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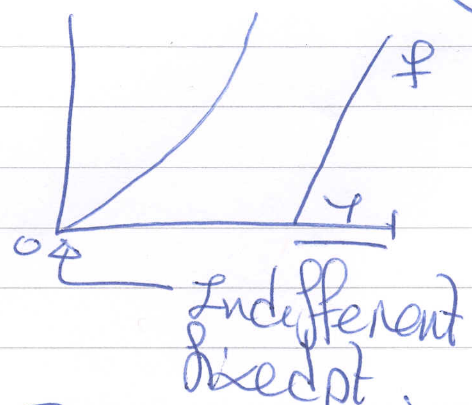
Non uniformly expanding maps

Example



eg $x \mapsto x + x^{1+1/\beta}$.

$\phi : Y \to \mathbb{Z}^+$.
$\phi(x) = \inf\{n \mid f^n x \in Y\}$

$F(x) = f^{\phi(x)}(x)$ $\quad F : Y \to Y$, uniformly expanding

$\mu(\phi > a) \simeq \frac{1}{a^\beta}$

$\rho(n) = \int_X v . w \circ f^n \, d\mu - \int_X v \int_X w \simeq \frac{1}{n^{\beta-1}}$

v Hölder, $w \in L^\infty$
Optimal Hu '04.

Liverani - Saussol - Vaienti '99 stoch approach.
Young '99 coupling
Maume-Deschamps '01 } operator
Sarig '02 } renewal
Gouezel '04. } systems
lower + upper bds.

M + T operator renewal sequences
+ dynamical truncation.

Tower maps: Good map $F: Y \to Y$, μ_Y.
Discrete roof function
$\phi: Y \to \mathbb{Z}^+$, $\phi \in L^1$, $\bar{\phi} = \int \phi \, d\mu_Y$,

$\Delta = Y^\phi = \{(y,\ell) \in Y \times \mathbb{Z} : 0 \leq \ell \leq \phi(y)\}/\sim$

$(x, \phi(y)) \sim (Fy, 0)$.

$f: \Delta \to \Delta$, $f(x,0) = (y, \ell+1) \bmod \sim$

$\mu_\Delta = \dfrac{\mu_Y \times \text{counting}}{\bar{\phi}}$

Operator renewal sequences

$f: \Delta \to \Delta$, transfer operator L

$\int_\Delta v . w \circ f \, d\mu = \int_\Delta Lv . w \, d\mu_\Delta$

$F: Y \to Y$, transfer operator R

$T_n = 1_Y L^n 1_Y$, $n \geq 0$. Return to Y

$R_n = 1_Y L^n 1_{\{\phi = n\}}$, $n \geq 1$ First return to Y

$T_n = \sum_{j=1}^{n} T_{n-j} R_j$, $T(z) = \sum_{n=1}^{\infty} T_n z^n$

$R(z) = \sum_{n=1}^{\infty} R_n z^n$.

Renewal eq^n: $\begin{cases} T(z) = I + T(z)R(z) \\ T(z) = (I - R(z))^{-1} \end{cases}$

Asymptotics of R_n $\longrightarrow$ regularity of $R(z)$

$\swarrow$

regularity of $T(z)$

$\swarrow$

$$\boxed{\begin{aligned} &\text{Note: } R_n = R\,1_{\{\phi = n\}} \\ &R(1) = R \end{aligned}}$$

asymptotics of T_n

$\searrow$ asymptotics of L^n.

Hypothesis: $\begin{cases} B = B(Y) \xhookrightarrow{\text{embed}} L^\infty(Y) \\ 1 \in B \\ \bullet \sum_1^\infty \|R_n\| < \infty \end{cases}$

Thus $R(z)$, $T(z)$ are analytic on $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

$R(z)$ continuous on $\overline{\mathbb{D}}$

- $1 \notin sp(R(z))$ $\forall z \in \overline{\mathbb{D}} - \{1\}$.

  $T(z) = (I - R(z))^{-1}$ cont on $\overline{\mathbb{D}} - \{1\}$.

- 1 simple isolated in the spectrum of $R(1)$.

P = spectral projection: $Pv = \int_Y v \, d\mu_Y$

$\|R_n\| \leq C\mu(\phi = n)$. — usual hypothesis, but this won't be used.

$B(\Delta)$

$v : \Delta \to \mathbb{R}$.

$v_n(y) = v(y, \phi(n) - n)$.

$\|v\| = \sup_n \|v_n\|$

<u>Example 1</u> : $\phi \in L^p$, $\sum j^{p-1} \left(\sum_{\ell \geq j} \|R_\ell\| \right) < +\infty$ for some $p > 1$.

$\forall q > 0$, $\exists \delta > 0$ s.t.

$$\rho(n) \ll \sum_{j \geq} \mu(\phi \geq j) + n\mu(\phi > \delta n) + O(1/n^q)$$

Example : $\mu(\phi \geq j) = O(1/n^\beta)$, $\beta > 1$

Then $\rho(n) = O(1/n^\beta)$. $v \in B(\Delta)$, $w \in L^\infty(\Delta)$

$$\mu(\phi \geq j) = O\left(\frac{\ell(n)}{n^\beta}\right) \quad p > 1, \ \ell(n) \text{ slowly varying.}$$

<u>Cor</u> : $\phi \in L^2$ and $\sum j^{p-1} \left(\sum_{\ell > j} \|R_\ell\| \right) < \infty$ some $p > 1$

$\Rightarrow \rho(n)$ is summable.

<u>Thm</u> (MT) $\phi \in L^p$ and $\sum j^{p-1} (\sum -) < \infty$, $p > 1$

$$\rho(n) = \sum_{j > n} \mu(\phi > j) \int v \int w + O(\mu(\phi > \delta n)) + O(1/n^q)$$

$v \in B(Y)$, $w \in L^\infty(Y)$.

and $\rho_{v,w}(n) \leq \sum_{j>n/2} \mu(\phi>j) + \mu(\phi>\delta_n) + O(1/n^\beta)$

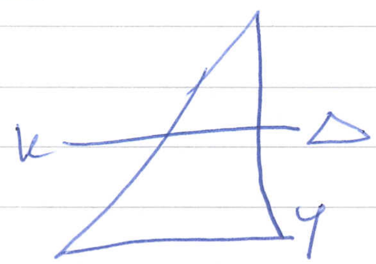
Question: If $\phi \in L^1$, does there exist a uniform decay rate:

$$\rho_{v,w}(n) \leq \|v\| \cdot |w|_\infty d_n, \quad d_n \downarrow 0$$

Thm 8 (MT): If $\|R_n\| < C\mu(\phi=n)$ and $\phi \in L^1$ and $\mu(\phi>n) = O\left(\frac{1}{n \log n}\right)$

$\Rightarrow \exists d_n.$

Dynamical truncation:



Fix $k \geq 1$. Let $\phi' = \min\{\phi, k\}$

Keep F, Y fixed

Construct $\Delta' = Y^{\phi'}$, $f' : \Delta' \to \Delta'$, μ_0'.

$$\rho(n) - \rho'(n) << \sum_{j>k} \mu(\phi>j) + n(\phi>k).$$

L'^n converges at rate $C(k) e^{-n a(k)}$

Thm 1: $C(k) = C$, $a(k) = \frac{\varepsilon \log k}{k}$

Since use renewal theory: $F = (f')^{\phi'}$ independent of k.

Finally $T'(z) = (I - \underbrace{R'(z)}_{\text{polynomial}})^{-1}$.

$R' = \sum R'_n z^n$, $R'_n = R'[\phi' = n]$