The Frenkel-Kontorova model for quasi-periodic environments of Fibonacci type

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Outline

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- I. Minimizing configurations and minimizing measures in Aubry-Mather theory in the periodic case
- II. Quasi-periodic environments of Fibonacci type
- III. A few results on Aubry-Mather theory in the quasi-periodic case

I. Minimizing configurations

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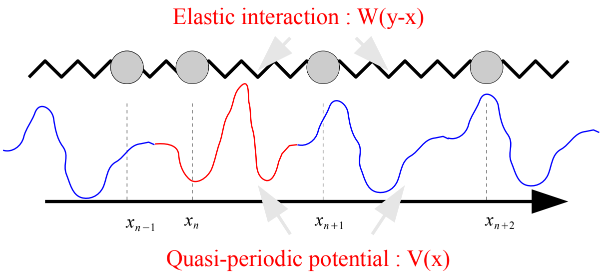

- Consider a chain of atoms in $\mathbb{R}$: x_n position of the nth atom
- Each atom is in interaction with its nearest neighbours and with an external potential
- The energy at each site is $E(x_n, x_{n+1}) = W(x_{n+1} - x_n) + V(x_n)$

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Definition: A configuration $\{x_n\}_{n\in\mathbb{Z}}$ is minimizing in the Aubry sense if

$$\begin{aligned} E(x_n, \ldots, x_{n+k}) &:= \sum_{i=0}^{k-1} E(x_{n+i}, x_{n+i+1}) \\ &\leq E(y_n, \ldots, y_{n+k}) \end{aligned}$$

whenever $x_n = y_n$ and $x_{n+k} = y_{n+k}$, for all $n \in \mathbb{Z}$ and $k \geq 1$

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- **The environment $V(x)$ is periodic**: (of period 1)

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- **A more general framework:** $E_\lambda(x,y) = E_0(x,y) - \lambda(y-x)$

$$E_0(x,y) \text{ is of class } C^2$$

$$E_0 \text{ is periodic:} \quad E_0(x+1,y+1) = E_0(x,y)$$

$$E_0 \text{ is superlinerar:} \quad \lim_{\|y-x\|\to+\infty} \frac{E_0(x,y)}{\|y-x\|} = +\infty$$

$$E_0 \text{ is twist:} \quad \frac{\partial^2 E_0}{\partial x \partial y} < -\alpha < 0$$

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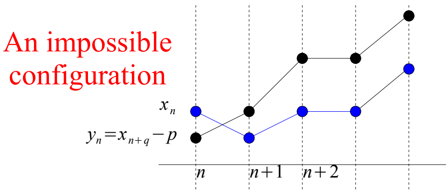
2) All recurrent minimizing configuration admits a rotation number
3) The main idea in the proof: a translation by an integer of a minimizing configuration is still minimizing and cannot cross itself



– It is not any more true in the quasi-periodic case

Mather minimizing measures

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Remark: If $\{x_n\}_{n\in\mathbb{Z}}$ is minimizing in the Aubry sense then

$$\frac{\partial E}{\partial y}(x_{n-1}, x_n) + \frac{\partial E}{\partial x}(x_n, x_{n+1}) = 0, \quad \forall\ n$$

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Definition: A minimizing configuration can be seen as a particular orbit of a dynamical system called *Euler-Lagrange dynamics*.
Let $v_n = x_{n+1} - x_n$,

$$\Phi_{EL} = \begin{cases} \mathbb{T}^1 \times \mathbb{R} & \to \quad \mathbb{T}^1 \times \mathbb{R} \\ (x_n, v_n) & \to \quad (x_{n+1} = x_n + v_n, v_{n+1} = v_n + V'(x_{n+1}) \end{cases}$$

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Definition: A configuration $\{x_n\}_n$ is minimizing in the Mather sense if

$$(x_n,v_n) \in \mathrm{Supp}(\mu_{min}), \quad \forall\ n\in\mathbb{Z}, \qquad \text{where } \mu_{min} \text{ is minimizing}$$

$$\mu_{min} = \arg\min\Big\{ \int_{\mathbb{T}^1\times\mathbb{R}} E(x,x+v)\,d\mu(x,v)\ :\ \mu \text{ is a } \Phi_{EL}\text{-inv prob}\Big\}$$

A small part of Mather theory

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Theorem (Mather 1991): (the periodic case) Recall
$E_\lambda(x,y) = E_0(x,y) - \lambda(y-x)$

1) Any configuration minimizing E_λ in the Mather sense is minimizing in the Aubry sense
2) For configurations minimizing in the Aubry sense, minimizing E_λ is equivalent to minimizing E_0
3) Any recurrent minimizing configuration in the Aubry sense is minimizing E_λ in the Mather sense for any λ related to the rotation number ω

$$\omega = -\frac{d\bar{E}}{d\lambda}(\lambda)$$

$$\bar{E} := \min\left\{ \int E(x, x+v)\, d\mu(x,v) \ : \ \mu \ \Phi_{EL}\text{-inv} \right\}$$

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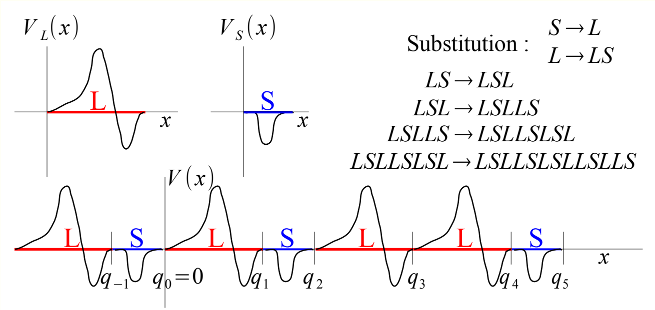

– $\mathbb{R}$ is partitioned into segments of two kinds: long and short
– the external potential admits two forms: $V_L(x)$ and $V_S(x)$
– $\underline{\Omega} =$ the closure of all the shifts of the Fibonacci word

$$\ldots, LSL, LS \big| LS, L, LS, LSL, LSLLS, \ldots$$

Existence of rotation number

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Notations: $\underline{\Omega}$ is the compact set of Fibonacci words. $\underline{\Omega}$ is compact minimal and uniquely ergodic. Each $\underline{\omega} \in \underline{\Omega}$ gives a quasi-periodic potential $V_{\underline{\omega}}(x)$.
As before, the total energy per site is

$$E_{\underline{\omega}}(x,y) = W(y-x) + V_{\underline{\omega}}(x), \qquad W''(x) < -\alpha < 0$$

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Theorem (Gambaudo, Guiraud, Petite, 2006): We fixe an environment $\underline{\omega} \in \underline{\Omega}$.

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$$\rho = \lim_{m-n \to +\infty} \frac{x_m - x_n}{m-n}$$

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Question:
What about minimizing configurations in the Mather sense?
What plays the role of $\mathbb{T}^1 \times \mathbb{R}$ in the periodic case?

The space of quasi-periodic environments

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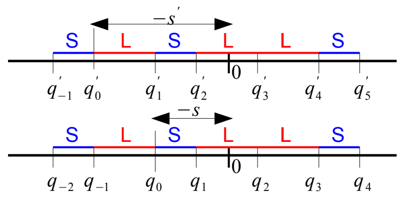
Extension of $\underline{\Omega}$:



– the origin does not play any role. We consider the set of all shifts

$$\Omega = \underline{\Omega} \times \mathbb{R}/\sim \quad \Leftrightarrow \quad \begin{cases} \text{different parametrizations but} \\ \text{same sequence of impurities} \end{cases}$$

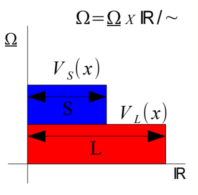
– Ω is a suspension over $\underline{\Omega}$ built with a return map of length L or S
– In the periodic case $L = S$ and $\Omega = \mathbb{T}^1$
– In the quasi-periodic case Ω plays the role of $\mathbb{T}^1$

Mather measures

Mather measures

A global potential V: Recall $(\Omega, \{\phi^t\})$ denotes the minimal Fibonacci flow

$$\begin{aligned} V_\omega(x) &= V \circ \phi^x(\omega) \\ E_\omega(x,y) &= W(y-x) + V_\omega(x) \\ &= L(\phi^x(\omega), y-x) \\ L(\omega, v) &= W(v) + V(\omega) \end{aligned}$$



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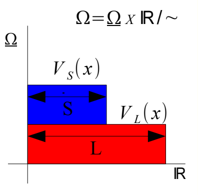

Minimizing measures in the Mather sense:

– There is no way we can define an equivalent Euler-Lagrange map Φ_{EL}

– In the periodic case: $\Phi_{EL}(x,v) = (x+v, \ldots)$, $\quad x \in \mathbb{T}^1$, $v \in \mathbb{R}$

– A measure μ is holonomic if

$$\int f(\omega)\, d\mu(\omega, v) = \int f \circ \phi^v(\omega)\, d\mu(\omega, v), \qquad \forall\ f \in C^0(\Omega)$$

– A measure μ_{min} is minimizing in the Mather sense if

$$\mu_{min} = \arg\min \left\{ \int_{\Omega \times \mathbb{R}} L(\omega, v)\, d\mu(\omega, v) \ : \ \mu \text{ is holonomic} \right\}$$

III. A few results in Aubry-Mather theory

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Results:

– Optimal segments $\{x_k\}_k$ (minimizing $E(x_0, \ldots, x_n)$) have uniform bounded gaps

$$\exists\ C \text{ s.t. } \{x_k\}_{k=0}^n \text{ is optimal } \Rightarrow |x_{k+1} - x_k| < C$$

– The Mather set is compact and non empty

– The lowest mean energy can be computed using either minimizing configurations or minimizing measures $\quad (L(\omega, v) = W(v) + V(\omega))$

$$\begin{aligned}\bar{E} &= \lim_{n \to +\infty} \min_{\omega, x_0, \ldots, x_n} \frac{1}{n} E_\omega(x_0, \ldots, x_n) \\ &= \min_{\mu \text{ holonomic}} \int L(\omega, v)\, d\mu(\omega, v) = \int L\, d\mu_{min}\end{aligned}$$

– If $\tilde{M}$ denotes the projection of the Mather set on Ω, then $\tilde{M}$ has a non empty intersection with any orbit of length long enough of the Fibonacci flow.