Ward

Uniformities for commuting maps
($\equiv \mathbb{Z}^d$-actions)

* diverse, easy to see in "entropy rank 1"

Motivation:

If $T: X \to X$ is a very nice map {circle doubling, hyperbolic autos}

then ① $\frac{1}{n} \log |\mathrm{Fix}(T^n)| \to h > 0$

② $\exists$ class S of smooth fns

s.t. $\left| \int f \, d(\text{uniform on } \mathrm{Fix}(T^n)) - \int f \, d\binom{\text{natural}}{\text{measure}} \right| < C(f) \cdot \lambda^n$

$\lambda < 1$, $\forall f \in S$

③ $\forall f, g \in S$: $\left| \int f \cdot g \circ T^n \, d\binom{\text{natural}}{\text{measure}} - \int f \, d\mu \int g \, d\mu \right|$ (μ = natural measure)

$< C(f,g) \cdot \lambda^n$, $\lambda < 1$

④ $h_{top}(T) > 0$.

Let S, T commute: $\begin{cases} \underline{n} = (n,m) \longrightarrow S^n T^m \\ C(f,g) \to C(f,g,\underline{n}) \end{cases}$

Q: Can we replace n by $\|\underline{n}\|$

Examples: X - compact connected gp, abelian
T - replaced by α, an action of $\mathbb{Z}^d$ by automorphism
α - acts expansively.
$h_{top}(\alpha^{\underline{n}}) < +\infty$, $\forall\, \underline{n}$

Choose $\times 2, \times 3$ (invertible ext^n of it)

or 2 commuting hyperbolic autos of $\mathbb{T}^3$ (generating a mixing action)

All the matrices here are diagonal

$$(n,m) \longmapsto 2^n 3^m \equiv \begin{pmatrix} 2^n3^m & 0 & 0 \\ 0 & 2^n3^m & 0 \\ 0 & 0 & 2^n3^m \end{pmatrix}$$

Why is this difficult?

Look at ①

$6^n - 1$

13 53 102 215

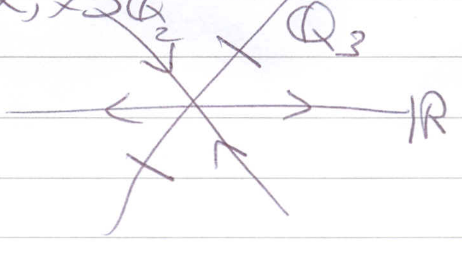
1 17 35

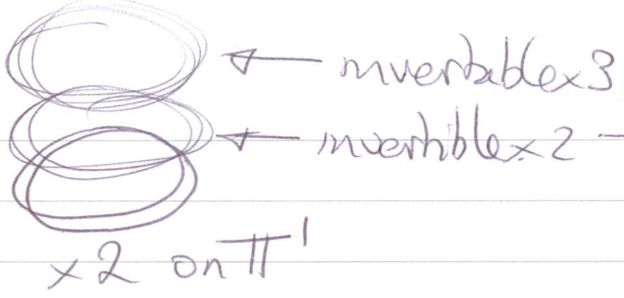
1 8 11 23 47 95.

~~∞ 1 1 7 5 31~~

$$(1,0) \longrightarrow \times 2 \equiv \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

local geometry of $\times 2, \times 3$





skew product "kills" some periodic points.

$$|\mathrm{Fix}(\alpha^{(n,0)})| = (2^n-1) \times |2^n-1|_3$$

$$1 \geq |2^n-1|_3 \gg 1/n$$

Similarly for vertical maps
All other directions are expansive.

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7	1	5	7	1
13	5	1	1	1
5	7	1	1	∞

Starting question: Is $\lim\limits_{\|\underline{n}\| \to \infty} \frac{1}{\|\underline{n}\|} \log |\mathrm{Fix}(\alpha^{\underline{n}})| > 0$?

<u>Answer</u> = Yes: Algebra + Baker Thm.

2 & 3 Also have a positive answer

<u>2 steps</u>: 1) Formulate appropriately
2) Algebra
3) Baker like step.

1) Looks like this: Construct an exhaustive sequence
$H_1 \subset H_2 \subset \cdots \nearrow \hat{X}$ (Bases)
$\psi(k) \nearrow +\infty$ such that
$H_k \cap (\alpha^{\underline{n}} - Id) H_k = \{0\}$ for $\|\underline{n}\| > \psi(n)$

$H_k = [-k, k] \cap \mathbb{Z}$

$H_k \cap (2^n - 1) H_k = \{0\}$ if n so large

Entropy rank one: "eigenvalues" are a list of numbers:

$(n, m) \mapsto \times 2^n 3^m$ governed by

$(|2^n 3^m|_\infty, |2^n 3^m|_2, |2^n 3^m|_3)$

Entropy rank > 1:- $\{x \in \mathbb{T}^{\mathbb{Z}^2} \mid 3x_{\underline{n}} + x_{\underline{n}+(1,0)} + x_{\underline{n}+(0,1)} = 0 \pmod 1 \ \forall \underline{n}\}$