Putnam

CBMS lecture notes.

Problem (R. Bowen)

(M, f) = Axiom A, Ω = basic set

Is there: $N \geq 0$, $H_N(\Omega, f|\Omega)$ finite dim^l vector space

$\sigma(f|\Omega)_N \in \mathrm{aut}(\,)$.

s.t. Lefschetz formula: $\sum_N (-1)^N \mathrm{Tr}((f|\Omega)_N^p)$ $p \geq 1$

$= \#\{x \in \Omega \mid f^p(x) = x\}$.

(Motivation?: Manning's proof of rationality of ζ-function).

Changes: $N \in \mathbb{Z}$, finite rank abelian groups

$H \otimes \mathbb{R}$ = finite dimensional vector space functor

Change in dynamics: D. Ruelle's def^n of Smale space

(X, d) cpt metric, ϕ = homeomorphic hyperbolic.

Special case: Shifts of finite type.

Krieger (or Bowen-Franks).

G = finite graph, k-vertices
A = $k \times k$ adjacency matrix.
Σ_G

Krieger's invariant: $D^s(\Sigma_G, \sigma) = \varinjlim \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \cdots$
if $\det A = \pm 1$ $\cong \mathbb{Z}^k$.

$\mathbb{Z}^k$
A

General Case

$(X, \varphi) \xleftarrow{\pi} (\Sigma, \sigma)$ Bowen's Thm.
irred. Smale space; (Σ, σ) SFT.

For $N \geq 0$: $\Sigma_N(\pi) = \{(e_0, \dots, e_N) \in \Sigma^{N+1} \mid \pi(e_0) = \dots = \pi(e_N)\}$.
SFT.

Associate $D^s(\Sigma_N(\pi))$

Need $\delta_n : \Sigma_N(\pi) \longrightarrow \Sigma_{N-1}(\pi)$
delete entry n.

$D^s(\ ) \xrightarrow{?} D^s(\ )$ Unfortunately not!

<u>Defn</u>: $\pi : (Y, \psi) \longrightarrow (X, \phi)$ factor map on Smale spaces
s-bijective if $\pi|$ stable eq. class of y
$\longrightarrow$ stable eq. class of $\pi(y)$
bijection for all $y \in Y$.

$$(\Sigma,\sigma) \xrightarrow{\pi} (\Sigma',\sigma)$$

s-bijective $\Longrightarrow$ $\exists\, D^s(\Sigma) \xrightarrow{\pi^s} D^s(\Sigma')$ | finite to one

u-bijective $\Longrightarrow$ $\exists\, D^s(\Sigma) \xleftarrow[\pi^{s*}]{} D^s(\Sigma')$

General case: Use "Better" Bowen's Thm.

$(X,\phi) \xleftarrow[\pi_s]{\text{s-bij}^n} (Y,\psi)$ Smale sp.

imed Smale sp

loc. unstable tot. disconnected.

u-bijective

(Z,ζ) loc stable sets tot. disc.

$\xleftarrow[\pi]{}$ s-bijective

$\Sigma \subset Y \times Z$ SFT.

[Ex $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ – Markov Partition with 3 sets gives factor map.]

$L;\ M \geq 0$

$$\Sigma_{L,M}(\pi) = \{(y_0,\dots,y_L, z_0,\dots,z_M) \mid \pi_s(y_l) = \pi_u(z_m),\ \forall l, m\}$$

SFT: $\delta_{l,}: \Sigma_{LM} \to \Sigma_{L-1\,M}$ erase y_l; (s-bij^n)

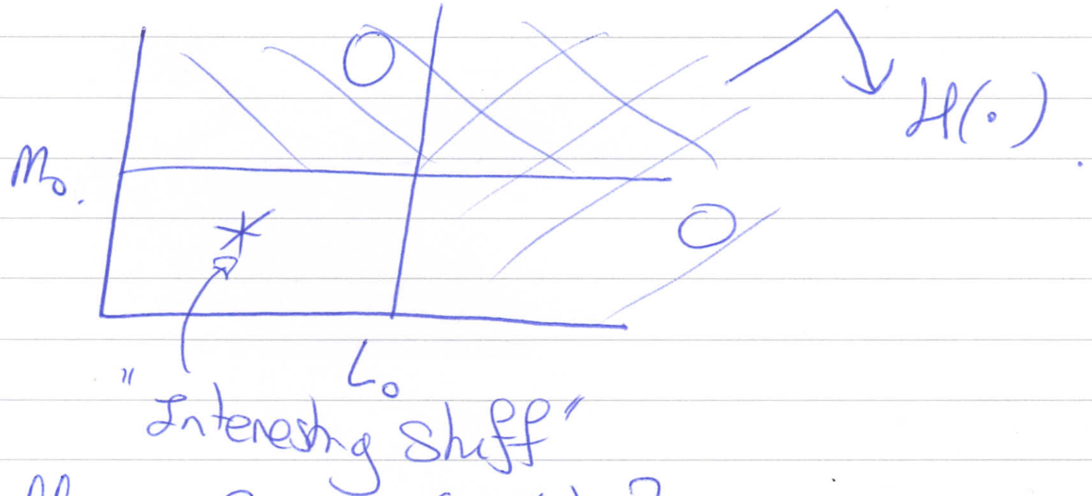
$\delta_{,m}: \Sigma_{LM} \to \Sigma_{L,M-1}$ erase z_m (u-bij^n)

$$\begin{tikzcd}
D^s(\Sigma_{0,1}) & D^s(\Sigma_{11}) \arrow[l] & \\
D(\Sigma_{0,0}) \arrow[u] & D^s(\Sigma_{1,0}) \arrow[l] \arrow[u] & D^s(\Sigma_{20}) \arrow[l] \arrow[u]
\end{tikzcd}$$

$\sum_l (-1)^l \delta_{l,}^s$ $\quad$ $\sum (-1)^m \delta_{,m}^{s*}$

"Direct sum" groups on diagonals.

$S_{L+1} \times S_{M+1}$ "reduce"



<u>Recall</u>: Given (X,ϕ) Choose π Compute $\Big\}$ 1. Depends on (X,ϕ) <u>not</u> on π. $H_N^S(X,\phi)$

2. Covariant for s-bij^n
Contrav for u-bij^n

3. All the groups are finite rank.
Only finitely many $\neq 0$.

4. Lefschetz formula.

<u>Ex 0</u>: $X = \Sigma$ SFT. Let $Y = Z = \Sigma$.

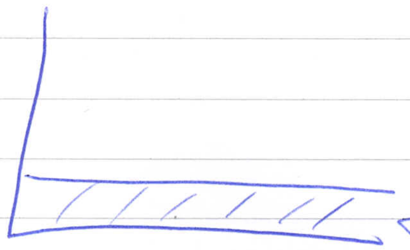
$M_0 = L_0 = 1$; $(D^s(\Sigma))$

$$H_N^S = \begin{cases} D^s(\Sigma) & N = 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Ex 1</u>: m^∞ solenoid

X $\quad \mathbb{T} \xleftarrow{z^m \leftarrow z} \mathbb{T} \leftarrow \cdots$ $\quad Z = X$

$X \longleftarrow \Sigma$ full m-shift (s-bijective) Y



$$\begin{matrix} 0 & 0 & 0 \\ \mathbb{Z}[1/m] \leftarrow \mathbb{Z} & & 0 \\ 0 & & \end{matrix}$$

← Single row. (T. Bazett)

$$H^s_N = \begin{cases} \mathbb{Z}[1/m] & N=0 \\ \mathbb{Z} & N=1 \\ 0 & \text{otherwise.} \end{cases}$$

<u>ex2</u>: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \mathbb{T}^2$
[+T. Bazett]

Then (Y,ψ) = DA system
(Z,ζ) = inverse.
Σ – from M.P with 3 rectangles

$$\begin{tikzcd} {} \\ \mathbb{Z}^2 \arrow[u] & \mathbb{Z} \arrow[l, hook'] \\ \mathbb{Z}^3 \arrow[u] & \mathbb{Z}^2 \arrow[l, hook'] \arrow[u] & 0 \arrow[l, hook'] \end{tikzcd}$$

$$H^s_N = \begin{cases} \mathbb{Z} & N=\pm 1 \\ \mathbb{Z}^2 & N=0 \\ 0 & \text{else} \end{cases}$$

<u>Problems</u>: • What are elts of groups
(like manifolds Čech vs. deRham)

Connection with operator algebras.

<u>Conjecture</u>: $\check{H}(X) \hookrightarrow H^s(X,\phi) \otimes H^u(X,\phi)$