Aspects of numerical analysis in the optimal control of nonlinear PDEs
II: state constraints and problems with quasilinear equations

Fredi Tröltzsch

Technische Universität Berlin

Inverse Problems and Optimal Control for PDEs

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Outline

- Motivating industrial applications
- Elliptic problems with linear state equation
- Semilinear elliptic state equation
- State-constrained control problems
- The case of quasilinear elliptic equations
- Error estimates

Outline

1. Pointwise state constraints
   - The control problem and necessary conditions
   - A test example
   - An open problem for SSC

2. Quasilinear elliptic control problems
   - The problem and well-posedness of the state equation
   - Optimality conditions
   - Approximation by finite elements

The optimal control problem

Let real bounds $\alpha < \beta, \quad y_a < 0 < y_b$ be given.

Problem with control and state constraints:

$$(P) \qquad \min J(y,u) := \frac{1}{2} \int_\Omega (y(x) - y_d(x))^2\, dx + \frac{\lambda}{2} \int_\Omega (u(x))^2\, dx$$

$$\begin{aligned} -\Delta y(x) + d(y(x)) &= u(x) \quad \text{in } \Omega \\ y(x) &= 0 \qquad \text{on } \Gamma, \end{aligned}$$

$$\alpha \le u(x) \le \beta, \quad \text{a.e. in } \Omega,$$

$$y_a \le y(x) \le y_b \quad \text{for all } x \in \bar{\Omega}.$$

Lagrangian function

It holds $y_u = G(u)$, $G : L^2(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$, $n \leq 3$. Therefore, the state-constrained problem can be written as follows:

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$$\mathcal{L}(u, \mu_a, \mu_b) := f(u) + \int_{\bar{\Omega}} (y_a - G(u))d\mu_a + \int_{\bar{\Omega}} (G(u) - y_b)d\mu_b.$$

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- $\mu_a \geq 0$, $\mu_b \geq 0$ in the sense of $C(\bar{\Omega})^*$,
- and the following complementarity conditions are satisfied:

$$\int_{\bar{\Omega}} (y_a - G(\bar{u}))d\mu_a = 0 = \int_{\bar{\Omega}} (G(\bar{u}) - y_b)d\mu_b.$$

Adjoint equation with measures

$$\mathcal{L}(u,\mu_a,\mu_b) \quad = \quad f(u) + \int_{\bar\Omega} (y_a - G(u))\,d\mu_a + \int_{\bar\Omega} (G(u) - y_b)\,d\mu_b$$

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\mathcal{L}(u,\mu_a,\mu_b) &= f(u) + \int_{\bar\Omega}(y_a - G(u))\,d\mu_a + \int_{\bar\Omega}(G(u)-y_b)\,d\mu_b \\
\frac{\partial \mathcal{L}}{\partial u}(\bar u,\mu_a,\mu_b)\,v &= f'(\bar u)\,v + \int_{\bar\Omega}(G'(\bar u)v)\,d(\mu_b-\mu_a) \\
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This new adjoint state $\bar\varphi$ is the weak solution of an adjoint elliptic equation. The first rigorous mathematical explanation of this fact was given by E. Casas.

Reference: E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM J. Control and Optimization 1986.

Necessary optimality conditions

Theorem (Karush-Kuhn-Tucker conditions)

Let $\bar{u}$ be locally optimal for (P) and let $\bar{y}$ the associated state. Assume that a linearized Slater condition is satisfied: $\exists \tilde{u} \in U_{ad}$ *such that*

$$y_a < \Big(G(\bar{u}) + G'(\bar{u})(\tilde{u} - \bar{u})\Big)(x) < y_b \quad \forall x \in \bar{\Omega}.$$

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Then there exist nonnegative regular Borel measures $\mu_a,\ \mu_b$ *on* $\bar{\Omega}$ *and an adjoint state* $\bar{\varphi} \in W^{1,s}(\Omega) \quad \forall s < n/(n-1)$ *such that*

$$\begin{aligned} -\Delta\bar{\varphi} + d'(\bar{y})\bar{\varphi} &= \bar{y} - y_d + \mu_b - \mu_a \\ \bar{\varphi}|_\Gamma &= 0, \end{aligned}$$

$$\int_\Omega (\bar{\varphi} + \lambda\bar{u})(u - \bar{u})\, dx \geq 0 \qquad \forall u \in U_{ad},$$

$$\int_{\bar{\Omega}} (\bar{y} - y_b)\, d\mu_b = \int_{\bar{\Omega}} (\bar{y} - y_a)\, d\mu_a = 0.$$

Two main numerical approaches

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- Reduce the problem to a control-constrained one by penalization:

$$\min_{u \in U_{ad}} f(u) + \rho \int_{\Omega} \left\{ ((y_a - y)_+)^2 + ((y - y_b)_+)^2 \right\} \, dx, \quad \rho >> 0$$

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- If no control constraints are given, you may also regularize as follows:

$$y_a \le y(x) \le y_b \quad \longrightarrow \quad y_a \le \varepsilon\, u(x) + y(x) \le y_b, \quad \varepsilon > 0 \text{ small}$$

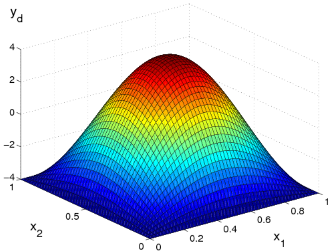
  $\rightarrow$ Lavrentiev type regularization.

Measures? A numerical example

Problem with semilinear equation

$$\min \frac{1}{2}\|y-y_d\|^2+\frac{\lambda}{2}\|u\|^2$$
$$-\Delta y + y + y^3 = u \quad \text{in } \Omega$$
$$\partial_\nu y = 0 \quad \text{on } \Gamma$$
$$-1 \le y(x) \le 1 \quad \text{in } \Omega$$

in $\Omega = (0,1)^2$, $\qquad y_d = 8\,\sin(\pi\, x_1)\,\sin(\pi\, x_2) - 4$



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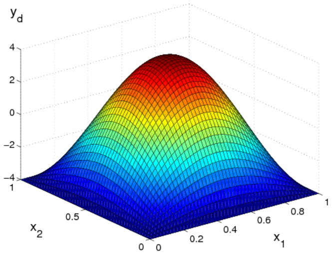

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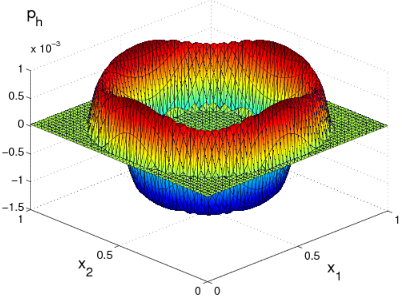
in $\Omega = (0,1)^2$, $\qquad y_d = 8\,\sin(\pi\, x_1)\,\sin(\pi\, x_2) - 4$

Computations: Christian Meyer, by regularization $-1 \le \varepsilon u + y \le 1$

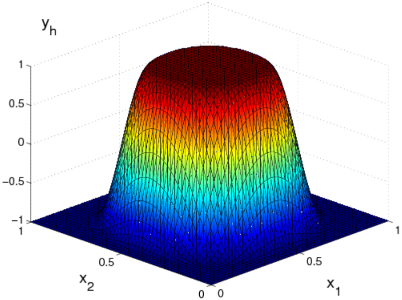
Numerical Technique: SQP + primal dual active set strategy

Test run

Data: $\lambda = 10^{-5}$, $\varepsilon = 10^{-4}$



Control u



State y

Lagrange multipliers μ_a, μ_b

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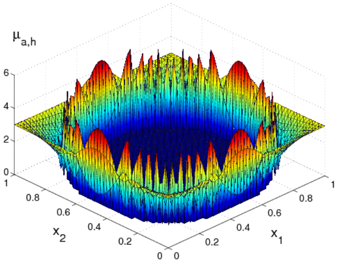
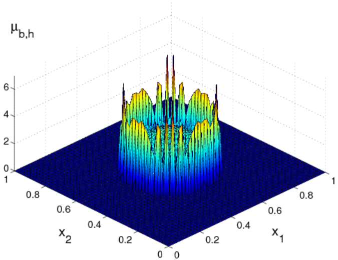


μ_a



μ_b

Sufficient second-order conditions

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General form of second-order sufficient conditions (SSC):

The pair $(\bar{y}, \bar{u})$ satisfies the KKT conditions and there exists $\delta > 0$ such that

$$\mathcal{L}''_{(y,u)}(\bar{y}, \bar{u}, \bar{p}, \mu_a, \mu_b)(y, u)^2 \geq \delta \|u\|^2_{L^2}$$

for all (y, u) belonging to the so-called critical cone (accounts for linearization and active state and control constraints).

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For state-constraints, the difficulty is to show that such SSC are really sufficient for local optimality.

On open problem

We are not able to set up second-order sufficient optimality conditions for important cases of elliptic and parabolic control problems.

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$$\frac{\partial \mathcal{L}}{\partial u}(u, \mu_a, \mu_b)\, v = f'(u)\, v + \int_{\bar{\Omega}} G'(u)\, v\, d(\mu_b - \mu_a).$$

We need the continuity of $\mathcal{L}''$ with respect to v in the L^2-norm, in particular for the second part.

$$\Big| \int_{\bar{\Omega}} \underbrace{G'(u)\, v}_{z}\, d(\mu_b - \mu_a) \Big| \le c \|v\|_{L^2(\Omega)}.$$

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We have

$$\Big| \int_{\bar\Omega} z\, d(\mu_b - \mu_a) \Big| \le \|z\|_{C(\bar\Omega)} \|\mu_b - \mu_a\|_{C(\bar\Omega)^*},$$

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hence we need $\|z\|_{C(\bar\Omega)} \le c \|v\|_{L^2(\Omega)}$, *where*

$$-\Delta z + d'(\bar y) z = v.$$

However, the mapping $v \mapsto z$ is not continuous from $L^2(\Omega)$ to $C(\bar\Omega)$ for $n > 3$.

Conclusion

- We cannot establish the standard SSC for elliptic distributed control problems with pointwise state constraints, if $n = \dim \Omega > 3$. Even with stronger requirements, this problem cannot be fully resolved.
- This happens already for $n > 2$ in elliptic boundary control, if the state constraints are imposed in the whole domain.
- In parabolic distributed control we cannot have more than $n = 1$.
- There are no SSC for parabolic boundary control problems with state constraints in the whole domain.

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Quasilinear control problem

We substitute $\Delta y(x)$ by $\operatorname{div}\,[a(x, y(x))\,\nabla y(x)]$.

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Remark:

Even if $y \mapsto a(x, y)$ is monotone, the state equation is not of monotone type!

Assumptions on a

The function $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function,

$$\exists \alpha_0 > 0 \text{ such that } a(x, y) \geq \alpha_0 \text{ for a.e. } x \in \Omega \text{ and all } y \in \mathbb{R}$$

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The function $a(\cdot, 0)$ belongs to $L^\infty(\Omega)$ and for any $M > 0$ there exist a constant $C_M > 0$ such that for all $|y_1|, |y_2| \leq M$

$$|a(x,y_2) - a(x,y_1)| \leq C_M |y_2 - y_1| \text{ for a.e. } x \in \Omega.$$

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Remarks:

- Instead of $d(y)$, a more general function $d(x,y)$ can be considered under associated assumptions.

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$$\exists \alpha_0 > 0 \;\text{ such that }\; a(x,y) \geq \alpha_0 \text{ for a.e. } x \in \Omega \;\text{ and all } y \in \mathbb{R}$$

The function $a(\cdot, 0)$ belongs to $L^\infty(\Omega)$ and for any $M > 0$ there exist a constant $C_M > 0$ such that for all $|y_1|, |y_2| \leq M$

$$|a(x,y_2) - a(x,y_1)| \leq C_M |y_2 - y_1| \;\text{ for a.e. } x \in \Omega.$$

Remarks:

- Instead of $d(y)$, a more general function $d(x,y)$ can be considered under associated assumptions.
- We shall also need the derivatives $\frac{\partial a}{\partial y}(x,y)$ and $\frac{\partial^2 a}{\partial y^2}(x,y)$.

Well-posedness of the state equation

Define: $p > n$ and $q > n/2$.

Theorem

Under our assumptions, for any element $u \in W^{-1,p}(\Omega)$, the quasilinear state equation has a unique solution $y_u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover there exists $\mu \in (0,1)$ independent of u such that $y_u \in C^\mu(\bar{\Omega})$ and for any bounded set $U \subset W^{-1,p}(\Omega)$

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C^\mu(\bar{\Omega})} \leq C_U \quad \forall u \in U$$

for some constant $C_U > 0$. Finally, if $u_k \to u$ in $W^{-1,p}(\Omega)$, then $y_{u_k} \to y_u$ in $H_0^1(\Omega) \cap C^\mu(\bar{\Omega})$.

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a) **Existence:** Depending on $M > 0$, we introduce the truncated function a_M by

$$a_M(x,y) = \begin{cases} a(x,y), & |y| \le M \\ a(x,+M), & y > +M \\ a(x,-M), & y < -M. \end{cases}$$

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$$\begin{aligned} -\mathrm{div}\,[a_M(x,y)\,\nabla y] + d_M(y) &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma \end{aligned}$$

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Stampacchia truncation method $\Rightarrow$

$$\|y_M\|_{L^\infty(\Omega)} \leq c_\infty,$$

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Hölder regularity of y: results of Gilbarg and Trudinger.

b) **Uniqueness:** First surprise: Very delicate!

Application of a comparison principle; we use ideas of Douglas/Dupont/Serrin (1971) and Křížek/Liu (2003).

□

$W^{1,p}$-regularity

Assume slightly higher regularity of a, Γ and u:

Theorem

Assume in addition that $a : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and Γ is of class C^1. Then the state equation has a unique solution $y_u \in W_0^{1,p}(\Omega)$. Moreover, for any bounded set $U \subset W^{-1,p}(\Omega)$, there exists a constant $C_U > 0$ such that

$$\|y_u\|_{W_0^{1,p}(\Omega)} \leq C_U \quad \forall u \in U.$$

If $u_k \to u$ in $W^{-1,p}(\Omega)$ then $y_{u_k} \to y_u$ strongly in $W_0^{1,p}(\Omega)$.

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If $u_k \to u$ in $W^{-1,p}(\Omega)$ then $y_{u_k} \to y_u$ strongly in $W_0^{1,p}(\Omega)$.

Follows from $W^{1,p}(\Omega)$-results for linear elliptic equations; Giaquinta (1993) and Morrey (1966).

Notice that $\hat{a}(x) = a(x, y_u(x))$ is continuous in $\bar{\Omega}$ and $u - d(y_u) \in W^{-1,p}(\Omega)$.

$W^{2,p}$-regularity

Assume more smoothness of a:

$$|a(x_1, y_1) - a(x_2, y_2)| \leq c_M \, \{|x_1 - x_2| + |y_1 - y_2|\}$$

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Theorem

Let this additional assumption be satisfied and Γ be of class $C^{1,1}$. Then for any $u \in L^q(\Omega)$, the quasilinear equation has one solution $y_u \in W^{2,q}(\Omega)$. Moreover, for any bounded set $U \subset L^q(\Omega)$, there exists a constant $C_U > 0$ such that

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Main trick of the proof: Expand the divergence term $a(x, y)$ and divide by a:

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$$-\Delta y = \underbrace{\frac{1}{a}}_{L^\infty} \Big\{ \underbrace{u - d(y)}_{L^q} + \sum_{j=1}^{n} \underbrace{\partial_j a(x, y)}_{L^\infty} \underbrace{\partial_j y}_{L^q} + \underbrace{\frac{\partial a}{\partial y}}_{L^\infty} \underbrace{|\nabla y|^2}_{L^q} \Big\},$$

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The $C^{1,1}$-smoothness of Γ permits to apply a result by Grisvard (1985) to get $y \in W^{2,q}(\Omega)$. The case $n/2 < q < n$ follows by some embedding results. $\square$

Differentiability of G

Since $n \leq 3$, $q = 2 > n/2$ is satisfied.

Therefore, $G : u \mapsto y_u$ is continuous from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$.
The choice $q = 2$ is possible in the theorems below.

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Additional assumption:

The function a is of class C^2 with respect to the second variable and, $\forall\, M > 0$ $\exists\, D_M > 0$ such that

$$\left|\frac{\partial a}{\partial y}(x,y)\right| + \left|\frac{\partial^2 a}{\partial y^2}(x,y)\right| \leq D_M \;\text{ for a.e. } x \in \Omega \;\text{ and all } |y| \leq M.$$

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Differentiability will hold, if the linearized equation defines an isomorphism in the associated spaces.

Theorem

Given $y \in W^{1,p}(\Omega)$*, for any* $v \in H^{-1}(\Omega)$ *the linearized equation*

$$\begin{aligned} -\mathrm{div}\left[a(x,y)\nabla z + \frac{\partial a}{\partial y}(x,y) z \,\nabla y\right] + d'(y)\, z &= v \ \ \text{in } \Omega \\ z &= 0 \ \ \text{on } \Gamma \end{aligned}$$

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Steps of the proof:

a) The uniqueness is shown by a comparison principle as for the state equation.

Idea of proof

b) A homotopy with respect to $t \in [0,1]$ is considered:

$$\begin{aligned} -\mathrm{div}\left[a(x,y)\nabla z + t\frac{\partial a}{\partial y}(x,y)z\,\nabla y_u\right] + d'(y)\,z &= v \ \text{ in } \Omega \\ z &= 0 \ \text{ on } \Gamma. \end{aligned}$$

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- For $t = 0$: Apply the Lax-Milgram Theorem.
  There exists a unique solution $z_0 \in H_0^1(\Omega)$ for every $v \in H^{-1}(\Omega)$.

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- $t_{max} := \sup S$. First, it is shown $t_{max} \in S$ and second $t_{max} = 1$. □

Theorem

Let all previous assumptions be satisfied. Then $G : W^{-1,p}(\Omega) \to W_0^{1,p}(\Omega)$, $G : u \mapsto y_u$, *is of class* C^2.

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For all $v_1, v_2 \in W^{-1,p}(\Omega)$ *the function* $z_{v_1,v_2} = G''(u)[v_1,v_2]$ *is the unique solution in* $W_0^{1,p}(\Omega)$ *of*

$$-\mathrm{div}\left[a(x,y_u)\nabla z + \frac{\partial a}{\partial y}(x,y_u) z \nabla y_u\right] + d'(y_u)\, z = -d''(y_u) z_{v_1} z_{v_2}$$

$$+\mathrm{div}\left[\frac{\partial a}{\partial y}(x,y_u)(z_{v_1}\nabla z_{v_2} + \nabla z_{v_1} z_{v_2}) + \frac{\partial^2 a}{\partial y^2}(x,y_u) z_{v_1} z_{v_2} \nabla y_u\right] \quad \text{in } \Omega$$

$$z = 0 \quad \text{on } \Gamma.$$

respectively, where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

Other spaces for G'

Additional assumption: $\forall\ M > 0\ \exists c_M > 0$ such that

$$\left| \frac{\partial^j a}{\partial y^j}(x_1, y_1) - \frac{\partial^j a}{\partial y^j}(x_2, y_2) \right| \le d_M \ \{|x_1 - x_2| + |y_1 - y_2|\}$$

for all $x_i \in \bar{\Omega}$, $y_i \in [-M, M]$, $i = 1, 2$ and $j = 1, 2$.

Theorem

Let all previous assumptions be satisfied and Γ be of class $C^{1,1}$. Then the control-to-state mapping $G : L^q(\Omega) \to W^{2,q}(\Omega)$, $G(u) = y_u$, is of class C^2 for all $q > n/2$.

Adjoint equation

With theses prerequisites, first-order necessary and second-order sufficient optimality conditions can be shown. Take $q := 2$ in the sequel

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Adjoint equation: Associated with u, the adjoint state $\varphi_u \in H^2(\Omega) \cap H_0^1(\Omega)$ is obtained from

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\begin{aligned}
-\mathrm{div}\,[a(x,y_u)\nabla\varphi] + \frac{\partial a}{\partial y}(x,y_u)\nabla y_u \cdot \nabla\varphi + d'(y_u)\varphi &= y_u - y_d && \text{in } \Omega \\
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Reduced gradient: Define as before $f(u) := J(y_u, u) = J(G(u), u)$.

$$\boxed{f'(u)\,v = \int_\Omega \left(\varphi_u(x) + \lambda\, u(x)\right) v(x)\,dx}$$

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Riesz identification: $\quad f'(u) \cong \varphi_u + \lambda\, u$

First-order necessary condition

Theorem

If $\bar{u}$ is locally optimal for (P) (in the sense of L^2) and $\bar{\varphi} := \varphi_{\bar{u}}$ is the associated adjoint state, then

$$\int_\Omega (\bar{\varphi} + \lambda \, \bar{u})(u - \bar{u}) \, dx \geq 0 \quad \forall u \in U_{ad}.$$

This is equivalent to the projection formula

$$\bar{u}(x) = \mathbb{P}_{[\alpha,\beta]} \left(-\frac{\bar{\varphi}(x)}{\lambda} \right) \qquad a.e.\ in\ \Omega.$$

This result gives different options for the numerical treatment.

The nonsmooth optimality system

Optimality system

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\begin{aligned}
-\operatorname{div}\,[a(x,y)\,\nabla y] + d(y) &= \mathbb{P}_{[\alpha,\beta]}(\lambda^{-1}\varphi) \\
-\mathrm{div}\,[a(x,y)\nabla\varphi] + \frac{\partial a}{\partial y}(x,y)\nabla y\cdot\nabla\varphi + d'(y)\varphi &= y - y_d
\end{aligned}
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(in Ω subject to homogeneous Dirichlet boundary condition.)

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Numerical options:

- Semismooth Newton method
- Direct solution of the system by COMSOL Multiphysics

Both methods were tested by V. Dhamo (TU Berlin) – very good experience.

Second-order derivative of f

For error estimates and the local convergence of numerical methods we need again second-order sufficient optimality conditions.

Theorem

Under our previous assumptions, the functional $f : L^2(\Omega) \to \mathbb{R}$ is of class C^2. We have

$$J''(u)v_1 v_2 = \int_\Omega \Big\{ z_{v_1} z_{v_2} + \lambda v_1 v_2 - \varphi_u \, d''(u) z_{v_1} z_{v_2}$$

$$-\nabla\varphi_u \Big[\frac{\partial a}{\partial y}(x, y_u)(z_{v_1} \nabla z_{v_2} + \nabla z_{v_1} z_{v_2}) + \frac{\partial^2 a}{\partial y^2}(x, y) z_{v_1} z_{v_2} \nabla y_u \Big] \Big\} \, dx$$

where $\varphi_u \in W_0^{1,p}(\Omega) \cap W^{2,q}(\Omega)$ is the adjoint state associated with u and $z_{v_i} = G'(u)v_i$.

Second-order sufficient optimality condition

Theorem

Assume that $\bar{u} \in U_{ad}$ satisfies the first-order necessary optimality conditions with the associated adjoint state $\bar{\varphi} \in W_0^{1,p}(\Omega)$.
Let there exist $\delta, \tau > 0$ such that

$$f''(\bar{u})v^2 \geq \delta \|v\|^2_{L^2(\Omega)} \quad \forall v \in C_{\bar{u}}^\tau$$

where

$$C_{\bar{u}}^\tau = \left\{ v \in L^2(\Omega) : v(x) = \begin{cases} \geq 0 & \text{if } \bar{u}(x) = \alpha \\ \leq 0 & \text{if } \bar{u}(x) = \beta \\ = 0 & \text{if } |\bar{\varphi}(x) + \lambda \bar{u}(x)| > \tau \end{cases} \quad \text{for a.e. } x \in \Omega \right\}.$$

Then $\bar{u}$ is locally optimal in the sense of $L^2(\Omega)$.

Remarks

- No two-norm discrepancy (quadratic structure of f).
- We discussed more general functionals of the form

$$f(u) = \int_\Omega L(x, y_u, u)\, dx.$$

  Here the two-norm discrepancy will occur in general.
- The condition $f''(\bar{u})v^2 > 0$ for all nonzero v of the critical cone is equivalent to the condition above under some additional requirements on the Hamiltonian.

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  Assume that $\overline{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are points of Γ.

Finite element approximation

Assumption: $\Omega \subset \mathbb{R}^n$ is open, convex and bounded $n \in \{2, 3\}$, with boundary Γ of class $C^{1,1}$. For $n = 2$, Ω is allowed to be polygonal instead of of class $C^{1,1}$.

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Piecewise linear approximation of the states:

$$Y_h = \{y_h \in C(\bar{\Omega}) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ on } \bar{\Omega} \setminus \Omega_h\}.$$

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Discretized state equation

$$\begin{cases} \text{Find } y_h \in Y_h \text{ such that, for all } z_h \in Y_h, \\ \displaystyle\int_{\Omega_h} [a(x, y_h(x)) \nabla y_h \cdot \nabla z_h + d(y_h(x))\, z_h]\, dx = \int_{\Omega_h} u z_h\, dx. \end{cases}$$

Local uniqueness of discretized states

- By the Brouwer fixed point theorem, the existence of solutions y_h to the discretized equation can be shown.
- We did not assume (global) boundedness of $a(x, y)$. To our surprise, we were not able to show uniqueness in this case. If a is bounded, then the uniqueness can be shown for all sufficiently small $h > 0$.
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- Therefore, in the unbounded case, we had to work with local uniqueness of y_h as in the setting of the implicit function theorem.

Assume for simplicity boundedness of a and that h is sufficiently small so that the mapping $u \mapsto y_h(u)$ is well defined:

Definition: For given $u \in U_{ad}$, $y_h(u)$ is the solution to the discretized equation.

Discretized optimal control problem

Under the same simplification as above, we define

$$f_h(u) = \frac{1}{2} \int\limits_{\Omega_h} (y_h(u) - y_d)^2 \, dx + \frac{\lambda}{2} \int\limits_{\Omega_h} u^2 \, dx.$$

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We considered the following sets U_{ad}^h:

- $U_{ad}^h = U_{ad} \quad \forall h > 0$ (variational discretization)
- All piecewise constant functions on Ω_h (constant on each triangle) with values in $[\alpha, \beta]$
- All piecewise linear functions on Ω_h with values in $[\alpha, \beta]$.

Theorem (Piecewise constant controls, L^2-estimate)

Let a locally optimal control $\bar{u}$ of (P) satisfy the second-order sufficient conditions introduced above and let U_{ad}^h be defined by piecewise constant functions. Assume that $\bar{u}_h$ is a sequence of locally optimal (piecewise constant) solutions to (P_h) that converges strongly in $L^2(\Omega)$ to $\bar{u}$. Then there is some constant $C > 0$ not depending on h such that

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Survey of other results:

- Same estimate in the L^∞-norm for piecewise constant controls
- Order h^2 for variational discretization (L^2 and L^∞)
- $\lim_{h\to 0} h^{-1}\|\bar{u}_h - \bar{u}\|_{L^2(\Omega_h)} = 0$ for piecewise linear controls
- L^2-estimate of order $h^{3/2}$ for piecewise linear controls under some standard structural assumption on the triangles, where the reduced gradient vanishes on a positive measure.

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In our problem, we have a sequence $\varepsilon_h \to 0$ such that

$$|[f_h'(u) - f'(u)]v| \leq \varepsilon_h \|v\|_{L^2(\Omega)}$$

for all $(u, v) \in U_{ad} \times L^2(\Omega)$ with $v = u_h - \bar{u}$ with $u_h \in U_{ad}^h$.

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Theorem

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$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq C \left[\varepsilon_h^2 + \|\bar{u} - u_h\|^2_{L^2(\Omega)} + f'(\bar{u})(u_h - \bar{u})\right]^{1/2} \quad \forall u_h \in U_{ad}^h,\ \forall h < h_0.$$

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Reference: E. Casas, F.T., *A general theorem on error estimates with application to a quasilinear elliptic optimal control problem*, submitted 2011.