Dimensions of certain self-similar measures

Nikita Sidorov

The University of Manchester

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That is,

$$\lim_{h \to 0} \frac{\log \nu_\lambda(x - h, x + h)}{\log h} \equiv H_\lambda \quad \text{for } \nu_\lambda\text{-a.e. } x.$$

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Theorem (Hare-S, 2010)

1. *We have $H_\lambda > 0.81$ for all Pisot $\lambda \in (1/2, 1)$.*
2. *For $\lambda^{-1} < 1.7$ we have $H_\lambda > 0.87$.*
3. *For the small Pisot numbers $\beta = \lambda^{-1}$ we have the following individual lower bounds:*

Garsia's entropy: a lower bound

Minimal polynomial of β	β	Depth	Lower Bnd for H_λ
$x^3 - x - 1$	1.3247	17	.88219
$x^4 - x^3 - 1$	1.3803	16	.87618
$x^5 - x^4 - x^3 + x^2 - 1$	1.4433	15	.89257
$x^3 - x^2 - 1$	1.4656	15	.88755
$x^6 - x^5 - x^4 + x^2 - 1$	1.5016	14	.90307
$x^5 - x^3 - x^2 - x - 1$	1.5342	15	.89315
$x^7 - x^6 - x^5 + x^2 - 1$	1.5452	13	.90132
$x^6 - 2x^5 + x^4 - x^2 + x - 1$	1.5618	15	.90719
$x^5 - x^4 - x^2 - 1$	1.5701	15	.88883
$x^8 - x^7 - x^6 + x^2 - 1$	1.5737	14	.90326
$x^7 - x^5 - x^4 - x^3 - x^2 - x - 1$	1.5900	15	.89908
$x^9 - x^8 - x^7 + x^2 - 1$	1.5912	14	.90023

Table: Lower bounds for Garsia's entropy for all Pisot numbers < 1.6

Question. Are multinacci parameters local maxima for the function $\lambda \mapsto H_\lambda$?

Two-dimensional model

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Now take any three non-collinear points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$ and put

$$f_j(\mathbf{x}) = \lambda \mathbf{x} + (1-\lambda)\mathbf{a}_j, \quad j = 1, 2, 3.$$

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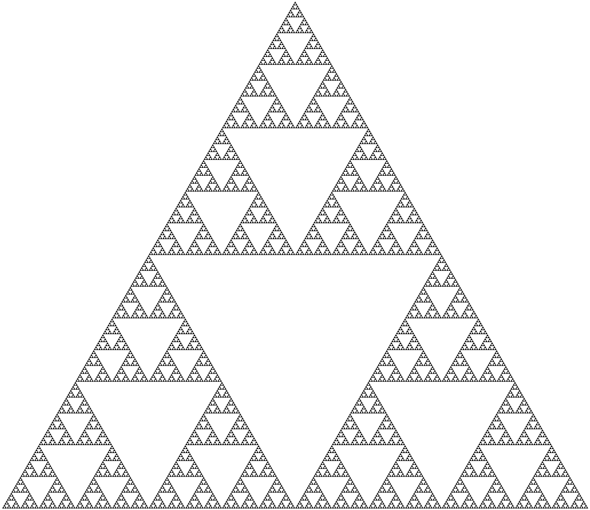
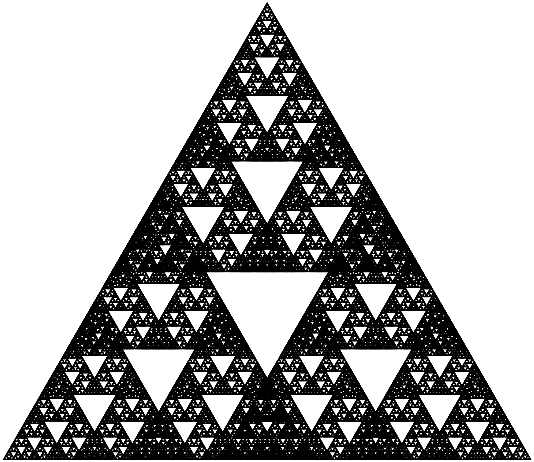
Let S_λ denote the attractor for this IFS.

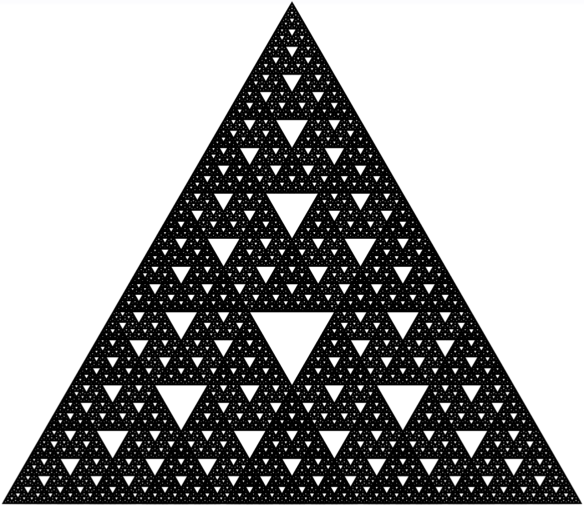
The most famous case is $\lambda = 1/2$:



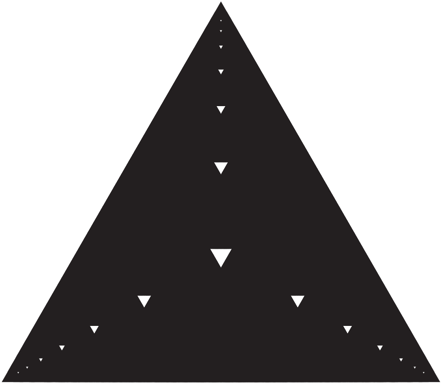
The Sierpiński Gasket



The fat Sierpiński Gasket for $\lambda = 0.59$
(zero Lebesgue measure?)



The Golden Gasket, $\lambda = \frac{\sqrt{5}-1}{2} \approx 0.618$.



The fat Sierpiński Gasket for $\lambda = 0.65$
(has a nonempty interior)

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Note that if $\lambda = 1/2$, then μ_λ is the normalized Hausdorff measure (for $s = \log 3/\log 2$), whence these are equal.