(+Pollicott, Giulietti, Liverani).

Ruelle ζ-function for smooth Anosov flows.

$$\zeta_S(z) = \prod_{\gamma} \prod_{n=0}^{\infty} \left(1 - e^{-(z+n)\,l(\gamma)}\right)$$

(Selberg ζ-function)

γ = closed geodesic of length $l(\gamma)$

$K=-1$: Zeros described by evs of Laplacian. – Analytic on $\mathbb{C}$.

This def^n makes sense for any Anosov flow.

<u>Thm</u>: If $\phi_t = C^\infty$ Anosov flow then $\zeta(z)$ is meromorphic in $\mathbb{C}$.

Consider Ruelle ζ-function

$$\begin{cases} \zeta_R(z) = \prod_{\gamma} \left(1 - e^{-z\,\lambda(\gamma)}\right)^{-1} \\ \zeta_S(z) = \prod_{k=0}^{\infty} \zeta_R(z+k)^{-1}. \end{cases}$$

Long history for maps & flows.

- Ruelle, Fried, Rugh – Analytic.
- Pollicott – extend to half-plane.

$\phi_t : M \to M$

$\mathcal{L}_t f(x) = (f \cdot |\det D\phi_t^{-1}|) \circ \phi_{-t}$

Dual of composition of functions with flow.

<u>Consider case of diffeos</u>.

$$\begin{cases} T : M \to M \\ \mathcal{L} f(x) = (f |\det DT^{-1}|) \circ T^{-1} \end{cases}$$

$$\det(\mathbb{1} - z\mathcal{L}) = e^{\mathrm{Tr}\log(\mathbb{1} - z\mathcal{L})}$$
$$= \exp \sum_{n=1}^{\infty} z^n \,\mathrm{Tr}(\mathcal{L}^n)/n$$

Consider $Kf(x)$

$$\overset{\shortparallel}{\int K(x,y) f(y)} \Bigg\} \mathrm{Tr}\, K = \int K(x$$

$$\mathrm{Tr}\,\mathcal{L}^n = \int \delta(x - T^n x)\, dx$$
$$= \sum_{x \in \mathrm{Fix}\, T^n} \frac{\int \delta(\xi)}{\det(\mathbb{1} - DT^n)}$$

$$\int \delta(x - T^n y) f$$
$$\shortparallel$$
$$\int \delta(x-y) \mathcal{L}^n$$
$$\shortparallel$$
$$\mathcal{L}^n f(x).$$

$$(\xi = x - T^n x) = \sum_{x \in \mathrm{Fix}\, T^n} \det(\mathbb{1} - DT^n)^{-1}$$

$$\zeta_R(z) = \exp - \sum_{\tau \text{ prime.}} (1 - e^{-\lambda(\tau) z})^{-1}$$

$$= \exp - \sum_{\tau} \sum_{n=1}^{\infty} e^{-\lambda(\tau) n z} / n.$$

$$= \exp - \sum_{m=1}^{\infty} \sum_{\mathrm{Fix}\, T^m} e^{nz} / n$$

$$\det(\mathbb{1}-A) = \sum_{n=1}^{\infty} (-1)^n \operatorname{Tr}(\Lambda^n A).$$

Thus on manifold one needs to look at the operator on forms

$(T^{-1})^* \omega = \mathcal{L}_k \omega \qquad \omega = k\text{-form}.$

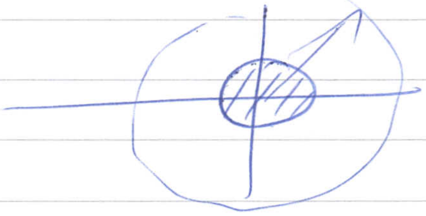
Finally, $\zeta_R(z) = \prod_{k=0}^{d} \det(\mathbb{1} - e^{-z}\mathcal{L}_k)^{(-1)^k}$

Thus determinants allow one to study the ζ function.

(One uses mollifiers to make the formulas more rigorous).

Banach spaces important. For T want

$sp(\mathcal{L})$



For flows

← move to left for different Banach spaces.

<u>Quasi-cpt</u>: $\mathcal{L} = A + R$

Told by Dolgopyat (attributed to Margulis).

$A \in GL(n, \mathbb{R})$.

$\underline{\delta} = \mathbb{R}^n \otimes \mathbb{R}^n$, $\underline{\delta}_{ij} := \delta_{ij}$

$$\langle \underline{\delta}, A \otimes A^T \underline{\delta} \rangle = \sum_{i,j,k,\ell} \delta_{ij} A_{ik} A_{\ell j} \delta_{k\ell}$$

$$= \sum_{ik} A_{ik} A_{ki} = \mathrm{Tr}(A^2).$$

$$\int \delta(x-y) \underbrace{\mathcal{L} \otimes \mathcal{L}^T}_{\text{like transfer operator for } T \times T^{-1}} \delta(x-y)$$

Banach spaces of anisotropic dist^n.

If $d = d_u + d_s$ then this is a legitimate expression.

Thus: $\mathrm{Tr}\, \mathcal{L}^{2n} = \mathrm{tr}\, A^{2n} + O(\sigma^n)$.

But: $(A+R) \otimes (A' + R') = A \otimes A + R \otimes A' + A \otimes R' + R \otimes R'$

~~[illegible]~~

lose because don't vanish. (perhaps).

For flows: $\det(1 - zX)^{-1}$ generator of flow

But

$$(1 - X)^{-1} = \int_0^\infty e^{-zt} \mathcal{L}_t \, dt = R(z).$$

(Laplace transform of semi-group).

Integrates along flow - removes problem of flow direction.

In fact $\quad \det(R(z,\omega)) = \dfrac{\det R(z)}{\det(\mathbb{1} - \omega R(z))}$

Study this term as before

- Need action of flow on forms.
- Need $R(z) \otimes R(z)'$

$$= \int_0^\infty dt \int_0^\infty ds \, e^{-z(t+s)} \mathcal{L}_t \otimes \mathcal{L}_s'$$

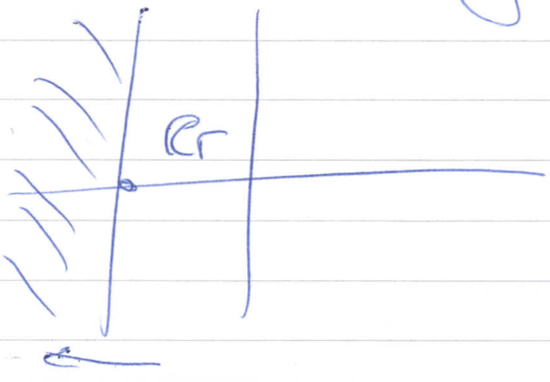
Transfer action of $\underbrace{\phi_t \times \phi_s}_{\mathbb{R}^2\text{-action}}$ | Quasi-cpt.

Use result on spectrum of C^r flow

$\phi_t : M \to M$, C^r Anosov.

$\exists$ Banach space

In which the generator



$+\infty$ as $r \to \infty$

Transfer operator

- forms expand (more expansion)
- Ind^n - get top entropy
- expansion from unstable bundle.