channels}\}$
- with dx_2 = counting measure

[Fornasier, Ramlau, Teschke (2008)]

TV-based image restoration

$$\Psi(u) = \int_{\Omega_1} |\nabla u| \, dx_1, \quad \Omega_2 = \{1, 2, \ldots, N\} \text{ for N-D images}$$

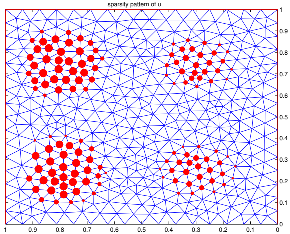
Parabolic Example with Spatial Sparsity

Parabolic example

$$\text{minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \beta\,\|u\|_{L^1(L^2)}$$

$$\text{s.t.} \quad \begin{cases} y_t - \frac{1}{10}\Delta y = u & \text{in } \Omega = \Omega_1 \times (0, T) \\ y = 0 & \text{on } \Gamma \times (0, T) \\ y(\cdot, 0) = 0 & \text{in } \Omega_1 \end{cases}$$

$$\text{and} \quad u_a \le u \le u_b \quad \text{a.e. in } \Omega$$



$n = 2$ sparse directions (space)

$N - n = 1$ non-sparse direction (time)

Summary

- use of $\|u\|_{L^1}$ induces sparse solutions
- it is often an appropriate measure of control cost
- applications in actuator placement problems
- presented 1st- and new 2nd-order optimality conditions
- used them to derive FE error estimates
- extension to directional sparsity concept

References I

E. Casas, R. Herzog, and G. Wachsmuth.
Optimality conditions and error analysis of semilinear elliptic control problems with L^1 cost functional.
Technical report, 2010.

C. Clason and K. Kunisch.
A duality-based approach to elliptic control problems in non-reflexive Banach spaces.
ESAIM: Control, Optimisation, and Calculus of Variations, in print.
doi: 10.1051/cocv/2010003.

M. Fornasier, R. Ramlau, and G. Teschke.
The application of joint sparsity and total variation minimization algorithms in a real-life art restoration problem.
Advances in Computational Mathematics, 31(1–3):301–329, 2009.
URL http://dx.doi.org/10.1007/s10444-008-9103-6.

M. Hinze.
A variational discretization concept in control constrained optimization: The linear-quadratic case.
Computational Optimization and Applications, 30(1):45–61, 2005.

References II

G. Stadler.
Elliptic optimal control problems with L^1-control cost and applications for the placement of control devices.
Computational Optimization and Applications, 44(2):159–181, 2009.
URL `http://dx.doi.org/10.1007/s10589-007-9150-9`.

G. Vossen and H. Maurer.
On L^1-minimization in optimal control and applications to robotics.
Optimal Control Applications and Methods, 27(6):301–321, 2006.