Two dynamical extensions of the Nielsen-Thurston theory of surface diffeomorphsims

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University of Geneva

Warwick, 16 April 2012

Surface diffeomorphisms

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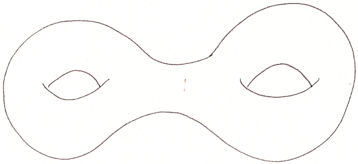

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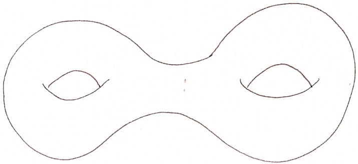

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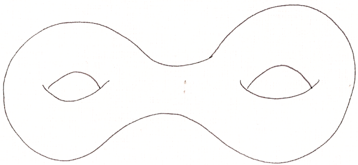

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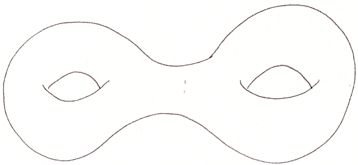

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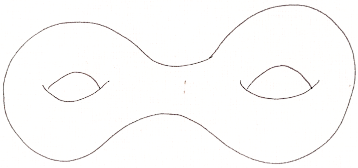

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- Nielsen, 1927-1945 used lifts to $\mathbb{H}^2$and $\partial\mathbb{H}^2$.
- Thurston, 1976, used Teichmüller theory

The spectral theorem

Let

$$\mathcal{S} = \text{isotopy classes of simple closed curves on M}$$

and for a Riemannian metric ρ, $l_\rho(\beta) := \inf_{\beta' \sim \beta} \mathit{length}_\rho(\beta')$.

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Theorem 5 in Thurston's seminal 1976 preprint:

Theorem

For any diffeomorphism f of M, there is a finite set $1 \leq \lambda_1 < \lambda_2 < ... < \lambda_K$ of algebraic integers such that for any $\alpha \in \mathcal{S}$ there is a λ_i such that for any Riemannian metric ρ,

$$\lim_{n \to \infty} l_\rho(f^n \alpha)^{1/n} = \lambda_i.$$

The map f is isotopic to a pseudo-Anosov map iff $K = 1$ and $\lambda_1 > 1$.

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Compare with matrices $\lim_{n \to \infty} \|A^n v\|^{1/n} = |\lambda_v|$.

Part I

Statements of new results.

Random spectral theorem

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- Random walks, iid, Kaimanovich-Masur 1996,
- Duchin 2003,
- K.-Margulis 2005
- Rivin, Kowalski, Maher, 2008-2012

Holomorphic self-maps

Let $\mathcal{T}(\mathrm{M})$ be the Teichmüller space. Theorem of Royden, 1971:

$$\mathbb{C}\text{−Aut}(\mathcal{T}) \cong \mathrm{MCG} := \mathrm{Diff}_+(\mathrm{M})/\mathrm{Diff}_0(\mathrm{M}).$$

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Let $f : \mathcal{T}(M) \to \mathcal{T}(M)$ be a holomorphic map. Then there is a $\lambda \geq 1$ and a point P in the Gardiner-Masur compactification such that for any $x \in \mathcal{T}$ and curve $\beta \in \mathcal{S}$ with $E_P(\beta) > 0$

$$Ext_{f^n x}(\beta)^{1/n} \to \lambda.$$

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Examples of holomorphic self-maps of $\mathcal{T}$:

- Thurston's skinning map in three-dimensional topology
- Thurston's pull-back map in complex dynamics

Wolff-Denjoy theorem

Theorem

Let $f : \mathcal{T} \to \mathcal{T}$ be holomorphic. Then either every orbit is bounded, or every orbit leaves every compact set and there are associated points P in the Gardiner-Masur boundary . If P is uniquely ergodic, then every orbit converges to P and for some $\lambda \geq 1$ and any $x \in \mathcal{T}(M)$

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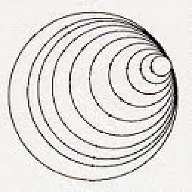

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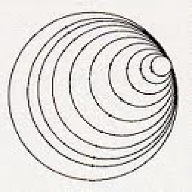

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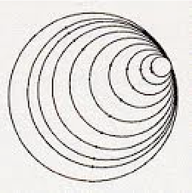

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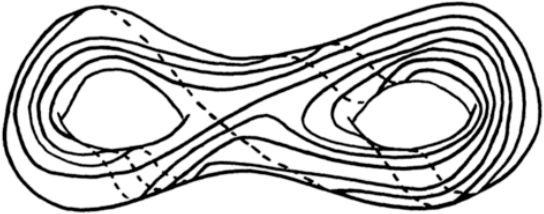
Question: In the bounded orbit case does f always have a fixed point in $\mathcal{T}$?

Part II

Definitions and Proofs

Simple closed curves

Let $\mathcal{S}$ denote the isotopy classes of simple closed curves on M not isotopically trivial.



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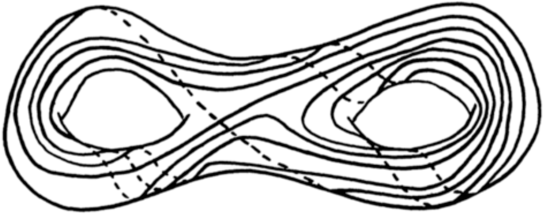

FIGURE 1. A typical simple closed curve on a surface is complicated, from the point of view of someone tracing out the curve.

Can embedd $\mathcal{S}$ into $P(\mathbb{R}^{\mathcal{S}}_{\geq 0})$ via the intersection number

$$\alpha \mapsto i(\alpha, \cdot).$$

projectivized. The closure $\mathcal{PMF}$ is homeomorphic to a sphere of dim 6g-7 and points are projective measured foliations.

Thurston compactification

Thurston also showed that embedding $\mathcal{T}(\mathrm{M})$ into $P(\mathbb{R}^{\mathcal{S}}_{\geq 0})$ via

$$x \mapsto l_x(\cdot)$$

projectivized, taking closure gives a ball, with boundary $\mathcal{PMF}$:

$$\mathcal{T}(M) \hookrightarrow B^{6g-6} \quad \mathcal{PMF} = \partial \mathrm{B}^{6\mathrm{g}-6}.$$

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"Using the theory of foliations of surfaces" $\Rightarrow$ "spectral theorem".

Teichmüller and Thurston metrics

Kerckhoff's formula for *Teichmüller* distance:

$$d(x,y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{Ext_y(\alpha)}{Ext_x(\alpha)},$$

where $Ext_x(\alpha)$ is the extremal length:

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Let

$$E_x(\alpha) = \frac{Ext_x(\alpha)^{1/2}}{K_x^{1/2}},$$

where K_x is the q-c dilation of the Teichmüller map from x_0 to x.
Miyachi noted that E_x extends continuously to a function defined on the Gardiner-Masur compactification $\overline{\mathcal{T}}^{GM}$ of $\mathcal{T}$.

Proof of Theorem 2

Let $f : \mathcal{T}(M) \to \mathcal{T}(M)$ be holomorphic. By Royden, $d =$Kobayashi, hence f is $1 - Lip$. Define

$$l = \lim_{n\to\infty} \frac{1}{n} d(f^n x_0, x_0).$$

For any point $P \in \overline{\mathcal{T}}^{GM}$ define following Liu and Su

$$h_P(x) = \log \sup_{\beta} \frac{E_P(\beta)}{Ext_x(\beta)^{1/2}} - \log \sup_{\alpha} \frac{E_P(\alpha)}{Ext_{x_0}(\alpha)^{1/2}}.$$

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Given a sequence $\epsilon_i \searrow 0$ we set $b_i(n) = d(f^n x_0, x_0) - (l - \epsilon_i)n$. Since these numbers are unbounded, we can find a subsequence such that $b_i(n_i) > b_i(m)$ for any $m < n_i$ and by sequential compactness we may moreover assume that $f^{n_i}(x_0) \to P \in \overline{\mathcal{T}}^{GM}$.

Proof of Theorem 2, II

By a result of Liu and Su identifying the horoboundary compactification of $(\mathcal{T}, d)$ with the Gardiner-Masur compactification we have for any $k \geq 1$ that

$$h_P(f^k x_0) = \lim_{i \to \infty} d(f^k x_0, f^{n_i} x_0) - d(x_0, f^{n_i} x_0)$$

$$\leq \liminf_{i \to \infty} d(x_0, f^{n_i - k} x_0) - d(x_0, f^{n_i} x_0)$$

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$$\leq \liminf_{i\to\infty} b_i(n_i - k) + (l - \epsilon_i)(n_i - k) - b_i(n_i) - (l - \epsilon_i)n_i$$

$$\leq \liminf_{i\to\infty} -(l - \epsilon_i)k = -lk.$$

Proof of Theorem 2, conclusion

This means in terms of extremal lengths that for any $\beta \in \mathcal{S}$

$$Ext_{f^k x_0}(\beta) \geq E_P(\beta)^2 \left(\sup_\alpha \frac{E_P(\alpha)}{Ext_{x_0}(\alpha)^{1/2}} \right)^{-2} e^{2lk}.$$

On the other hand, in view of Kerchoff's formula one has an estimate from above:

$$e^{2d(f^k x_0, x_0)} = \sup_\alpha \frac{Ext_{f^k x_0}(\alpha)}{Ext_{x_0}(\alpha)} \geq \frac{Ext_{f^k x_0}(\beta)}{Ext_{x_0}(\beta)}.$$

In particular, provided $E_P(\beta) > 0$, the two estimates imply that

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(I'm leaving out the additional arguments required for the weak Wolff-Denjoy analog - uniquely ergodic.)

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$$\lim_{n \to \infty} l_\rho(f_n \alpha)^{1/n} = \lambda$$

for any $\alpha \in \mathcal{S}$ *such that* $i(\mu, \alpha) > 0$.

Proof of Theorem 1

Let $T : (\Omega, \mu) \to (\Omega, \mu)$ be an ergodic m.p.t., $\mu(\Omega) = 1$. Assume that $g : \Omega \to \mathit{Diff}^+(M) \twoheadrightarrow MCG(M)$ is measurable and

$$Z_n(\omega) := g(\omega)g(T\omega)...g(T^{n-1}\omega).$$

Notice here that we have shifted the order, so that in the terminology of the theorem, $f_n = Z_n^{-1}$ and the $g_i = \left(g(T^{i-1}\omega)\right)^{-1}$.

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$$\int_\Omega L(g(\omega)x_0, x_0) + L(x_0, g(\omega)x_0)d\mu(\omega) < \infty,$$

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$$L(Z_{n+m}(\omega)x_0, x_0) \leq L(Z_n(\omega)Z_m(T^n\omega)x_0, Z_n(\omega)x_0)) + L(Z_n(\omega)x_0, x_0)$$

$$= L(Z_m(T^n\omega)x_0, x_0) + L(Z_n(\omega)x_0, x_0).$$

Kingman $\Rightarrow l := \lim_{n\to\infty} \frac{1}{n} L(Z_n(\omega)x_0, x_0)$.

Proof of Theorem 1, cont

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$$h_\mu(x) = \log \sup_\alpha \frac{i(\mu,\alpha)}{l_x(\alpha)} - \log \sup_\beta \frac{i(\mu,\beta)}{l_{x_0}(\beta)},$$

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(well-defined) and for $x_n \to \mu$

$$h_\mu(x) = \lim_{n\to\infty} L(x, x_n) - L(x_0, x_n).$$

Following work of K. & Ledrappier: for $g \in MCG$ and h as above let $F(g, h) = -h(g^{-1}x_0)$. We note the following cocycle property:

$$F(g_1, g_2 h) + F(g_2, h) = -(g_2 \cdot h)(g_1^{-1}x_0) - h(g_2^{-1}x_0)$$

$$= -h(g_2^{-1}g_1^{-1}x_0) + h(g_2^{-1}x_0) - h(g_2^{-1}x_0) = F(g_1 g_2, h).$$

Note that moreover

$$L(gx_0, x_0) = -L(g^{-1}x_0, g^{-1}x_0) + L(x_0, g^{-1}x_0) = \max_{h\in H} F(g, h),$$

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$$F(Z_n(\omega)^{-1}, h) = \sum_{i=0}^{n-1} \bar{F}(\bar{T}^i(\omega, h)).$$

Moreover, we have
$\left|F(g^{\cdot 1}(\omega), h)\right| \leq \max\left\{L(x_0, g(\omega)x_0, L(g(\omega)x_0, x_0)\right\}$ so F is integrable.

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The proof now runs as in K.-Ledrappier, that is, construct a special measure that accounts for drift and projects to μ.. Birkhoff's ergodic theorem and a selecting measurable section. We get that for a.e. ω there is an $h = h^\omega$ such that

$$\lim_{n\to\infty} -\frac{1}{n} h(Z_n x_0) = l.$$

Proof of Theorem 1, conclusion

Letting $C_\mu^{-1} = \sup \frac{i(\mu,\beta)}{l_{x_0}(\beta)}$ we then obtain

$$\sup_\alpha \frac{i(\mu,\alpha)}{l_{Z_n x_0}(\alpha)} \leq C_\mu^{-1} e^{-(l-\epsilon)n},$$

which leads to that for every α we have

$$l_{Z_n x_0}(\alpha) \geq C_\mu i(\mu,\alpha) e^{(l-\epsilon)n}.$$

Note: more precise than the theorem!

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Part III

Concluding remarks and questions

Random products case

Questions

- Ray approximation in the Teichmüller metric:

$$\frac{1}{n} d(Z_n x_0, \gamma(l \cdot n)) \to 0.$$

- Version with several foliations μ and several λs.
- Central limit theorem
- Behaviour of $i(f_n \alpha, \beta)$
- Study of surface bundles

Holomorphic self-maps

- Is there a more refined Wolff-Denjoy theorem / extended Nielsen-Thurston classification ?
- Fixed point?
- Tighter relations to Thurston's pull-back map and Thurston obstruction

Thanks

Thanks!