The dynamics of circle maps

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Dynamics in the circle

- Circle Diffeomorphisms
- critical circle mappings
- covering maps, expanding circle maps
- multimodal maps

Some problems

- Conjugacy classes- conjugacy invariants
- Full families
- Structural stability: characterization, density
- Rigidity: lower regularity of conjugacy $\Rightarrow$ higher regularity
- bifurcation

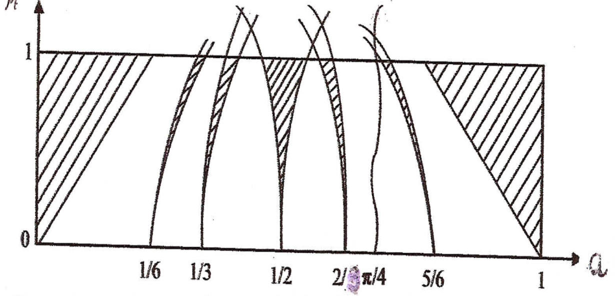
- Poincaré, Denjoy (1932), Arnold (1961), Herman, Yoccoz, ...
- Yoccoz, (1984), Swiatek (1988), Herman, deMelo-Strien(2000), Yampolski (2002), Khanin-Teplinsky (2007)
- Shub (1969), Mañe , Shub-Sullivan
- deMelo-Salomão Vargas (2010), Strien,

Lift to the real line

- Covering map $\phi\colon \mathbb{R} \to \mathbb{S}^1$, $\phi(t) = e^{2\pi it}$
  $\phi(t) = \phi(s) \Leftrightarrow t - s \in \mathbb{Z}$
- Lift of $f\colon \mathbb{S}^1 \to \mathbb{S}^1$:
  $F\colon \mathbb{R} \to \mathbb{R}$ such that $\phi \circ F = \phi \circ f$
  $F(t+1) = F(t) + d$ where $d \in \mathbb{Z}$
  topological degree of f
- G another lift of $f \Leftrightarrow F(t) - G(t) \in \mathbb{Z}$

The standard family: the Arnold´s tongues

- $F_{a,\mu}(t) = t + a + \frac{\mu}{2\pi}\sin(2\pi t)$
- $\mu < 1$ $f_{a,\mu}$ diffeomorphisms, $\mu = 0$ rigid rotation
- rotation number of a diffeo : $\rho(f) = \lim_{n\to\infty}\frac{F^n(x)-x}{n}$
- $\mu > 1$: rotation interval.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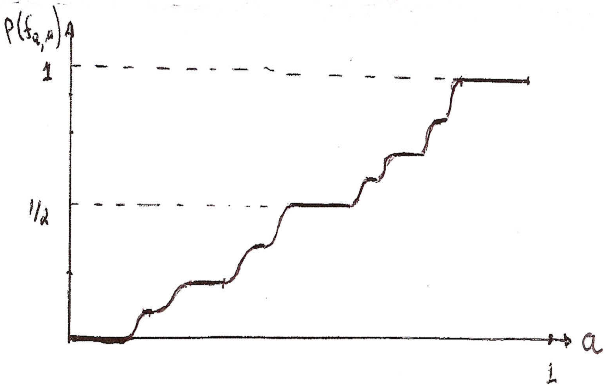
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A dynamical dichotomy

- Periodic behavior: $x \in P_f \subset S^1 \iff \exists$ neigh. V of x s.t. the orbit of each $y \in V$ is asymptotic to a periodic orbit.
- Chaotic behavior: $Ch_f = S^1 \setminus P_f$
- P_f is open and backward invariant. Each component that does not contain critical point is mapped onto another component. Otherwise is mapped into the closure of a component.

- Each component of P_f is eventually mapped in the closure of a periodic component.
- there is a finite number of periodic components.
- f diffeomorphism or critical circle map $\Longrightarrow$ either P_f is empty and f is top. conj. to irrational rotation or $P_f = S^1$

Full Family

- h is a strong semi-conjugacy between f and g if
  1. h is continuous, monotone and onto.
  2. $h \circ f = g \circ h$
  3. $h^{-1}(y)$ is either a unique point or an interval contained in P_f
- A family g_μ is $2m$-full if any $2m$-modal map f is strongly semi-conjugate to some g_μ.
- The standard family is 2-full (deMelo-Salomão-Vargas: $2m$-full trigonometric family).

The trigonometric family

- $f_\mu : \mathbb{S}^1 \to \mathbb{S}^1$
- $F_\mu : \mathbb{R} \to \mathbb{R}$ the lift of f_μ

  $$F_\mu(t) = dt + \mu_{2m} \sin(2\pi mt) + \sum_{j=0}^{m-1} (\mu_{2j} \sin(2\pi jt) + \mu_{2j+1} \cos(2\pi jt))$$

  where

  $\mu = (\mu_1, \ldots, \mu_{2m}) \in \Delta := \{\mu \in \mathbb{R}^{2m} \ :$
  $\mu_{2m} > 0$ and f_μ is $2m -$ multimodal $\}$
  and $d \in \mathbb{Z}$, the topological degree of f_μ.

Structural Stability and Hyperbolicity

- f is hyperbolic if $\exists$ N s.t $|df^N(x)| > 1$ $\forall x \in Ch_f$
- Koslovsky-Strien-Shen:
  structural stability is dense
  f structurally stable $\Rightarrow$ f is hyperbolic

Rigidity- diffeomorphisms

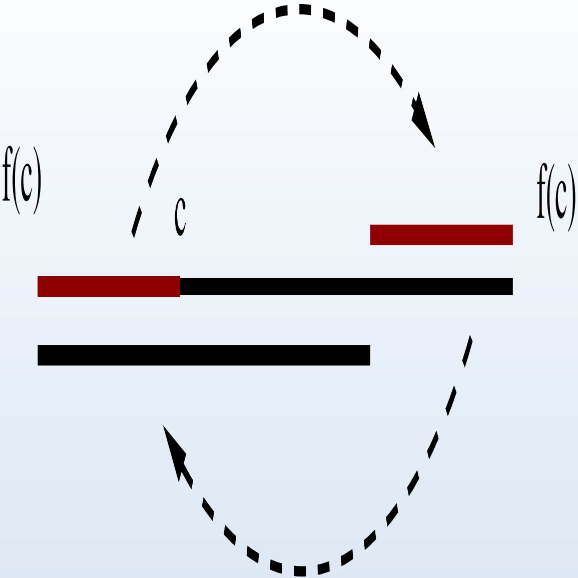
- Rigidity: lower regularity of conjugacy implies higher regularity
- Herman=Yoccoz: f smooth circle diffeomorphism whose rotation number is an irrational that satisfies a Diophantine condition is smoothly conjugate to a irrational rotation.
- Arnold: $\exists$ smooth diffeo, with irrational rotation number (of Liouville type) such that the conjugacy with a rigid rotation maps a set of zero Lebesgue measure in a set of full Lebesgue measure.

Rigidity of critical circle maps

- deMelo-deFaria, Yampolski, Khanin-Templiski, Guarino-deMelo: h C^0 conjugacy between two smooth critical circle maps with irrational rotation number $\Rightarrow$ h is C^1.

Main tool: renormalization

- Critical circle map $\longrightarrow$ commuting pair of interval maps
- commuting pair of interval maps $\longrightarrow$ smooth conjugacy class of critical circle maps.
- renormalization operator: first return map to smaller interval around the critical point+ affine rescale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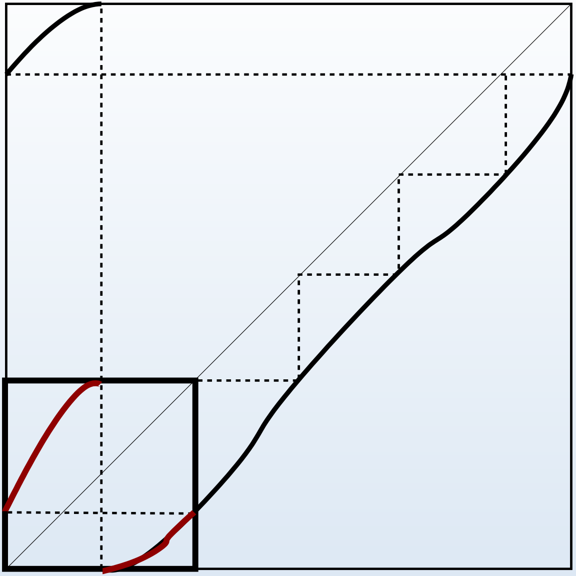

Complex Bounds

- Given an infinitely renormalizable commuting pair, there exists an integer n_0 such that if $n.n_0$ then the $n-th$ iterate of the commuting pair by the renormalization operator has a holomorphic extension to a holomorphic commuting pair that belongs to a compact family.

HOLOMORPHIC PAIR