Ergodic theorems beyond amenable groups

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Joint work with Lewis Bowen

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- (X, μ) an ergodic probability measure preserving action of Γ.
- Consider the uniform averages λ_t supported on Γ_t.
- Basic problem : Do these averages

$$\lambda_t f(x) = \frac{1}{|\Gamma_t|} \sum_{\gamma \in \Gamma_t} f(\gamma^{-1} x)$$

converge, for a given function f on X ? If so, what is their limit ?

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$$\lim_{n\to\infty} \frac{|KF_n \Delta F_n|}{|F_n|} = 0 \text{ for any finite } K \subset G\,,$$

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2. The doubling, and more generally, regularity condition :

$$|\cup_{m\le n} F_m^{-1} F_n| \le C_d |F_n|$$

implies a maximal inequality for the convolution operators λ_n define on $\ell^1(\Gamma)$ (Wiener, Calderon, Tempelman....)

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For non-amenable groups the volume growth is exponential, regularity fail, there are no Følner sets, and no transference.

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• A Borel map $\phi : B \to B$ is an inner automorphism of $\mathcal{R}$ if it is invertible with Borel inverse and its graph is contained in $\mathcal{R}$.

• If ν is $\mathcal{R}$-invariant then $\phi_* \nu = \nu$ for every ϕ in the group of inner automorpshims $\mathrm{Inn}(\mathcal{R})$.

Asymptotic invariance, Folner sets and doubling

- A set $\Phi \subset \mathrm{Inn}(\mathcal{R})$ generates $\mathcal{R}$ if for almost every pair $(b_1, b_2) \in \mathcal{R}$ (w.r.t. $\nu \times c$), there exists ϕ in the group generated by Φ such that $\phi(b_1) = b_2$.

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• Let $\mathcal{F} = \{\mathcal{F}_n(b)\}_{n=1}^{\infty}$, with $\mathcal{F}_n(b)$ a finite subset of the equivalence class of b. Furthermore $\{(b, b') : \ b' \in \mathcal{F}_n(b)\} \subset B \times B$ is a Borel subset of $\mathcal{R}$. We also assume $b \in \mathcal{F}_n(b)$ for every b and n.

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1. $\mathcal{F}$ is asymptotically invariant (or *Følner*) with respect to ν if there exists a countable set $\Phi \subset \mathrm{Inn}(\mathcal{R}(B))$ which generates $\mathcal{R}(B)$ and

$$\lim_{n\to\infty} \frac{|\mathcal{F}_n(b) \Delta \phi(\mathcal{F}_n(b))|}{|\mathcal{F}_n(b)|} = 0$$

for all $\phi \in \Phi$, ν-a.e. $b \in B$.

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for all $\phi \in \Phi$, ν-a.e. $b \in B$.
2. $\mathcal{F}$ satisfies the regularity condition with respect to ν if there is a constant $C_d > 0$ such that for ν-a.e. $b \in B$ and every $n \in \mathbb{N}$
$$\left| \bigcup \{\mathcal{F}_m(b') : \ m \leq n, \mathcal{F}_m(b') \cap \mathcal{F}_n(b) \neq \emptyset\} \right| \leq C_d |\mathcal{F}_n(b)|.$$

Pointwise ergodic theorem in L^1

- For a function f on B, consider the averaging operators $\mathbb{A}_n[\mathcal{F}; f]$

$$\mathbb{A}_n[\mathcal{F}; f](b) := \frac{1}{|\mathcal{F}_n(b)|} \sum_{b' \in \mathcal{F}_n(b)} f(b').$$

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Pointwise ergodic theorem.
If $\mathcal{F}$ is asymptotically invariant and satisfies the regularity condition then $\mathcal{F}$ is a pointwise ergodic sequence in L^1. i.e., for every $f \in L^1(B, \nu)$, $\mathbb{A}_n[\mathcal{F}; f]$ converges pointwise a.e. and in L^1-norm to $\mathbb{E}[f|\mathcal{R}]$ as $n \to \infty$.

Examples

• This result generalizes the classical ergodic theorem : Let $\mathcal{R} = \mathcal{O}_G$ be the orbit equivalence relation of a m.p. action of an amenable group G on (B, ν), $\{F_n\}_{n=1}^{\infty}$ regular Folner subsets in G. Then $\mathcal{F}_n(b) = \{gb\,;\, g \in F_n\}$ is asymptotically invariant and regular for the equivalence relation.

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• Hyperfiniteness. $\mathcal{R}$ is hyperfinite if $\mathcal{R} = \cup_n \mathcal{R}_n$ in the increasing union of subequivalence relations with finite classes. For $b \in B$, $\mathcal{R}_n(b)$ (the $\mathcal{R}_n$-equivalence class of b) form doubling Folner sequences for the relation $\mathcal{R}$ provided the union of automorphisms groups $Inn(\mathcal{R}_n)$ generate $\mathcal{R}$. In that case, $\mathcal{R}_n(b)$ satisfy

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• Extreme Besicovich property: If $\mathcal{F}_n(b)$ intersects $\mathcal{F}_m(b')$, then one of the two sets is contained in the other !

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Let B be any Borel space which is non-singular under the equivalence relation $\mathcal{R}$, with Radon-Nikodym derivative $\delta(b', b)$.

The ratio ergodic operators $\mathcal{Q}_{2n}[\cdot, \cdot]$ are defined, given $U, V \in L^1(B)$ with $V > 0$, by

$$\mathcal{Q}_{2n}[U, V](b) = \frac{\sum_{b' \in \mathcal{F}_n(b)} U(b')\delta(b', b)}{\sum_{b' \in \mathcal{F}_n(b)} V(b')\delta(b', b)} .$$

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Ratio ergodic theorem. For any $U, V \in L^1(B)$ with $\mathbb{E}[V|\mathcal{R}] > 0$, the sequence $\{\mathcal{Q}_{2n}[U, V]\}_{n=1}^{\infty}$ converges pointwise a.e. to the limit

$$\frac{\mathbb{E}[U|\mathcal{R}](b)}{\mathbb{E}[V|\mathcal{R}](b)} .$$

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• In the case of $\mathbb{Z}^d$-actions for $d > 1$ the ratio ergodic theorem is of recent vintage (Feldman 2007, Hochman 2009). It was shown by Hochman (2009) that the Besicovich property is necessary and sufficient for the validity of the ratio ergodic theorem in this case.

From amenable groups to amenable actions

• Non-amenable groups do not have Følner sets, but they do have amenable actions (in the sense of Zimmer) where the orbit equivalence relation does have a sequence of Følner sets (Connes-Feldman-Weiss 1980).

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• In particular, the action of a non-amenable group G on its Poisson boundary $Y = \partial G$ (Furstenberg 1963) is an amenable action (Zimmer 1978), and hence so is $X \times \partial G$. Note however that the orbit relation does not preserve the measure in the last three case.

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- Let us proceed to consider a natural example of a non-amenable group G where the amenable equivalence relation on $X \times \partial G$ has a natural subrelation with an invariant measure and natural Folner sets with the extreme Besicovich property.

The free group and its boundary

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- The boundary $\partial\mathbb{F}$ is the set of all geodesic rays emanating from the origin. Equivalently, the set of all sequences $\xi = (\xi_1, \xi_2, \ldots) \in \mathrm{S}^{\mathbb{N}}$ such that $\xi_{i+1} \neq \xi_i^{-1}$ for all $i \geq 1$. Thus $\partial\mathbb{F}$ is a subshift of finite type.

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• A metric d_∂ on $\partial\mathbb{F}$ is defined by $d_\partial\big((\xi_1, \xi_2, \ldots), (t_1, t_2, \ldots)\big) = \frac{1}{n}$ where n is the largest natural number such that $\xi_i = t_i$ for all $i < n$.

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- The probability measure ν on $\partial\mathbb{F}$ is the Markov measure satisfying for every finite sequence $t_1, \ldots, t_n$ with $t_{i+1} \neq t_i^{-1}$ for $1 \leq i < n$,

$$\nu\Big(\big\{(\xi_1, \xi_2, \ldots) \in \partial\mathbb{F} : \ \xi_i = t_i\,,\ 1 \leq i \leq n\big\}\Big) := (2r-1)^{-n+1}(2r)^{-1}.$$

Horofunctions and horospheres

- There is a natural action of $\mathbb{F}$ on $\partial\mathbb{F}$ by

$$(t_1 \cdots t_n)\xi := (t_1, \ldots, t_{n-k}, \xi_{k+1}, \xi_{k+2}, \ldots)$$

where $t_1, \ldots, t_n \in \mathrm{S}$, $t_1 \cdots t_n$ is in reduced form and k is the largest number $\leq n$ such that $\xi_i^{-1} = t_{n+1-i}$ for all $i \leq k$.

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• For $\xi \in \partial\mathbb{F}$ as above, define the horofunction $h_\xi : \mathbb{F} \to \mathbb{Z}$ by

$$h_\xi(g) := -\log_{2r-1}\left(\frac{d\nu \circ g^{-1}}{d\nu}(\xi)\right).$$

- For example, if $g = \xi_1 \cdots \xi_n$ then $h_\xi(g) = -n$. More generally, if $g = \xi_1 \cdots \xi_n t_1 \cdots t_m$ is in reduced form and $t_1 \neq \xi_{n+1}$ then $h_\xi(g) = m - n$. So $h_\xi(g) = 0$ iff $n = m$.

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• A horosphere is any level set of a horofunction. Let H_ξ denote the horosphere $H_\xi := h_\xi^{-1}(0)$. Then

$$H_\xi = \left\{ g \in \mathbb{F} : \frac{d\nu \circ g^{-1}}{d\nu}(\xi) = 1 \right\}.$$

• For example, if $g = \xi_1 \cdots \xi_n$ then $h_\xi(g) = -n$. More generally, if $g = \xi_1 \cdots \xi_n t_1 \cdots t_m$ is in reduced form and $t_1 \neq \xi_{n+1}$ then $h_\xi(g) = m - n$. So $h_\xi(g) = 0$ iff $n = m$.

• Alternatively, if $s_n = \xi_1 \xi_2 \cdots \xi_n \to \xi$ then

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• If $\xi = (\xi_1, \xi_2, \ldots)$ then $g \in H_\xi$ if and only if the reduced form of g is $g = \xi_1 \xi_2 \cdots \xi_n t_1 \cdots t_n$ for some $t_1, \ldots, t_n \in$ S such that $\xi_{n+1} \neq t_1$ (so $g^{-1}\xi = (t_n^{-1}, \ldots, t_1^{-1}, \xi_{n+1}, \ldots)$). H_ξ is called the *horosphere* passing through the identity e associated to ξ.

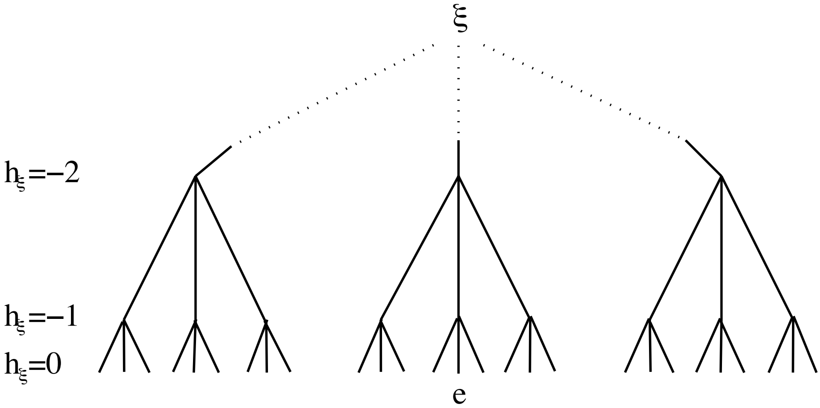

Figure: The “upper half space” model of the rank 2 free group.

The boundary action and associated relation

- The group $\mathbb{F}$ acts on horofunctions by $g \cdot h_\xi(f) = h_\xi(g^{-1}f)$ for any $g, f \in \mathbb{F}, \xi \in \partial\mathbb{F}$, and thus $\mathbb{F}$ acts on horospheres by

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• Let $\mathcal{R}_{\partial\mathbb{F}}$ be the horospherical equivalence relation on $\partial\mathbb{F}$ where $\xi \sim \eta$ if and only if there is a $g \in \mathbb{F}$ such that $g\xi = \eta$ and $\frac{d\nu \circ g}{d\nu}(\xi) = 1$.

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• In other words, $\eta = g\xi$ where $g^{-1} \in H_\xi$. Symmetry and transitivity follow from the cocycle equation.

Finite order automorphisms

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• Thus it is clear that $\mathcal{R}_{\partial\mathbb{F}}$ is an increasing union of finite equivalence relations $\mathcal{R}_n$. Just define ξ and ξ' to be $\mathcal{R}_n$ equivalent if they coincide from the n-th place onwards. In fact, the sets $\mathcal{F}_n(\xi) = \mathcal{R}_n(\xi)$ are Folner sets for $\mathcal{R}$.

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• To show that, define finite order automorphisms of $\mathcal{R}$, declaring bijections $\phi : \partial\mathbb{F} \to \partial\mathbb{F}$ to have *order n* if for any two boundary points $\xi, \xi' \in \partial\mathbb{F}$ with identical first *n* coordinates, $\phi(\xi) = \phi(\xi')$.

Proposition.

• For any $(\xi, \xi') \in \mathcal{R}_{\partial\mathbb{F}}$, there exists a map $\phi \in \mathrm{Inn}(\mathcal{R}_{\partial\mathbb{F}})$ such that $\phi(\xi) = \xi'$ and ϕ has order n for some $n < \infty$. Thus the set of finite order inner automorphisms of $\mathcal{R}_{\partial\mathbb{F}}$ is a generating set for the equivalence relation $\mathcal{R}_{\partial\mathbb{F}}$.

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• Furthermore, the sets $\mathcal{R}_n(\xi)$ are asymptotically invariant under finite-order automorphisms, and constitute Folner sets with the extreme Besicovich property.

Geometric interpretation

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• The horospherical ball $\mathcal{B}_{2n}(\xi)$ coincide with the equivalence class of ξ under the finite equivalence relation $\mathcal{R}_n$.

The amenable equivalence relation associated with a measure-preserving ergodic action

• Let $\mathbb{F}$ act on (X, λ) by m.p.t. , and define an equivalence relation $\mathcal{R}_{X\times\partial\mathbb{F}}$ on $X \times \partial\mathbb{F}$, with (x, ξ) equivalent to (x', ξ') if there exists a $g^{-1} \in H_\xi$ such that $gx = x'$ and $g\xi = \xi'$.

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• For $f \in L^1(X \times \partial\mathbb{F})$, denote by $\mathbb{E}[f|\mathcal{R}_{X \times \partial\mathbb{F}}]$ the conditional expectation of f on the σ-algebra of $\mathcal{R}_{X \times \partial\mathbb{F}}$-invariant sets.

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$$\tilde{\mathcal{B}}_n(x, \xi) := \{(gx, g\xi) \in X \times \partial\mathbb{F} : \; g^{-1} \in H_\xi, \; |g| \le n\}.$$

For $n \ge 0$ and $(x, \xi) \in X \times \partial\mathbb{F}$.

Main convergence result for horospherical ball averages

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Then for any $f \in L^1(X \times \partial\mathbb{F})$, the sequence $\{\mathbb{A}_{2n}[\tilde{\mathcal{B}}; f]\}_{n=1}^{\infty}$ converges pointwise a.e. and in L^1 norm to $\mathbb{E}[f|\mathcal{R}_{X \times \partial\mathbb{F}}]$.

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Proof : $\tilde{\mathcal{B}}_n(x, \xi)$ is asymptotically invariant and extremely Besicovich for $\mathcal{R}_{X \times \partial\mathbb{F}}$ because $\mathcal{B}_n(\xi)$ is for $\mathcal{R}_{\partial\mathbb{F}}$.

Ergodic theorems for free groups

• Clearly, we can view a function f on X as a function on $X \times \partial\mathbb{F}$, apply the averages $\mathbb{A}_{2n}[\tilde{\mathcal{B}}; f](x, \xi)$, and then integrate as ξ ranges over the boundary $\partial\mathbb{F}$, w.r.t *any* continuous probability density η.

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• This gives a new proof and generalizes the pointwise ergodic theorem for free groups in L^p, $p > 1$ (N 1994, N-Stein 1994).

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4. Show that the horospherical relation $\mathcal{R}_{\partial\Gamma}$ has an asymptotically invariant doubling or regular sequence,
5. Deduce a pointwise ergodic theorem in L^1 for the horospherical ball averages defined in the equivalence relation $\mathcal{R}_{X\times\partial\Gamma}$,

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Let Γ act discretely and co-compactly on a $CAT(-1)$ space M. Fix $m \in M$ with trivial stabilizer, and let $\Gamma_t = \Gamma \cdot m \cap B_t(m)$ be the discrete ball, and $A_t = \Gamma_{t+c} \setminus \Gamma_t$ be the discrete annuli of width c.

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von-Neumann and Birkhoff theorems for hyperbolic groups. There exists a sequence of weights λ_t supported on $A_t \subset \Gamma$, which forms a mean and pointwise ergodic sequence in L^p, $1 < p < \infty$.

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• It is an indication of the difficulty of this problem that in the convergence results just quoted, the limit is not identified.

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• It is also known for general hyperbolic group provided the action has strong mixing properties (Fujiwara-N 1998.)

Ratio ergodic theorems beyond Abelian groups : some surprises

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1. the operators $\pi_X(\lambda_t)$ may fail to converge even in the case where λ_t are ball averages w.r.t. a word metric and the action is an isometric action on a compact group preserving Haar measure.

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5. The limit measure ν may depend non-trivially on the family of sets B_t which are taken as the support of the measures β_t, even when the action is isometric.

6. Under a spectral gap assumption, the operators $\pi_X(\lambda_t)$ converge with a *uniform rate of convergence*, valid for almost all points. In isometric actions, the rate can be uniform over all points (for Hölder functions). This can happen in compact space and also in non-compact spaces, and it implies of course that equidistribution of orbits points, or their ratios, takes place at a uniform rate.

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Facts 1,2 above are implicit in Arnold-Krylov (1962) and Guivarc'h (1968), and noted explicitly by Bewley (1970). Fact 3, 4, 5 above were exhibited by Ledrappier 1999 and Ledrappier-Pollicott 2003 for lattices in $SL_2(\mathbb{R})$ acting on $\mathbb{R}^2$. Other contributions are by Maucourant, Oh. A major generalization is due to Gorodnik-Weiss 2006.