Invariant Forms, Pressure and Rigidity for Anosov Flows

Anosov Flows

- An Anosov flow on a manifold M is a smooth flow φ^t with
  - an invariant decomposition $TM = X \oplus E^u \oplus E^s$ (where $X = \dot{\varphi} \neq 0$ is the generator of the flow and E^u and E^s are called the unstable and stable subbundles) and
  - a Riemannian metric on M such that $D\varphi^t\restriction_{E^s}$ and $D\varphi^{-t}\restriction_{E^u}$ are contractions whenever $t > 0$.

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  - a Riemannian metric on M such that $D\varphi^t\restriction_{E^s}$ and $D\varphi^{-t}\restriction_{E^u}$ are contractions whenever $t > 0$.
- The *canonical 1-form* A of an Anosov flow φ^t is defined by $A(X) = 1$ and $E^u, E^s \subset \ker A$.
- A *canonical time-change* is defined using a closed 1-form α by replacing the generator X of the flow by the vector field $X/(1+\alpha(X))$, provided α is such that the denominator is positive.

Local Charts

Lemma

There exist local coordinates adapted to the invariant laminations, coordinate systems $\Psi\colon M \times (-\epsilon, \epsilon)^{2n+1} \to M$ such that $\Psi_p \Psi(p, \cdot)$ satisfies

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Longitudinal KAM cocycle

F - Hasselblatt Israel J. Math. (2003)

- Geometric description

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Theorem

Let M be a 3-manifold, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent :

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- Paternain,Dairbekov-Paternain for applications of this to magnetic flows.

Longitudinal KAM cocycle, higher dimensions

Fang - Foulon- Hasselblatt JMD 2010

- Let φ be a C^∞ flow on a closed manifold M. Denote by X the generating vector field of φ. The flow φ is said to be *transversely symplectic* if there exists a C^∞ closed 2-form ω on M such that $\mathrm{Ker}\omega = \mathbb{R}X$. The closed 2-form ω is said to be the *transverse symplectic form* of φ. It is easy to see that ω is φ-invariant.

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- Let (N, g) be a closed C^∞ Riemannian manifold and Ω a C^∞ closed 2-form of N. Let α denote the C^∞ 1-form on TN obtained by pulling back the Liouville 1-form of T^*N via the Riemannian metric. For $\lambda \in \mathbb{R}$, the *twisted symplectic structure* ω_λ is defined as

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- denotes the canonical projection. Let $H \colon TN \to \mathbb{R}$ be the Hamiltonian function defined as

$$H(v) = \frac{1}{2}g(v, v)$$

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- φ^λ is a transversely symplectic flow with respect to $\omega_\lambda \mid_{SN}$, which is said to be the *magnetic flow* of the pair $(g, \lambda\Omega)$.

Longitudinal KAM cocycle, higher dimensions

- An Anosov flow is said to be *uniformly quasiconformal* if

$$K_i(x,t) := \frac{\|d\varphi^t\restriction_{E^i}\|}{\|d\varphi^t\restriction_{E^i}\|^*} \tag{1}$$

is bounded on $\{u,s\} \times M \times \mathbb{R}$, where $\|A\|^* := \min_{\|v\|=1} \|Av\|$ is the *conorm* of a linear map A.

Longitudinal KAM cocycle, higher dimensions

Theorem (Fang)

Let M be a compact Riemannian manifold and $\varphi \colon \mathbb{R} \times M \to M$ a transversely symplectic Anosov flow with $\dim E^u \geq 2$ and $\dim E^s \geq 2$. Then φ is quasiconformal if and only if φ is up to finite covers C^∞ orbit equivalent either to the suspension of a symplectic hyperbolic automorphism of a torus, or to the geodesic flow of a closed hyperbolic manifold.

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Theorem (Fang)

Let φ be a C^∞ volume-preserving quasiconformal Anosov flow. If $E^s \oplus E^u \in C^1$ and $\dim E^u \geq 3$ and $\dim E^s \geq 2$ (or $\dim E^s \geq 3$ and $\dim E^u \geq 2$), then φ is up to finite covers and a constant change of time scale C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical time change of the geodesic flow of a closed hyperbolic manifold.

Longitudinal KAM cocycle, higher dimensions

Theorem (Fang - F - Hasselblatt 2010)

Let M be a compact Riemannian manifold of dimension at least 5, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a uniformly quasiconformal transversely symplectic C^k Anosov flow.
Then $E^u \oplus E^s$ is Zygmund-regular and there is an obstruction to higher regularity that defines the cohomology class of a cocycle we call the longitudinal KAM-cocycle. This obstruction can be described geometrically as the curvature of the image of a transversal under a return map, and the following are equivalent :

1. *$E^u \oplus E^s$ is "little Zygmund"*
2. *The longitudinal KAM-cocycle is a coboundary.*
3. *$E^u \oplus E^s$ is Lipschitz-continuous.*
4. *φ is up to finite covers, constant rescaling and a canonical time-change C^k-conjugate to the suspension of a symplectic Anosov automorphism of a torus or the geodesic flow of a real hyperbolic manifold.*

Invariant Forms

- To show that 3 implies 4 we study the *canonical 1-form* of the time-change of a geodesic flow or of the suspension of an infranilmanifoldautomorphism, and because we only have Lipschitz-continuity at our disposal, we need to explore how smooth-rigidity results can be pushed to the lowest conceivable regularity. This requires two main results

Invariant Forms

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Theorem (F - Hasselblatt 2010)

Let M be a compact locally symmetric space with negative sectional curvature and suppose A is a Lipschitz continuous 1-form such that dA is invariant under the geodesic flow. Then A is C^∞, and indeed dA is a constant multiple of the exterior derivative of the canonical 1-form for the geodesic flow.

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Note that the Lipschitz assumption ensures that dA is defined almost everywhere and essentially bounded (V. M. Goldshtein, V. I. Kuzminov, I. A. Shvedov : *Differential forms on a Lipschitz manifold*, (1982)). This is all we use. For comparison, we state an earlier result of Hamenstädt :

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Theorem (Hamenstadt)

If the Anosov splitting of the geodesic flow of a compact negatively curved manifold is C^1 and A is a C^1 1-form such that dA is invariant, then dA is proportional to the canonical 1-form of the geodesic flow.

Invariant Forms

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Corollary

Let M be a compact locally symmetric space with negative sectional curvature and consider a time-change whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is C^∞, and the time-change is a canonical time-change.

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Corollary

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Theorem (F - Hasselblatt 2010)

Let ψ be a hyperbolic automorphism of a torus or a infranilmanifold $\Gamma\backslash M$. Then any essentially bounded invariant 2-form is almost everywhere equal to an M-invariant (hence smooth) closed 2-form.
If, in addition, the form is exact, then it vanishes almost everywhere.

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Let ψ be a hyperbolic automorphism of a torus or a infranilmanifold $\Gamma\backslash M$ and consider a time-change of the suspension whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is C^∞, and the time-change is a canonical time-change.

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Theorem

Let (N, g) be a n-dimensional closed negatively curved Riemannian manifold and Ω a C^∞ closed 2-form of N. For small $\lambda \in \mathbb{R}$, let φ^λ be the magnetic Anosov flow of the pair $(g, \lambda\Omega)$. Suppose that $n \geqslant 3$ and φ^λ is uniformly quasiconformal. Then g has constant negative curvature and $\lambda\Omega = 0$. In particular, the longitudinal KAM-cocycle of φ^λ is a coboundary.

Finsler manifolds of negative curvature

Smooth Finsler metrics

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Smooth Finsler metrics

- Let (M, F) be a C^∞ closed Finsler manifold of negative flag curvature.
- Let φ be its geodesic flow defined on the homogeneous bundle HM.
- The lift of this Finsler structure to the universal covering space defines a possibly non-symmetric distance $\widetilde{d}$ on $\widetilde{M}$.
- We study the large scale metric geometry of $\widetilde{d}$

Finsler manifolds of negative curvature

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- Let $\pi : HM = TM_0/\mathbb{R}^+ \rightarrow M$ be the *homogeneous bundle*
- Recall that the generator X of the geodesic flow is a Reeb field of a contact form A on HM
  - $dA(X,.) = 0$
  - $A(X) = 1$

Dynamic invariants of Finsler geodesic flows

Theorem

Let (M, F) be a closed C^∞ Finsler manifold of negative flag curvature. Then its geodesic flow $\varphi : HM \to HM$ is Anosov. In addition the stable and unstable distributions of φ are both transverse to $V(HM)$.

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- It is well-known that contact Anosov flows are topologically transitive .
- There exists on HM a unique continuous φ-invariant 1-form λ_φ such that

$$\lambda_\varphi(X) = 1 \text{ and } \lambda_\varphi(E^{ss}) = \lambda_\varphi(E^{su}) = 0,$$

  which is said to be the *canonical* 1-*form* of φ.
- $A = \lambda_\varphi$

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  - If φ is in addition volume-preserving, we denote by ν the unique φ-invariant Lebesgue probability measure.
  - $h_{top}(\varphi) \geqslant h_{vol}(\varphi)$.

Pressure

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- More generally, let G be a Hölder continuous function on N. We define the topological pressure of φ with respect to G by

$$\mathrm{P}(\varphi, G) = \sup\{\mathrm{h}_\mu(\varphi) + \int_N G d\mu : \ \mu \text{ is a } \varphi-\text{invariant probability}\}.$$

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- By the well-known variational principle (see [HK]) there exists again a unique ergodic fully supported φ-invariant probability measure for which the supremum in the definition of $\mathrm{P}(\varphi, G)$ is attained. This measure is called the *Gibbs measure* of φ with respect to G. Clearly, $\mathrm{P}(\varphi, 0) = \mathrm{h}_{\mathrm{top}}(\varphi)$ and the Gibbs measure of φ with respect to the function zero is just the Bowen-Margulis measure.

Pressure

Pressure

- Two continuous functions G_1 and G_2 are said to be φ-cohomologous if $G_1 - G_2 = U'$ for some U which is continuously differentiable with respect to φ. If G_1 and G_2 are both Hölder continuous then they have the same Gibbs measure if and only if $G_1 - G_2$ is φ-cohomologous to a constant, c say. In this case we have $\mathrm{P}(\varphi, G_1) = \mathrm{P}(\varphi, G_2) + c$.

Cohomological pressure and cohomological Gibbs number

Cohomological pressure and cohomological Gibbs number

- Let $\varphi : N \to N$ be a topologically transitive C^∞ Anosov flow generated by X. We denote by $\mathrm{H}^1(N, \mathbb{R})$ the first de Rham cohomology group of N. Let us recall firstly the Schwartzman's definition of a winding cycle. If μ is a φ-invariant probability measure then the μ-winding cycle is a map $\Phi_\mu : \mathrm{H}^1(N, \mathbb{R}) \to \mathbb{R}$ defined by

$$\Phi_\mu(\alpha) = \int_N \alpha(X) d\mu,$$

where α is a closed C^∞ 1-form. Since μ is a φ-invariant, it is easy to see that Φ_μ is a well-defined map.

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- We define $\Lambda : \mathrm{H}^1(N, \mathbb{R}) \to \mathbb{R}$ by $\Lambda(\alpha) = \mathrm{P}(\varphi, \alpha(X))$, i.e. the topological pressure of φ with respect to the function $\alpha(X)$. Immediately from the definition we obtain the relationship

$$\Lambda(\alpha) = \sup\{\mathrm{h}_\mu(\varphi) + \Phi_\mu(\alpha) : \ \mu \text{ is } \ \varphi\text{−invariant}\}$$

  and hence that if $\mathrm{d}f$ is an exact form then $\Lambda(\alpha) = \Lambda(\alpha + df)$. Thus Λ is well-defined.

Cohomological pressure and cohomological Gibbs number

Definition

Following [Sharp], we define the *cohomological pressure* of φ, $\mathrm{P}(\varphi)$ by $\mathrm{P}(\varphi) = \inf\{\Lambda(\alpha) : [\alpha] \in \mathrm{H}^1(N, \mathbb{R})\}$.

Theorem

([Sharp], Theorem 1) Let $\varphi : N \to N$ be a topologically transitive C^∞ Anosov flow. Then the following two statements are equivalent :
(i) There exists a fully supported φ-invariant probability measure μ such that $\Phi_\mu \equiv 0$;
(ii) The function $\Lambda : \mathrm{H}^1(N, \mathbb{R}) \to \mathbb{R}$ is bounded below (i.e. $P(\varphi) > -\infty$) and there exists a unique cohomological class $[\alpha] \in \mathrm{H}^1(N, \mathbb{R})$ for which the infimum is attained.
If any (and hence both) of the above statements are true then we have

$$\mathrm{P}(\varphi) = \sup\{\mathrm{h}_\mu(\varphi) : \mu \textit{ is } \varphi - \textit{invariant with } \Phi_\mu \equiv 0\}$$

and $\Phi_{\mu_\alpha} \equiv 0$, where μ_α denotes the Gibbs measure of $\alpha(X)$.

Cohomological pressure and cohomological Gibbs number

Definition

The cohomology class of α as in (ii) is said to be the *Gibbs class* of φ, and the Gibbs measure of φ with respect to $\alpha(X)$ is said to be the *cohomological Gibbs measure* for φ. The *cohomological Gibbs number* of φ is defined as

$$G(\varphi) = \int_N \alpha(X) d\mu_{BM} = \Phi_{\mu_{BM}}([\alpha]).$$

Remark If (M, F) is reversible, for example in the Riemmanian case, then it is easy to verify (see [Pa3]) that $\Phi_{\mu_{BM}} \equiv 0$. So cohomological pressure and cohomological Gibbs number are interesting only for non-reversible Finsler manifolds of negative flag curvature.

Proposition

Let φ be a contact C^∞ Anosov flow. Then we have

$$\mathrm{h_{top}}(\varphi) \geqslant \mathrm{P}(\varphi) \geqslant \mathrm{h_{vol}}(\varphi).$$

Canonical time changes

Definition

For any C^∞ Anosov flow $\varphi : N \to N$ generated by X, a *canonical time change* of φ is the flow generated by $\dfrac{X}{1-\alpha(X)}$, where α is a closed C^∞ 1-form on N such that $1 > \alpha(X)$. We denote by φ^α the flow of $\dfrac{X}{1-\alpha(X)}$.

Canonical time changes

Proposition

Let $\varphi : N \to N$ be a contact C^∞ Anosov flow generated by X. Let α be a closed C^∞ 1-form on N such that $1 > \alpha(X)$. Then we have $\mathrm{P}(\varphi) = \mathrm{P}(\varphi^\alpha)$.

Proposition

Let φ be a contact C^∞ Anosov flow with $\Phi_{\mu_{BM}} \equiv 0$. Let α be a closed C^∞ 1-form on N such that $1 > \alpha(X)$. Then the Gibbs class of φ^α is $[-\mathrm{h}_{\mathrm{top}}(\varphi) \cdot \alpha]$.

Canonical time changes

Proposition

Let φ be a topologically transitive C^∞ Anosov flow and let G be a Hölder continuous function on N. Let f be any positive C^∞ function on N. Then we have

$$\mathrm{P}(\varphi, G) = \mathrm{P}(\varphi^f, \frac{G}{f} - \mathrm{P}(\varphi, G) \cdot \frac{1-f}{f}).$$

In addition the Gibbs measure of φ with respect to G is equivalent to that of φ^f with respect to the function $\frac{G}{f} - \mathrm{P}(\varphi, G) \cdot \frac{1-f}{f}$.

Anosov splitting regularity of Finsler geodesic flows

Theorem

([Ha], Theorem B) Let φ be the geodesic flow of a closed negatively curved Riemannian manifold. If the Anosov splitting of φ is C^2, then the topological entropy of φ coincides with its metric entropy.

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G. Paternain (see [Pa2]) : Let g be a locally symmetric Riemannian metric on M and θ be a small closed but non-exact C^∞ 1-form on M. Let $F = \sqrt{g} - \theta$ be the Randers metric and φ be its geodesic flow.

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- The Anosov splitting of φ is C^∞ .
- φ is generated by $\dfrac{X}{1 - \pi^*\theta(X)}$.
- the Gibbs class of φ is not trivial
- $\mathrm{h_{top}}(\varphi) > \mathrm{P}(\varphi) = \mathrm{h_{vol}}(\varphi)$.

Anosov splitting regularity of Finsler geodesic flows

Theorem

(Fang - Foulon 2009) Let (M, F) be a closed C^∞ Finsler manifold of negative flag curvature and φ its geodesic flow. If the Anosov splitting of φ is C^2, then the cohomological pressure of φ coincides with its metric entropy.

Ingredients for the proof

Definition

We say that φ is *$d\lambda$-transitive* if any continuous exact 2-form is a constant multiple of dA, where A denotes the potential of the metric F.

Proposition

Let φ be a contact C^∞ Anosov flow such that E^{ss} and E^{su} are both orientable. If φ is $d\lambda$-transitive and its Anosov splitting is C^2, then the cohomological pressure of φ coincides with its metric entropy.

So the key point is to show

Proposition

Let φ be the geodesic flow of a closed C^∞ Finsler manifold (M, F) of negative flag curvature. If the Anosov splitting of φ is C^1, then φ is $d\lambda$-transitive.

Action of the fundamental group

Let $\pi_1(M)$ be the fundamental group of M. For any $\gamma \in \pi_1(M)$, γ acts naturally on $\widetilde{M}$ and preserves the lifted Finsler metric $\widetilde{F}$. Thus γ acts naturally and Hölder continuously on the boundaries.

Definition

Let X be a topological space and $\Phi : X \to X$ be a homeomorphism. Then Φ is said to have a *north-south dynamic* if Φ fixes exactly two points $\{a, b\} \subseteq X$ and for any $x \in X - \{a, b\}$, $\Phi^n(x) \to a$ and $\Phi^{-n}(x) \to b$ as $n \to +\infty$.

Proposition

Let $\gamma \in \pi_1(M)$. If γ is not trivial, then the γ-action on $\partial^s\widetilde{M}$ (respectively on $\partial^u\widetilde{M}$) has a north-south dynamic.