Principal Algebraic Group Actions

Klaus Schmidt

Vienna

Warwick, April 2012

Principal Actions

Let Γ be a countably infinite discrete group. An *algebraic* Γ*-action* is a homomorphism $\alpha\colon \gamma \mapsto \alpha^\gamma$ from Γ to the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

Let Γ be a countably infinite discrete group. An *algebraic Γ-action* is a homomorphism $\alpha\colon \gamma \mapsto \alpha^\gamma$ from Γ to the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

Example: Let $X = \mathbb{T}^\Gamma$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let λ be the left shift-action on X, defined by $(\lambda^\gamma x)_\theta = x_{\gamma^{-1}\theta}$ for every $x = (x_\theta)_{\theta\in\Gamma} \in X$.
The right shift-action $\gamma \mapsto \rho^\gamma$ of Γ on X is given by $(\rho^\gamma x)_\theta = x_{\theta\gamma}$. The actions λ and ρ commute.

Let Γ be a countably infinite discrete group. An *algebraic Γ-action* is a homomorphism $\alpha\colon \gamma \mapsto \alpha^\gamma$ from Γ to the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

Example: Let $X = \mathbb{T}^\Gamma$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let λ be the left shift-action on X, defined by $(\lambda^\gamma x)_\theta = x_{\gamma^{-1}\theta}$ for every $x = (x_\theta)_{\theta\in\Gamma} \in X$.
The right shift-action $\gamma \mapsto \rho^\gamma$ of Γ on X is given by $(\rho^\gamma x)_\theta = x_{\theta\gamma}$. The actions λ and ρ commute.

Let $f = \sum_{\gamma\in\Gamma} f_\gamma \gamma \in \mathbb{Z}[\Gamma]$. Define a group homomorphism $\rho^f\colon X \longrightarrow X$ by $\rho^f = \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma$ (this is effectively right convolution by $f^* = \sum_{\gamma\in\Gamma} f_\gamma \gamma^{-1}$). Then ρ^f commutes with λ.

Principal Actions

Let Γ be a countably infinite discrete group. An *algebraic Γ-action* is a homomorphism $\alpha\colon \gamma \mapsto \alpha^\gamma$ from Γ to the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

Example: Let $X = \mathbb{T}^\Gamma$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let λ be the left shift-action on X, defined by $(\lambda^\gamma x)_\theta = x_{\gamma^{-1}\theta}$ for every $x = (x_\theta)_{\theta\in\Gamma} \in X$.
The right shift-action $\gamma \mapsto \rho^\gamma$ of Γ on X is given by $(\rho^\gamma x)_\theta = x_{\theta\gamma}$. The actions λ and ρ commute.

Let $f = \sum_{\gamma\in\Gamma} f_\gamma \gamma \in \mathbb{Z}[\Gamma]$. Define a group homomorphism $\rho^f : X \longrightarrow X$ by $\rho^f = \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma$ (this is effectively right convolution by $f^* = \sum_{\gamma\in\Gamma} f_\gamma \gamma^{-1}$). Then ρ^f commutes with λ.

Let $X_f = \ker(\rho^f)$, $\alpha_f = \lambda|_{X_f}$. This is the *principal Γ-action defined by f*.

Principal Actions Of $\mathbb{Z}$

For $\Gamma = \mathbb{Z}$, every $f = \sum_{n \in \mathbb{Z}} f_n n \in \mathbb{Z}[\mathbb{Z}]$ can be viewed as the Laurent polynomial $\sum_{n \in \mathbb{Z}} f_n u^n$. After multiplication by a power of u (which doesn't change X_f) we may assume that $f = \sum_{k=0}^{n} f_k u^k$ with nonzero f_0 and f_n. Then

$$X_f = \{x = (x_m) \in \mathbb{T}^{\mathbb{Z}} : f_0 x_m + \cdots + f_n x_{m+n} = 0 \text{ for all } m \in \mathbb{Z}\}.$$

For $\Gamma = \mathbb{Z}$, every $f = \sum_{n\in\mathbb{Z}} f_n n \in \mathbb{Z}[\mathbb{Z}]$ can be viewed as the Laurent polynomial $\sum_{n\in\mathbb{Z}} f_n u^n$. After multiplication by a power of u (which doesn't change X_f) we may assume that $f = \sum_{k=0}^{n} f_k u^k$ with nonzero f_0 and f_n. Then

$$X_f = \{x = (x_m) \in \mathbb{T}^{\mathbb{Z}} : f_0 x_m + \cdots + f_n x_{m+n} = 0 \text{ for all } m \in \mathbb{Z}\}.$$

If $f_n = |f_0| = 1$, α_f is (conjugate to) the toral automorphism given by the companion matrix

$$A_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -f_0 & -f_1 & -f_2 & \cdots & -f_{n-1} \end{pmatrix}$$

Principal Actions Of $\mathbb{Z}$

For $\Gamma = \mathbb{Z}$, every $f = \sum_{n\in\mathbb{Z}} f_n n \in \mathbb{Z}[\mathbb{Z}]$ can be viewed as the Laurent polynomial $\sum_{n\in\mathbb{Z}} f_n u^n$. After multiplication by a power of u (which doesn't change X_f) we may assume that $f = \sum_{k=0}^{n} f_k u^k$ with nonzero f_0 and f_n. Then

$$X_f = \{x = (x_m) \in \mathbb{T}^{\mathbb{Z}} : f_0 x_m + \cdots + f_n x_{m+n} = 0 \text{ for all } m \in \mathbb{Z}\}.$$

If $f_n = |f_0| = 1$, α_f is (conjugate to) the toral automorphism given by the companion matrix

$$A_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -f_0 & -f_1 & -f_2 & \cdots & -f_{n-1} \end{pmatrix}$$

Dynamical properties like ergodicity or expansiveness are determined by the roots of f, and the entropy of α_f is given by

$$h(\alpha_f) = \log|f_n| + \sum_{\{c : f(c)=0\}} \log^+|c| = \int_0^1 \log|f(e^{2\pi i s})|\, ds.$$

Other Examples Of Principal $\mathbb{Z}$-actions

- $f = u - 2$. The automorphism α_f factors onto multiplication by 2 on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. α_f is expansive and has entropy $\log 2$.

Other Examples Of Principal $\mathbb{Z}$-actions

- $f = u - 2$. The automorphism α_f factors onto multiplication by 2 on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. α_f is expansive and has entropy $\log 2$.
- $f = 2u - 3$. The automorphism α_f is 'multiplication by $3/2$' on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (it is the shift on the space of sequences $(x_n)_{n\in\mathbb{Z}} \in \mathbb{T}^{\mathbb{Z}}$ which satisfy $2x_{n+1} = 3x_n$ for every n). α_f is expansive and has entropy $\log 3$.

Other Examples Of Principal $\mathbb{Z}$-actions

- $f = u - 2$. The automorphism α_f factors onto multiplication by 2 on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. α_f is expansive and has entropy $\log 2$.
- $f = 2u - 3$. The automorphism α_f is 'multiplication by $3/2$' on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (it is the shift on the space of sequences $(x_n)_{n \in \mathbb{Z}} \in \mathbb{T}^{\mathbb{Z}}$ which satisfy $2x_{n+1} = 3x_n$ for every n). α_f is expansive and has entropy $\log 3$.
- f is cyclotomic (e.g., $f = u^2 + u + 1$). In this case α_f is of finite order and hence nonergodic with entropy 0. For $f = u^2 + u + 1$, $\alpha_f^3 = \mathrm{Id}$.

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables:
$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{n} = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$.

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables:
$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{n} = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$.

- For $d \geq 2$, α_f is always ergodic.

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables:
$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{n} = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$.

- For $d \geq 2$, α_f is always ergodic.
- α_f is expansive if and only if $V_{\mathbb{C}}(f) = \{\mathbf{c} \in (\mathbb{C} \smallsetminus \{0\})^d : f(\mathbf{c}) = 0\}$ contains no points whose coordinates all have absolute value 1.

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables:
$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{n} = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$.

- For $d \geq 2$, α_f is always ergodic.
- α_f is expansive if and only if $V_{\mathbb{C}}(f) = \{\mathbf{c} \in (\mathbb{C} \smallsetminus \{0\})^d : f(\mathbf{c}) = 0\}$ contains no points whose coordinates all have absolute value 1.
- The entropy $h(\alpha_f)$ is given by the *logarithmic Mahler measure* of f:

$$h(\alpha_f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d})| \, dt_1 \cdots dt_d$$

(Lind-S-Ward, 1990).

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables: $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{n} = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$.

- For $d \geq 2$, α_f is always ergodic.
- α_f is expansive if and only if $V_{\mathbb{C}}(f) = \{\mathbf{c} \in (\mathbb{C} \smallsetminus \{0\})^d : f(\mathbf{c}) = 0\}$ contains no points whose coordinates all have absolute value 1.
- The entropy $h(\alpha_f)$ is given by the *logarithmic Mahler measure* of f:

$$h(\alpha_f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})| \, dt_1 \cdots dt_d$$

  (Lind-S-Ward, 1990).
- The entropy of α_f is zero if and only if f is a product of 'generalized cyclotomic polynomials', i.e., of polynomials of the form $c(u^{\mathbf{n}})$, where c is cyclotomic (Lind-S-Ward, 1990).

Principal Actions Of $\mathbb{Z}^d$

For $\Gamma = \mathbb{Z}^d$ we write $f \in \mathbb{Z}[\Gamma]$ as a Laurent polynomial in d variables: $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}}\mathbf{n} = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$.

- For $d \geq 2$, α_f is always ergodic.
- α_f is expansive if and only if $V_{\mathbb{C}}(f) = \{\mathbf{c} \in (\mathbb{C} \smallsetminus \{0\})^d : f(\mathbf{c}) = 0\}$ contains no points whose coordinates all have absolute value 1.
- The entropy $h(\alpha_f)$ is given by the *logarithmic Mahler measure* of f:

$$h(\alpha_f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})| \, dt_1 \cdots dt_d$$

  (Lind-S-Ward, 1990).
- The entropy of α_f is zero if and only if f is a product of 'generalized cyclotomic polynomials', i.e., of polynomials of the form $c(u^{\mathbf{n}})$, where c is cyclotomic (Lind-S-Ward, 1990).
- The action α_f is Bernoulli if and only if f is not divisible by a generalized cyclotomic polynomial (Rudolph-S, 1995).

Let Γ be a countably infinite discrete group.

The pairing $\langle f, x\rangle = e^{2\pi i \sum_{\gamma\in\Gamma} f_\gamma x_\gamma}$, $f = \sum f_\gamma \gamma \in \mathbb{Z}[\Gamma]$, $x = (x_\gamma) \in \mathbb{T}^\Gamma$, identifies $\mathbb{Z}[\Gamma]$ with the dual group of $\mathbb{T}^\Gamma$.

Under this identification,

$$X_f = (\mathbb{Z}[\Gamma]f)^\perp \qquad \text{and} \qquad \widehat{X}_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f,$$

and the automorphism α_f^γ of X_f is dual to left multiplication by γ^{-1} on $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$.

Ergodicity Of Principal Actions

Let Γ be a countably infinite discrete group.

The pairing $\langle f, x\rangle = e^{2\pi i \sum_{\gamma\in\Gamma} f_\gamma x_\gamma}$, $f = \sum f_\gamma \gamma \in \mathbb{Z}[\Gamma]$, $x = (x_\gamma) \in \mathbb{T}^\Gamma$, identifies $\mathbb{Z}[\Gamma]$ with the dual group of $\mathbb{T}^\Gamma$.

Under this identification,

$$X_f = (\mathbb{Z}[\Gamma]f)^\perp \qquad \text{and} \qquad \widehat{X}_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f,$$

and the automorphism α_f^γ of X_f is dual to left multiplication by γ^{-1} on $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$.

The principal action α_f is ergodic if and only if the orbit of every nonzero $a \in \widehat{X}_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ is infinite.

Ergodicity Of Principal Actions

Let Γ be a countably infinite discrete group.

The pairing $\langle f, x\rangle = e^{2\pi i \sum_{\gamma\in\Gamma} f_\gamma x_\gamma}$, $f = \sum f_\gamma \gamma \in \mathbb{Z}[\Gamma]$, $x = (x_\gamma) \in \mathbb{T}^\Gamma$, identifies $\mathbb{Z}[\Gamma]$ with the dual group of $\mathbb{T}^\Gamma$.

Under this identification,

$$X_f = (\mathbb{Z}[\Gamma]f)^\perp \qquad \text{and} \qquad \widehat{X}_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f,$$

and the automorphism α_f^γ of X_f is dual to left multiplication by γ^{-1} on $\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$.

The principal action α_f is ergodic if and only if the orbit of every nonzero $a \in \widehat{X}_f = \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ is infinite.

In other words, α_f is nonergodic if and only if there exist an $a = h + \mathbb{Z}[\Gamma]f \in \mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f$ with $h \notin \mathbb{Z}[\Gamma]f$, and a finite-index subgroup $\Delta \subset \Gamma$, such that $h - \delta h \in \mathbb{Z}[\Gamma]f$ for every $\delta \in \Delta$.

Ergodicity And Cohomology

If f is not a right zero-divisor in $\mathbb{Z}[\Gamma]$, the non-ergodicity of α_f is equivalent to saying that $h - \delta h = c(\delta)f$ for every $\delta \in \Delta$, where $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ satisfies the cocycle equation

$$c(\delta\delta') = c(\delta) + \delta c(\delta'), \quad \delta, \delta' \in \Delta. \qquad (\star)$$

Ergodicity And Cohomology

If f is not a right zero-divisor in $\mathbb{Z}[\Gamma]$, the non-ergodicity of α_f is equivalent to saying that $h - \delta h = c(\delta)f$ for every $\delta \in \Delta$, where $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ satisfies the cocycle equation

$$c(\delta\delta') = c(\delta) + \delta c(\delta'), \quad \delta, \delta' \in \Delta. \tag{$\star$}$$

A cocycle of the form ($\star$) is a *coboundary* if there exists a $b \in \mathbb{Z}[\Gamma]$ such that $c(\delta) = b - \delta b$ for every $\delta \in \Delta$.

Ergodicity And Cohomology

If f is not a right zero-divisor in $\mathbb{Z}[\Gamma]$, the non-ergodicity of α_f is equivalent to saying that $h - \delta h = c(\delta)f$ for every $\delta \in \Delta$, where $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ satisfies the cocycle equation

$$c(\delta\delta') = c(\delta) + \delta c(\delta'), \quad \delta, \delta' \in \Delta. \tag{$\star$}$$

A cocycle of the form $(\star)$ is a *coboundary* if there exists a $b \in \mathbb{Z}[\Gamma]$ such that $c(\delta) = b - \delta b$ for every $\delta \in \Delta$.

If c in $(\star)$ were a coboundary, then $h - \delta h = c(\delta)f = bf - \delta bf$ and hence (since f is not a right zero-divisor) $h = bf \in \mathbb{Z}[\Gamma]f$ – contrary to our assumption about h.

If f is not a right zero-divisor in $\mathbb{Z}[\Gamma]$, the non-ergodicity of α_f is equivalent to saying that $h - \delta h = c(\delta)f$ for every $\delta \in \Delta$, where $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ satisfies the cocycle equation

$$c(\delta\delta') = c(\delta) + \delta c(\delta'), \quad \delta, \delta' \in \Delta. \tag{$\star$}$$

A cocycle of the form ($\star$) is a *coboundary* if there exists a $b \in \mathbb{Z}[\Gamma]$ such that $c(\delta) = b - \delta b$ for every $\delta \in \Delta$.

If c in ($\star$) were a coboundary, then $h - \delta h = c(\delta)f = bf - \delta bf$ and hence (since f is not a right zero-divisor) $h = bf \in \mathbb{Z}[\Gamma]f$ – contrary to our assumption about h.

Conclusion: if f is not a right zero-divisor, then α_f is nonergodic if and only if there exists a finite index subgroup $\Delta \subset \Gamma$ and a cocycle $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ which is not a coboundary, but such that cf *is* a coboundary.

Theorem. If Γ is finitely generated and amenable, $\Delta \subset \Gamma$ a finite-index subgroup and $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ a cocycle, then there exists a bounded map $v\colon \Gamma \longrightarrow \mathbb{Z}$ such that $c(\delta) = v - \lambda^{\delta} v$ for every $\delta \in \Delta$.

Theorem. If Γ is finitely generated and amenable, $\Delta \subset \Gamma$ a finite-index subgroup and $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ a cocycle, then there exists a bounded map $v\colon \Gamma \longrightarrow \mathbb{Z}$ such that $c(\delta) = v - \lambda^{\delta} v$ for every $\delta \in \Delta$.

Corollary. If Γ is finitely generated, amenable, and not virtually cyclic, then every cocycle $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ is a coboundary.

Amenability

Theorem. If Γ is finitely generated and amenable, $\Delta \subset \Gamma$ a finite-index subgroup and $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ a cocycle, then there exists a bounded map $v\colon \Gamma \longrightarrow \mathbb{Z}$ such that $c(\delta) = v - \lambda^{\delta} v$ for every $\delta \in \Delta$.

Corollary. If Γ is finitely generated, amenable, and not virtually cyclic, then every cocycle $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ is a coboundary.

Virtual cyclicity has to be excluded: if $\Gamma = \mathbb{Z}$ and $v = (v_n)$ with $v_n = 1$ for $n \geq 0$ and $v_n = 0$ for $n < 0$, then the equation $c(m) = v - \lambda^m v$ defines a cocycle $c\colon \mathbb{Z} \longrightarrow \mathbb{Z}[\mathbb{Z}]$ which is not a coboundary.

Theorem. If Γ is finitely generated and amenable, $\Delta \subset \Gamma$ a finite-index subgroup and $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ a cocycle, then there exists a bounded map $v\colon \Gamma \longrightarrow \mathbb{Z}$ such that $c(\delta) = v - \lambda^{\delta} v$ for every $\delta \in \Delta$.

Corollary. If Γ is finitely generated, amenable, and not virtually cyclic, then every cocycle $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ is a coboundary.

Virtual cyclicity has to be excluded: if $\Gamma = \mathbb{Z}$ and $v = (v_n)$ with $v_n = 1$ for $n \geq 0$ and $v_n = 0$ for $n < 0$, then the equation $c(m) = v - \lambda^m v$ defines a cocycle $c\colon \mathbb{Z} \longrightarrow \mathbb{Z}[\mathbb{Z}]$ which is not a coboundary.

In general, nontrivial cocycles $c\colon \Delta \longrightarrow \mathbb{Z}[\Gamma]$ exist if and only if the group Γ has more than one end: for every k, the 'level set' $L_k = \{\gamma \in \Gamma : v_\gamma = k\}$ of v must be almost left invariant under Δ, since $v - \lambda^{\delta} v \in \mathbb{Z}[\Gamma]$. If all but one of these level sets are finite, then $v - k \in \mathbb{Z}[\Gamma]$ for some k and c is a coboundary. If two of these level sets are infinite, Δ (and hence Γ) has more than one end (by definition).

Theorem. Let Γ be a countably infinite discrete group and $f \in \mathbb{Z}[\Gamma]$ an element which is not a right zero-divisor. Then α_f is ergodic if one of the following conditions hold.

- Γ is amenable and contains a finitely generated, infinite, and not virtually cyclic subgroup.
- Γ is infinitely generated (Hayes).
- Γ has property T (Li).
- Γ is the free group on $k \geq 2$ generators (Hayes).

Theorem. Let Γ be a countably infinite discrete group and $f \in \mathbb{Z}[\Gamma]$ an element which is not a right zero-divisor. Then α_f is ergodic if one of the following conditions hold.

- Γ is amenable and contains a finitely generated, infinite, and not virtually cyclic subgroup.
- Γ is infinitely generated (Hayes).
- Γ has property T (Li).
- Γ is the free group on $k \geq 2$ generators (Hayes).

The last of these statements is the most interesting one. If Γ is free, then every finite-index subgroup $\Delta \subset \Gamma$ is again free with more than one generator. Free groups have infinitely many ends, so that there exist many cocycles which are not coboundaries. *However, these nontrivial cocycles have the property that cf is not a coboundary*.

Zero-Divisors?

Conjecture (Kaplansky, around 1940): If Γ is an infinite torsion-free group then $\mathbb{Z}\Gamma$ has no nontrivial zero-divisors (i.e., $ab \neq 0$ whenever $a, b \in \mathbb{Z}\Gamma$ are nonzero).

The Zero Divisor Conjecture is known to hold if Γ is virtually abelian, elementary amenable, or orderable. (Γ is *elementary amenable* if it lies in the smallest class of groups which contains the finite and the abelian groups, and which is closed under taking subgroups, quotients, extensions, and directed unions).

Zero-Divisors?

Conjecture (Kaplansky, around 1940): If Γ is an infinite torsion-free group then $\mathbb{Z}\Gamma$ has no nontrivial zero-divisors (i.e., $ab \neq 0$ whenever $a, b \in \mathbb{Z}\Gamma$ are nonzero).

The Zero Divisor Conjecture is known to hold if Γ is virtually abelian, elementary amenable, or orderable. (Γ is *elementary amenable* if it lies in the smallest class of groups which contains the finite and the abelian groups, and which is closed under taking subgroups, quotients, extensions, and directed unions).

If Γ has an element of finite order $x^n = 1$, then $\mathbb{Z}\Gamma$ has zero-divisors: $(x-1)(1+x+\cdots+x^{n-1}) = 0$.
If $uv = 0$, but u and v are nonzero, one calls u a left zero-divisor and v a right zero-divisor.

Zero-Divisors?

Conjecture (Kaplansky, around 1940): If Γ is an infinite torsion-free group then $\mathbb{Z}\Gamma$ has no nontrivial zero-divisors (i.e., $ab \neq 0$ whenever $a, b \in \mathbb{Z}\Gamma$ are nonzero).

The Zero Divisor Conjecture is known to hold if Γ is virtually abelian, elementary amenable, or orderable. (Γ is *elementary amenable* if it lies in the smallest class of groups which contains the finite and the abelian groups, and which is closed under taking subgroups, quotients, extensions, and directed unions).

If Γ has an element of finite order $x^n = 1$, then $\mathbb{Z}\Gamma$ has zero-divisors: $(x-1)(1+x+\cdots+x^{n-1}) = 0$.
If $uv = 0$, but u and v are nonzero, one calls u a left zero-divisor and v a right zero-divisor.

Interesting fact: If Γ is amenable and $u \in \mathbb{Z}\Gamma$ is a left zero-divisor, then it is also a right zero-divisor. The proof uses von Neumann algebra techniques.

Zero-Divisors?

Conjecture (Kaplansky, around 1940): If Γ is an infinite torsion-free group then $\mathbb{Z}\Gamma$ has no nontrivial zero-divisors (i.e., $ab \neq 0$ whenever $a, b \in \mathbb{Z}\Gamma$ are nonzero).

The Zero Divisor Conjecture is known to hold if Γ is virtually abelian, elementary amenable, or orderable. (Γ is *elementary amenable* if it lies in the smallest class of groups which contains the finite and the abelian groups, and which is closed under taking subgroups, quotients, extensions, and directed unions).

If Γ has an element of finite order $x^n = 1$, then $\mathbb{Z}\Gamma$ has zero-divisors: $(x-1)(1+x+\cdots+x^{n-1}) = 0$.
If $uv = 0$, but u and v are nonzero, one calls u a left zero-divisor and v a right zero-divisor.

Interesting fact: If Γ is amenable and $u \in \mathbb{Z}\Gamma$ is a left zero-divisor, then it is also a right zero-divisor. The proof uses von Neumann algebra techniques.

This shows that there can be very complicated divisors of zero!

Let Γ be countably infinite, discrete, and amenable, and that $f \in \mathbb{Z}[\Gamma]$.

Let Γ be countably infinite, discrete, and amenable, and that $f \in \mathbb{Z}[\Gamma]$.

Theorem (Deninger-S, 2007): α_f is expansive if and only f is invertible in $\ell^1(\Gamma)$.

Let Γ be countably infinite, discrete, and amenable, and that $f \in \mathbb{Z}[\Gamma]$.

Theorem (Deninger-S, 2007): α_f is expansive if and only f is invertible in $\ell^1(\Gamma)$.

Theorem: If Γ is amenable and f is not a zero-divisor, then

$$h(\alpha_f) = \log \det{}_{\mathcal{N}\Gamma}(\rho_f), \tag{1}$$

where the last term is the *Fuglede-Kadison determinant* of f, acting by right convolution on $\ell^2(\Gamma)$, and viewed as an element of the (left-equivariant) group von Neumann algebra $\mathcal{N}\Gamma$.

Let Γ be countably infinite, discrete, and amenable, and that $f \in \mathbb{Z}[\Gamma]$.

Theorem (Deninger-S, 2007): α_f is expansive if and only f is invertible in $\ell^1(\Gamma)$.

Theorem: If Γ is amenable and f is not a zero-divisor, then

$$h(\alpha_f) = \log \det_{\mathcal{N}\Gamma}(\rho_f), \tag{1}$$

where the last term is the *Fuglede-Kadison determinant* of f, acting by right convolution on $\ell^2(\Gamma)$, and viewed as an element of the (left-equivariant) group von Neumann algebra $\mathcal{N}\Gamma$.

The proof of (1) is due to Deninger (2006, assuming expansiveness + technical conditions), Deninger-S (2007, assuming expansiveness + residual finiteness of Γ) and Li-Thom (2012, general case).

Assume that Γ is residually finite (i.e., that there exists a decreasing sequence $(\Delta_n)_{n\geq 1}$ of finite-index subgroups with $\bigcap_n \Delta_n = \{1\}$).

Assume that Γ is residually finite (i.e., that there exists a decreasing sequence $(\Delta_n)_{n\geq 1}$ of finite-index subgroups with $\bigcap_n \Delta_n = \{1\}$).

Theorem (Deninger-S, 2007): If Γ is amenable and α_f is expansive, then

$$h(\alpha_f) = \lim_{\Delta \searrow \{1\}} \frac{1}{|\Gamma/\Delta|} \log |\mathrm{Fix}_\Delta(X_f)| = \log \det{}_{\mathcal{N}\Gamma}(\rho_f). \tag{2}$$

Assume that Γ is residually finite (i.e., that there exists a decreasing sequence $(\Delta_n)_{n\geq 1}$ of finite-index subgroups with $\bigcap_n \Delta_n = \{1\}$).

Theorem (Deninger-S, 2007): If Γ is amenable and α_f is expansive, then

$$h(\alpha_f) = \lim_{\Delta \searrow \{1\}} \frac{1}{|\Gamma/\Delta|} \log |\mathrm{Fix}_\Delta(X_f)| = \log \det{}_{\mathcal{N}\Gamma}(\rho_f). \qquad (2)$$

For every finite-index subgroup $\Delta \subset \Gamma$, $|\mathrm{Fix}_\Delta(X_f)| = |\det \rho_f^{\Gamma/\Delta}|$. The convergence in (2) as $\Delta \searrow \{1\}$ follows from an approximation argument due to Lück (1994) and Schick (2001).

What Does Entropy Have To Do With Determinants?

Assume that Γ is residually finite (i.e., that there exists a decreasing sequence $(\Delta_n)_{n\geq 1}$ of finite-index subgroups with $\bigcap_n \Delta_n = \{1\}$).

Theorem (Deninger-S, 2007): If Γ is amenable and α_f is expansive, then

$$h(\alpha_f) = \lim_{\Delta \searrow \{1\}} \frac{1}{|\Gamma/\Delta|} \log |\mathrm{Fix}_\Delta(X_f)| = \log \det{}_{\mathcal{N}\Gamma}(\rho_f). \qquad (2)$$

For every finite-index subgroup $\Delta \subset \Gamma$, $|\mathrm{Fix}_\Delta(X_f)| = \left|\det \rho_f^{\Gamma/\Delta}\right|$. The convergence in (2) as $\Delta \searrow \{1\}$ follows from an approximation argument due to Lück (1994) and Schick (2001).

If α_f is nonexpansive, equality of the three terms in (2) is an open problem — even for $\Gamma = \mathbb{Z}^d$!