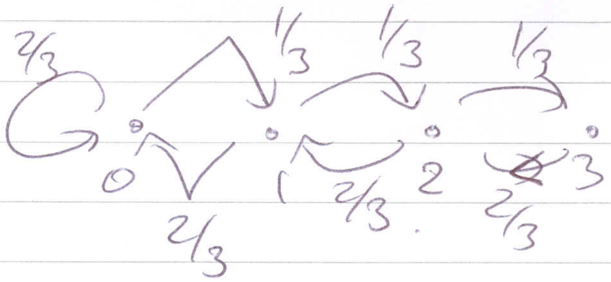
Keane

Ergodic Theory of the M/1/1 Queue.

Part of ideas – Neil O'Connell
– Jacek Serafin.

Countable state Markov chain

$\mathbb{N} = \{0,1,2,3,\dots\}$ $\quad P(k,k+1) = 1/3 \ (k\in\mathbb{N})$
$P(k+1,k) = 2/3 \ (k\in\mathbb{N})$.
$P(0,0) = 2/3$



$\left\{ \begin{array}{l} p(k) = 1/2^{k+1} \ (k\in\mathbb{N}) \\ \text{stationary state.} \end{array} \right.$

$\mathbb{Q} = \{ q = (\dots, q_{-1}, q_0, q_1, \dots) : q_t \in \mathbb{N},\ P(q_t, q_{t+1}) > 0 \ \forall t \}$

μ = Markov measure

$S : \mathbb{Q} \to \mathbb{Q}$ left shift.

<u>Proposition</u>: $(\mathbb{Q}, \mu, S)$ is isomorphic to the Bernoulli shift $BS(1/3, 2/3)$

<u>Proposition</u>: $X = \{-1,+1\}^{\mathbb{Z}}$, $\nu = (2/3, 1/3)^{\mathbb{Z}}$

T = shift. $\left\{ \begin{array}{l} \Phi : \mathbb{Q} \to X \\ \Phi(q) = x, \quad x_t = \begin{cases} -1 & \text{if } q_t > q_{t+1} \\ +1 & \text{otherwise} \end{cases} \end{array} \right.$

$\Phi(\mu) = \nu$. Isomorphism
(1-1 a.e., although ctble-1).

Note Φ not finitary

Question: Is (Q, μ, S) finitary isomorphic to $BS(1/3, 2/3)$?

Smorodinsky (+ k) – 1970s

$p = (p_1, \dots, p_m)$ $\quad$ $q = (q_1, \dots, q_n)$

$BS(p)$ $\quad$ $h(p) = h(q)$ $\quad$ $BS(q)$

$x = (\dots, x_{-1}, x_0, x_1, \dots)$ $\quad$ $y = (\dots, y_{-1}, y_0, y_1, \dots)$

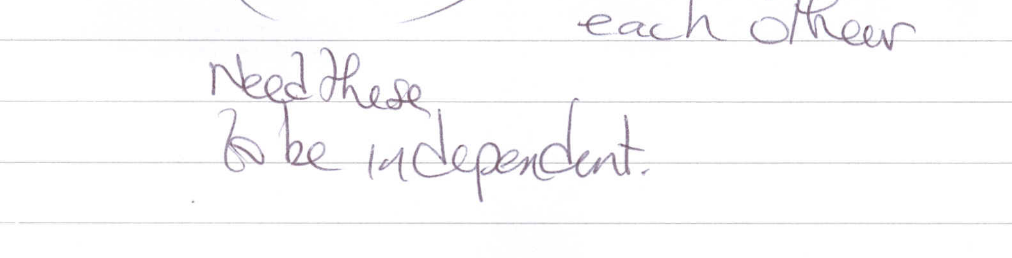
use markers – then fillers are independent

(Iterative "back + forth" procedure)

Need to get started:

Lemma: $\exists\, r = (r_1, \dots, r_0)$

$(p_1, \dots, p_m)$ $\quad$ $(q_1, \dots, q_n)$

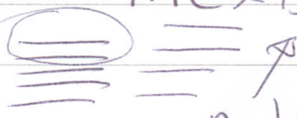


Perhaps union of states to be independent of each other

Need these to be independent.

MC = (finite) Markov chain BS

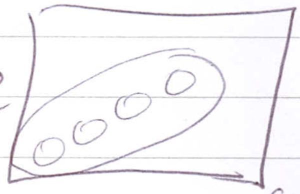
$MC \times \widetilde{BS}$



Another Bernoulli scheme.

Lists of states

MC



BS

(uses Perron Frobenius theory)

Now here is an approach for ctble state Markov chains.

$M_n = [0^k 1 2 N_n 1 0^k 1 0]$, N_n contains no 0^k

length n

$n \geq 0$

$M_n \cap T^{-j} M_n = \phi$, $0 \leq j \leq 2k+n+2, n \geq 0$

$\mu(\ ) = \dfrac{P^j(0,0)}{\mu(0)}$.