Equivalence relations and random graphs: an introduction to **graphical dynamics**

Vadim A. Kaimanovich

University of Ottawa

April 19, 2012

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Warwick

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Second letter of Newton to Leibniz (1676)

Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa

Given an equation involving any number of fluent quantities, to find the fluxions, and vice versa

It is useful to solve differential equations!

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It is useful to consider invariant measures!

▸ WHERE?

Classical examples

- smooth dynamics;
- measurable dynamics;
- symbolic dynamics.

graphical dynamics?!

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Definition

A **graph** Γ is determined by its set of **vertices** *(nodes)* V and its set of **edges** *(links)* E connected by an incidence relation (further "decoration" is possible!).

Structured "big" set $\Longrightarrow$ local structure $\Longrightarrow$ graph structure

How can one understand a collection of (large) finite objects?

finite objects	$\rightsquigarrow$	infinite objects	$\rightsquigarrow$	invariant measures
finite words	$\rightsquigarrow$	infinite words $A^{\mathbb{Z}}$	$\rightsquigarrow$	✓ information theory
finite graphs	$\rightsquigarrow$	infinite graphs $\mathcal{G}$	$\rightsquigarrow$	?

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Holonomy invariant measures on foliations ▸ Plante 1975

Measured equivalence relations (Feldman–Moore 1977)

(X, μ) — a Lebesgue probability space

$R \subset X \times X$ — a **Borel equivalence relation with at most countable classes** (examples: orbit equivalence relations of group actions, traces on transversals in foliations, etc.)

A **partial transformation** of R — a measurable bijection $\varphi : A \to B$ with graph $\varphi \subset R$

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The measure μ is R-**invariant** if $\varphi\mu_A = \mu_B$ for any partial transformation of R.

One can also talk about **quasi-invariant measures** and the associated **Radon–Nikodym cocycle**

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The (left) **counting measure** is

$$d\#_\mu(x, y) = d\mu(x) d\#_x(y) ,$$

where $\#_x$ is the counting measure on the fiber $\mathfrak{p}^{-1}(x)$ of the projection $\mathfrak{p} : R \to X$ (i.e., on the equivalence class of x).

The **involution** $[(x, y) \mapsto (y, x)]$ of $\#_\mu$ is the **right counting measure** $\#^\mu$, and μ is R-quasi-invariant $\iff$ $\#_\mu \sim \#^\mu$

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$$\mathcal{D}(x, y) = \frac{d\#^\mu}{d\#_\mu}(x, y) = \frac{d\mu(y)}{d\mu(x)}$$

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Definition (Plante 1975 - pseudogroups, Adams 1990)

$K \subset R$ — a **leafwise graph structure** on an equivalence relation R; (X, μ, R, K) — a **graphed equivalence relation**.

A discrete analogue of Riemannian foliations. Further “decoration” is possible! (edge length, labelling, colouring etc.). One can consider structures of higher dimensional **leafwise abstract simplicial complexes**.

Assume that

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A measure μ is R-invariant $\iff$ the restriction $\#_\mu|_K$ is involution invariant.

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Definition

The **simple random walk** on a (locally finite) graph Γ is the Markov chain with the transition probabilities

$$p(x, y) = \begin{cases} 1/\deg x, & x \sim y; \\ 0, & \text{otherwise.} \end{cases}$$

In the same way one defines the **simple random walk along classes of a graphed equivalence relation** (X, μ, R, K), cf. leafwise Brownian motion on foliations (Garnett 1983).

Theorem (K 1988, 1998)

A measure μ on a graphed equivalence relation (X, m, R, K) is R-invariant $\iff$ the measure $m = \deg \cdot \mu$ is **stationary** *and* **reversible** *with respect to the SRW on X.*

Idea of proof: Reversibility $\equiv$ involution invariance of the joint distribution of (x_0, x_1) $\equiv$ involution invariance of $\#_\mu|_K$.

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For continuous group actions the space of invariant measures is weak* closed $\Longrightarrow$ approximation by measures equidistributed on finite orbits (periodic points).

Definition (K)

A graphed equivalence relation (X, R, K) on a topological state space X is **continuous** if the map $x \mapsto \pi_x$ is continuous (with respect to the weak* topology on $M(X)$).

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Weaker form: a finite **stationary measure** for the leafwise simple random walk ($\equiv$ stationary scenery). Is the same as strong homogenization if the measure is, in addition, reversible.

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An *invariant measure* need **not** exist! Compactness of the state space implies existence of a *stationary* one (cf. Garnett's **harmonic measures** for foliations).

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- $\mathbb{Z}$-actions
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$$\pi : x \mapsto ([x]_K, x)$$

What can one say about the arising measures $\pi(\mu)$?

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$\mathcal{G} = \{(\Gamma, v) : v \text{ is a vertex of } \Gamma)\}$ – the space of (isomorphism classes) of locally finite **pointed** (rooted) infinite graphs.

$\mathcal{G} = \lim\limits_{\leftarrow} \mathcal{G}_r$ (pointed finite graphs of radius $\leq r$)

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The equivalence class of a graph Γ is the quotient

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A measure m on $\mathcal{R}$ is **unimodular** if the associated counting measure on $\mathcal{G}_{\bullet\bullet}$ is preserved by the involution (root switching).

For a finite graph the invariant measure is equidistributed on its equivalence class, whereas the unimodular measure is the quotient of the uniform measure on the graph itself.

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The space of unimodular measures on $\mathcal{G}$ is weak* closed — the space of invariant ones is **not**!

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The notions of invariance and unimodularity coincide for measures concentrated on **rigid** graphs (those with trivial automorphisms group). What happens in general?

$G = Iso(\Gamma)$ — group of isomorphisms of a graph Γ
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Definition (cf. Schlichting 1979, Trofimov 1985)

$\Delta(x, y) = |G_x y| / |G_y x|$ — the **modular cocycle** of Γ.

Δ determines a multiplicative cocycle of the equivalence relation $\mathcal{R}$ restricted to the subset $\mathcal{G}^0 \subset \mathcal{G}$ of graphs Γ with unimodular $\mathrm{Iso}(\Gamma)$.

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m is unimodular iff it is concentrated on $\mathcal{G}^0$ and its Radon–Nikodym cocycle is Δ.

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Are there purely non-atomic unimodular measures not equivalent to any invariant measure?

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(Dr. Strangelove or: How I Learned to Stop Worrying and Love the Bomb, **Stanley Kubrik** 1964)

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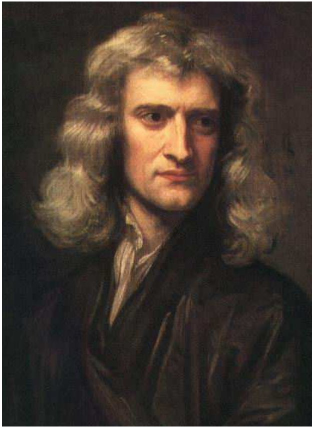

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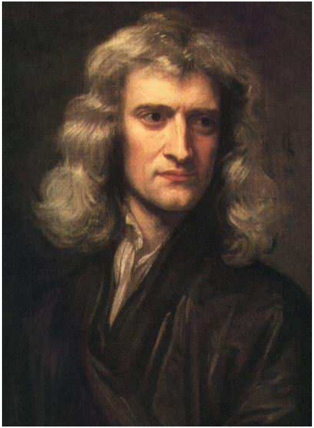

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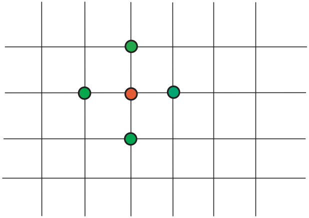
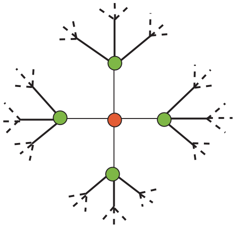


Euclidean lattice ($\mathbb{Z}^2$)



Bethe lattice *(free group $\mathcal{F}_2$)*

A is a finite alphabet

A^G — the **space of configurations**

The group G acts on $A^G = \{(a_g)\}_{g \in G}$ by translations

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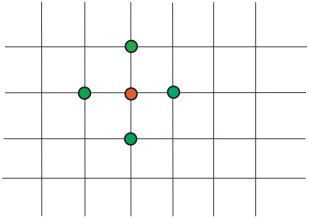

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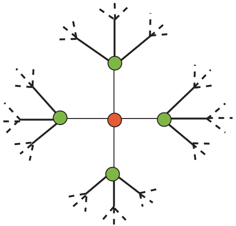

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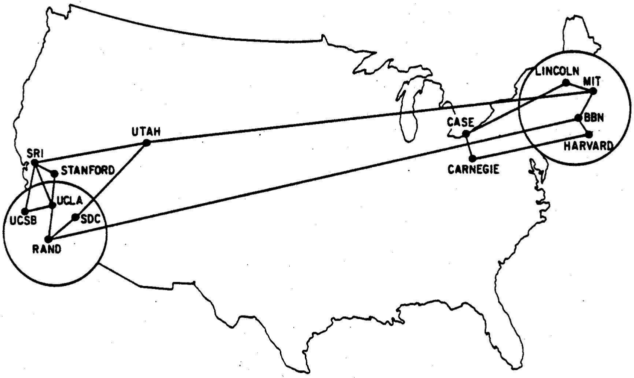

Arpanet in 1970

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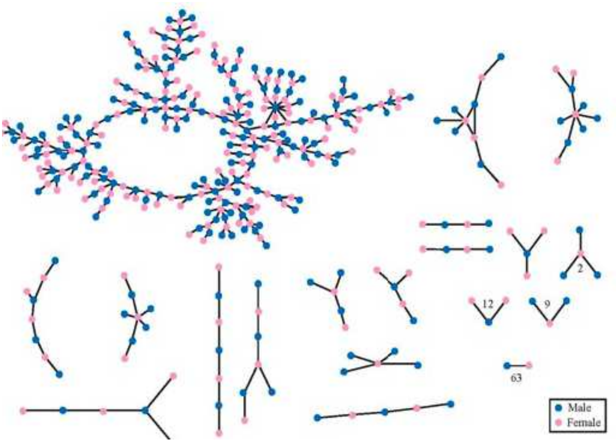
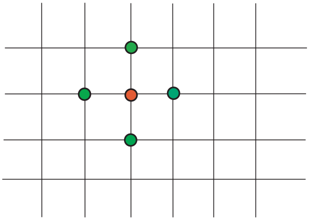


Dating in a high school

Return

“Roman” encoding $(1 \longleftrightarrow \mathrm{I},\ 2 \longleftrightarrow \mathrm{II},\ 3 \longleftrightarrow \mathrm{III})$

Return



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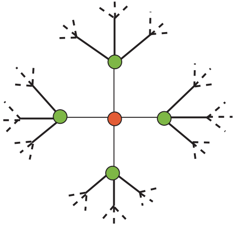

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Edges are labelled!

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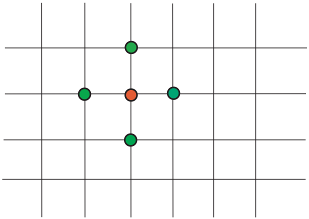

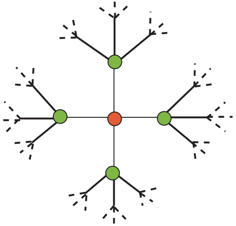

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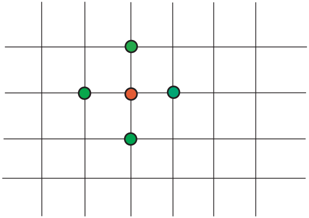

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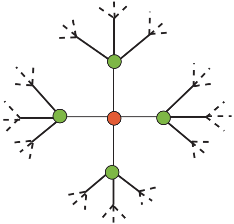

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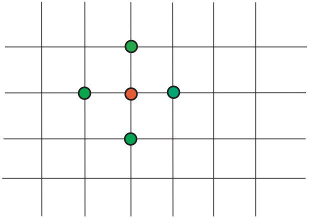

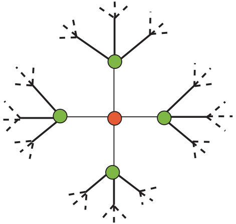

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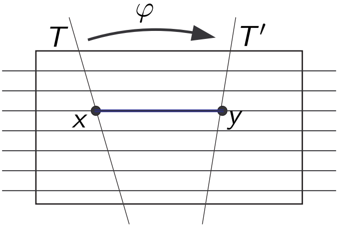
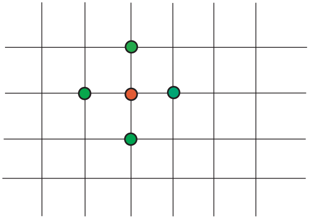
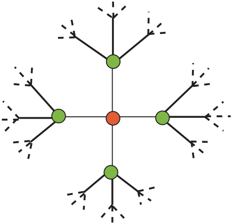


Holonomy invariant measures on foliations

◂ Return





random fields on vertices

vert.eps

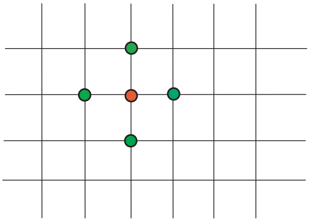
extreme case: site percolation

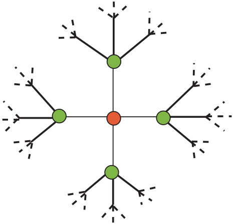
random fields on edges

edge.eps

extreme case: bond percolation

Return





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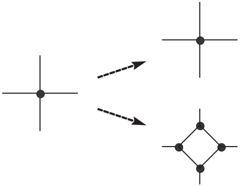

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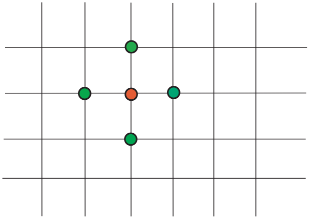

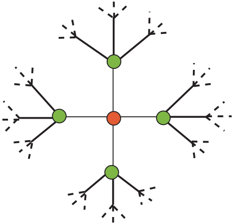

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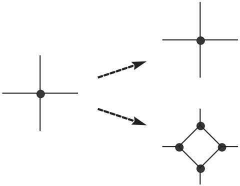

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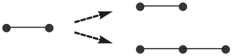

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For an action $G : X \circlearrowright$

$$X \ni x \mapsto \mathrm{Stab}_x = \{g \in G : gx = x\} \subset G \qquad (*)$$

In the presence of a generating set $K \subset G$ a subgroup $H \subset G$ determines the associated graph $\mathrm{Schreier}(X, G, K)$ on $X = G/H$ rooted at $o = \{H\} \in X$, and *vice versa*

If m is an invariant measure on X, then its image under $(*)$ is a G-invariant measure on subgroups of G ($\equiv$ an invariant measure on the space of Schreier graphs).

Definition (Vershik 2010)

An action $G : (X, m) \circlearrowright$ is **extremely non-free** if $(*)$ is a bijection (mod 0).

Extremely non-free actions of $G \equiv$ invariant measures on the space of Schreier graphs of $G \equiv$ stochastically homogeneous random Schreier graphs.

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A — finite alphabet, $T = T(A)$ — the rooted Cayley tree of finite words. Then any group $G \subset \mathrm{Iso}(T)$ preserves the **uniform measure** m on the **boundary** ∂T.

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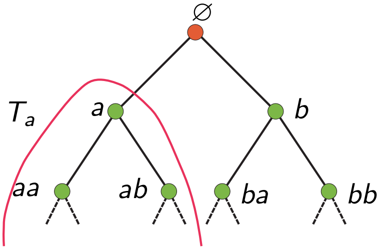

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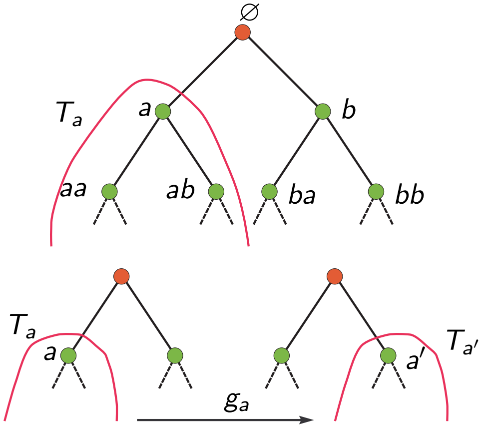

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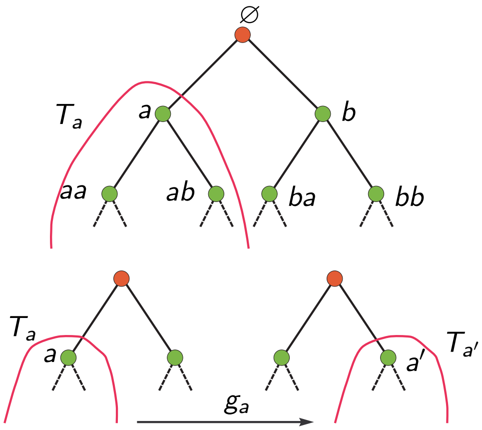

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Fractal sets arising from **Iterated Function Systems** (e.g., the **Sierpiński triangle**) give rise to the associated graphs:

serp.eps

"**Natural extension**" (analogous to the one used in dynamical systems) provides stochastic homogenization of such graphs.

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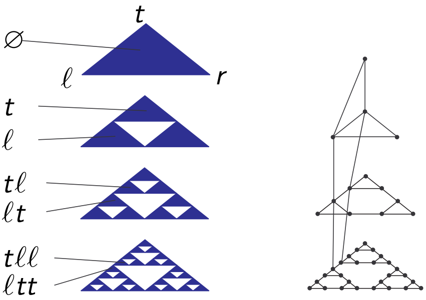

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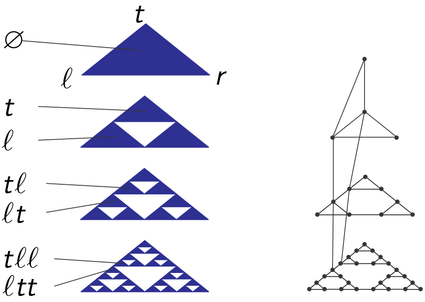

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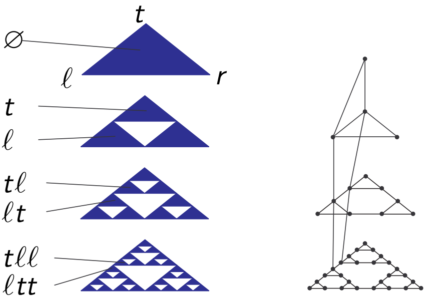

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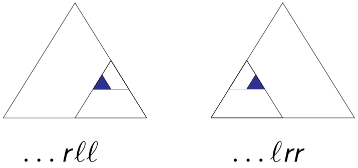

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The skew action $\alpha(\overline{\omega}, g) = (T\overline{\omega}, \alpha^{\omega_0} g)$ of the free group $\mathcal{F}_2 = \langle a, b\rangle$ (where $\alpha = a, b$) determines a **stochastically homogeneous Schreier graph** ("slowed down" Cayley tree).

Geometrically: $\chi = \#_a + \#_b - \#_{a^{-1}} - \#_{b^{-1}} : \mathcal{F}_2 \to \mathbb{Z}$ — the **signed letter counting character**. If $\omega_n = 0$, then any two edges with a common endpoint between $\chi^{-1}(n)$ and $\chi^{-1}(n+1)$ in the Cayley tree of $\mathcal{F}_2$ are "glued" together.

Another example: m — shift-invariant measure on bilateral infinite irreducible words in $\mathcal{F}_2$ (invariant measure of the **geodesic flow**), produces by "doubling" the associated **stochastically homogeneous Schreier graph** (or consider $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ instead of $\mathcal{F}_2$ — Elek 2011).

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Geometrically: $\chi = \#_a + \#_b - \#_{a^{-1}} - \#_{b^{-1}} : \mathcal{F}_2 \to \mathbb{Z}$ — the **signed letter counting character**. If $\omega_n = 0$, then any two edges with a common endpoint between $\chi^{-1}(n)$ and $\chi^{-1}(n+1)$ in the Cayley tree of $\mathcal{F}_2$ are "glued" together.

Another example: m — shift-invariant measure on bilateral infinite irreducible words in $\mathcal{F}_2$ (invariant measure of the **geodesic flow**), produces by "doubling" the associated **stochastically homogeneous Schreier graph** (or consider $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ instead of $\mathcal{F}_2$ — Elek 2011).

The associated action of the free group is **amenable** and **effective**.

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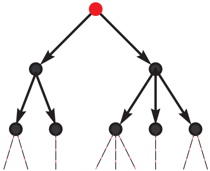
Realizations of a **branching** *(Galton-Watson)* **process** with offspring distribution $p = (0, p_1, p_2, \ldots, p_k)$ are rooted trees:



The arising measure **P** on rooted trees is **not** invariant (the root is statistically different from other vertices!).

Solution: consider **augmented GW trees**: add by force one offspring to the root, i.e., use $\widetilde{p} = (0, 0, p_1, p_2, \ldots)$ for the first generation, and p otherwise.

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GW2.eps

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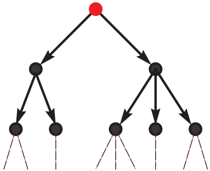

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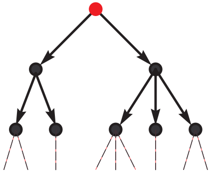

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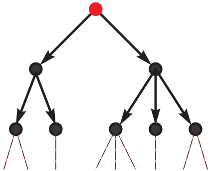

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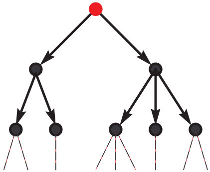

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part.eps

$$\widetilde{\mathbf{P}}(A) = p_1 \cdot p_2^2 \cdot 4p_1 p_2^3 = 4p_1^2 p_2^5 \ .$$

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$$3/2 = \widetilde{\mathbf{P}}(A')/\widetilde{\mathbf{P}}(A) = \deg o' / \deg o$$

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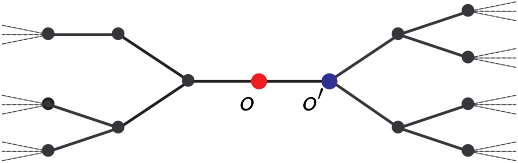

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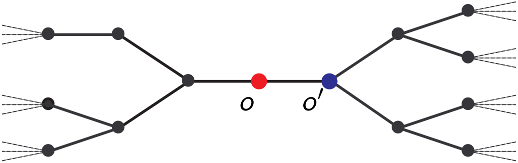

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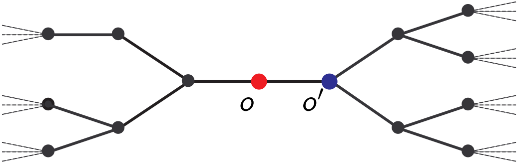

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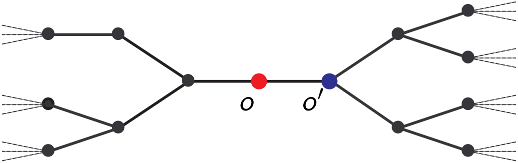

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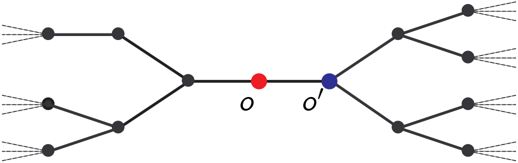

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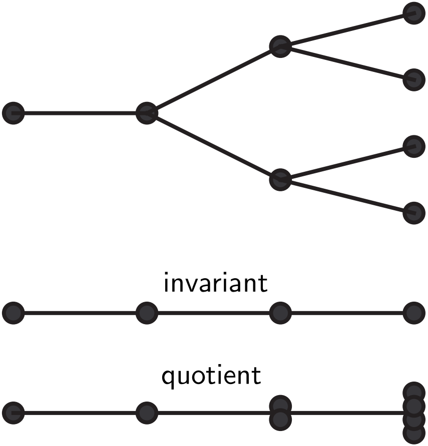
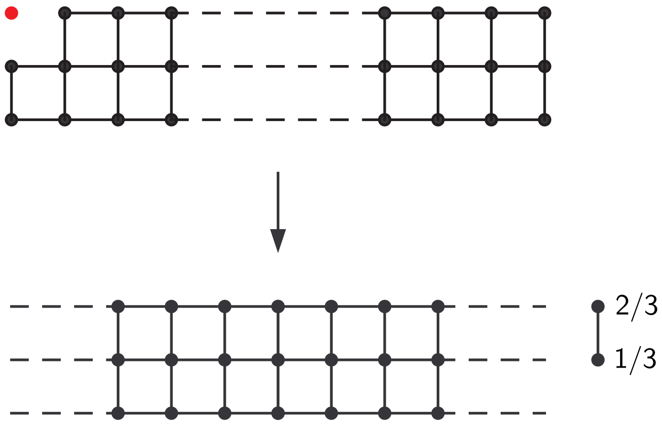


Invariant and quotient measures on the equivalence class of a finite graph

Return



Return