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$\Gamma = SL(2,\mathbb{Z})$, acts on $\mathbb{R}^2 = \{x = \binom{x_1}{x_2}, x_1, x_2 \in \mathbb{R}\}$.
$\gamma \in \Gamma$, γx

(S. Dani) $\overline{\Gamma x} \in \mathbb{R}^2$ iff x has irrational slope

Our motivation: Minkowski (Inhomog. Diophantine Approx.)
$\xi \in \mathbb{R} \setminus \mathbb{Q}$, $y \in \mathbb{R}$, $y \notin \mathbb{Z}\xi + \mathbb{Z}$.
Then $\exists\ \infty$-many $(p, q) \in \mathbb{Z}^2$, $q \neq 0$.
$|q\xi - p - y| \leq \frac{1}{4|\xi|}$

We want similar result for linear action of $SL(2,\mathbb{Z})$.

<u>Norms</u>: $|x| = \max_{i=1,2}$, $\left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\right| = \max_{i,j} |a_{ij}|$

$\gamma \binom{\xi}{1}$, $\quad |q'\xi + y' + y_2| \leq$ ——
$q\gamma' - q'p = 1$.

<u>Thm 1</u> (Laurent + N).
$x \in \mathbb{R}$ rational slope.
① $|\gamma x - 0| = |\gamma x| \leq \frac{|x|}{|\gamma|}$. $\quad \exists\ \infty$-many $\gamma \in \Gamma$ s.t. (eg Cont. Fracts).

② If $y = \binom{y_1}{y_2} \in \mathbb{R}^2 \setminus \{0\}$ with $y_1/y_2 \in \mathbb{Q}$
& $y_2 \neq 0 \Rightarrow \exists\ \infty$-many $\gamma \in \Gamma$ s.t.
$|\gamma x - y| \leq \frac{c}{|\gamma|^{1/2}}$

(iii) Otherwise, $\exists$ ∞-many matrices $\gamma \in \Gamma$ such that $|\gamma x - y| \leq c' / |\gamma|^{1/3}$

Number Theory.

Rk: (1) 1 is best exp
(2) 1/2 is best exp.
(3) 1/3 can prob. be improved.

Remark: F. Maucourant + B. Weiss. $\left(\frac{1}{144}\right)$

Horocycle flow.

Corollary. Inf. many $(q,p)=1$ s.t.
$|y\xi q - y| < c' / |q|^{n}$.

Pf: $x = \binom{\xi}{1}$, $y = \binom{q}{y}$.

Uniform approximation $\xi \in \mathbb{R} \setminus \mathbb{Q}$

Irrationality measure of ξ

$\omega(\xi) = \sup\{\omega \mid |q\xi - p| \leq 1/q^{\omega}$ has ∞-many sol. for $(p,q) \in \mathbb{Z}^2\}$

Let $x, y \in \mathbb{R}^2$: $\mu(x,y) = \sup\{\mu : |\gamma x - y| \leq 1/|\gamma|^{\mu}$ i.o.$\}$

Thm 2 (L, N).

(1) $\mu(x,0) = 1$
(2) $\mu(x,y) = \dfrac{\omega(\xi)}{\omega(\xi)+1}$ where $\xi = x_1/x_0$
(3) $\mu(x,y) \geq 1/3$

[ξ = irrational slope]

Thm 3 (L+N)

Let $y \in \mathbb{R}\setminus\mathbb{Q}$ be an irrational number $\omega(y) = 1$. Let $x \in \mathbb{R}^2$ irrational slope.

Then for infinitely many $\gamma \in \Gamma$.

$$|\gamma x - y| \le \frac{1}{|\gamma|^{1/2}} \quad \text{for a.e } y \text{ on the}$$

line $\mathbb{R}\begin{pmatrix} y \\ 1 \end{pmatrix}$

Remark (Ba. Weiss)

Let $x, y \in \mathbb{R}^2$. and $\gamma, \eta \in \Gamma$

$\mu(\gamma x, \eta\gamma) = \mu(x, y)$ so f^n μ is $\Gamma \times \Gamma$.

so for a.e. $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$.

$\mu(x, y)$ is const μ_0.

Conjecture $\mu_0 = 1/2$.

Another approach (Ledrappier N)

Let x = rational slope and $K = [0, r)^2$ where $r > 0$

$$cn \le \# \{\gamma : \gamma x \in K, |\gamma| < n\} \le Cn \ \forall x$$

$$\sum_{n \ge 1} \sum_{\substack{\gamma : \gamma x \in K \\ |\gamma| = n}} \frac{1}{n^\nu} < \infty \text{ iff } \nu > 1$$

If $\mu > 1/2$: $\bigcap_{m \ge 1} \bigcup_{n \ge m} \bigcup_{\substack{\gamma : \gamma x \in K \\ |\gamma| = n}} B(\gamma x, 1/n^\mu)$. is a null Leb measure set.

Ledrappier: $\Gamma \subset SL(2, \mathbb{R})$ with finite covolume.