Dimension and stability index for chaotically driven concave maps

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Introduction

- $S : \Theta \to \Theta$ invertible, $g : \Theta \to (0, \infty)$, $I = [0, a]$, $F_t : \Theta \times I \to \Theta \times I$,

$$F_t(\theta, x) = (S\theta, e^t g(\theta) h(x))$$

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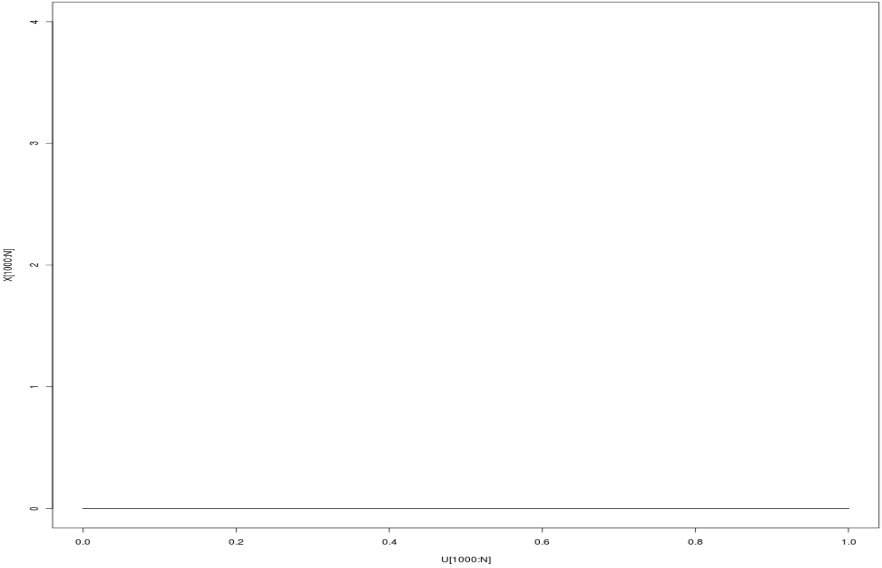
$g(u, v) = \tilde{g}(v) = 1 + \epsilon + \cos(2\pi v)$, hence $\varphi_t(\theta) = \tilde{\varphi}_t(v)$.

Pictures ($A = 0.49$, $\epsilon = 0.01$)



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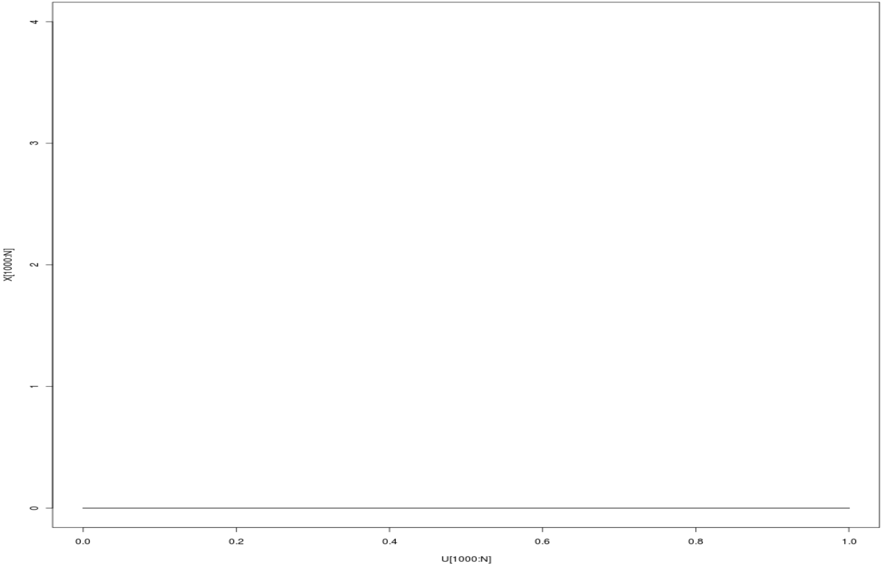

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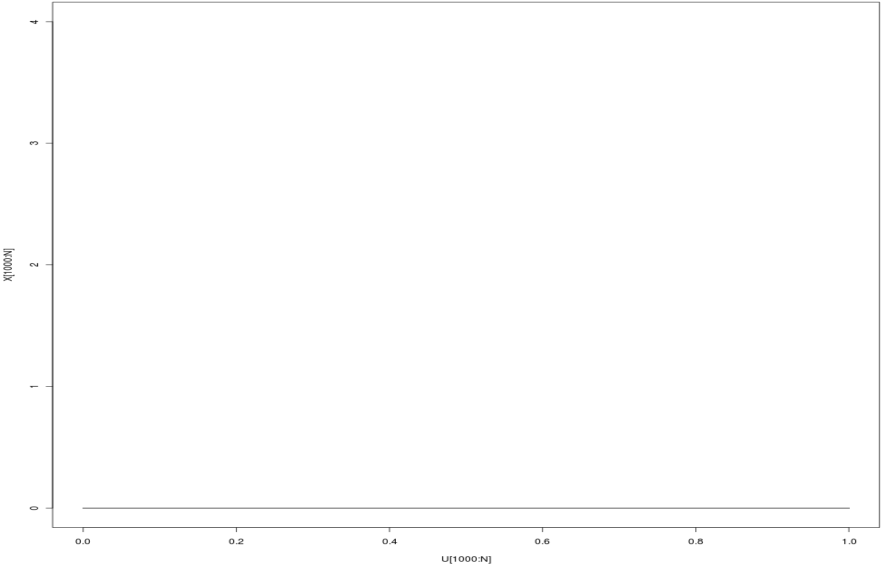

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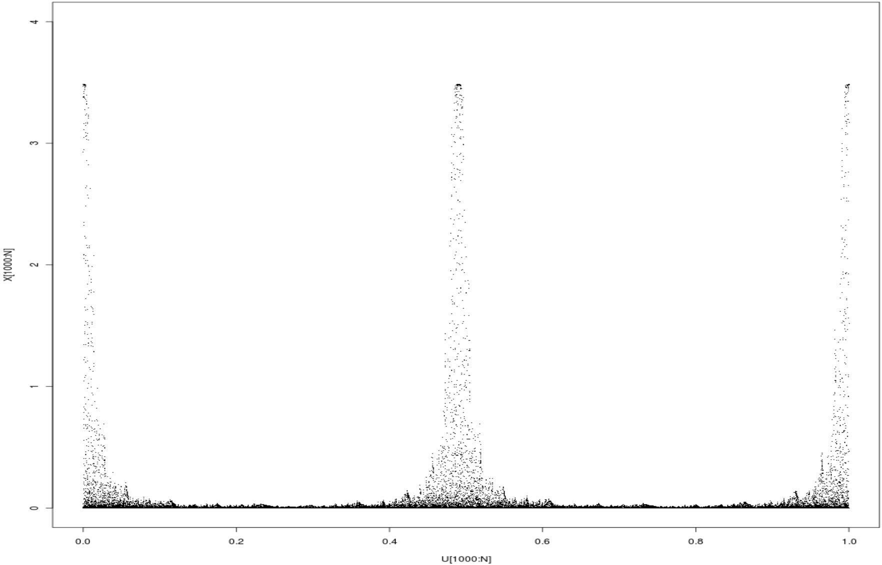

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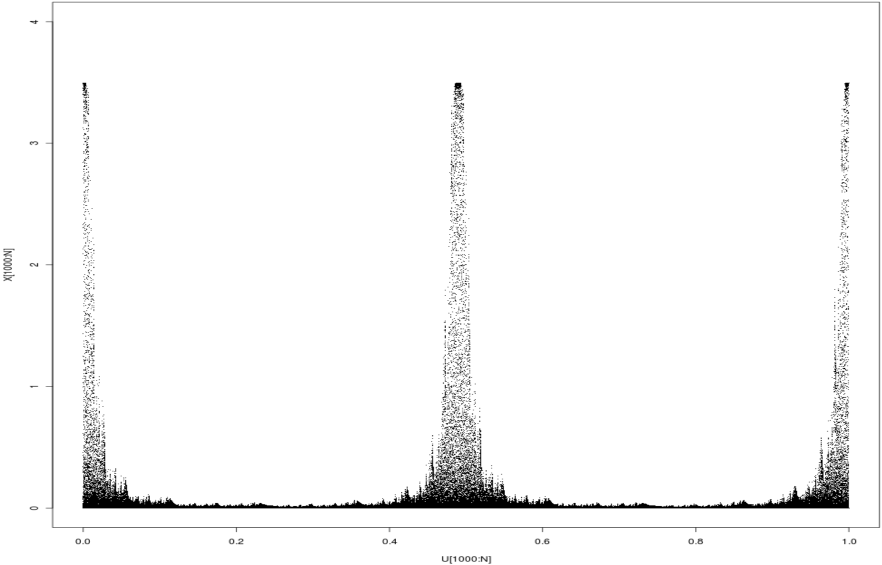

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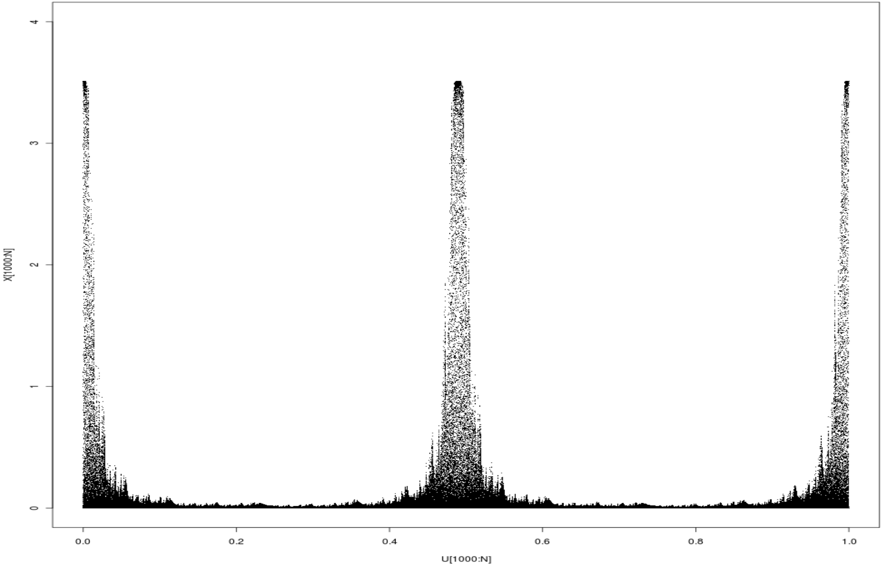

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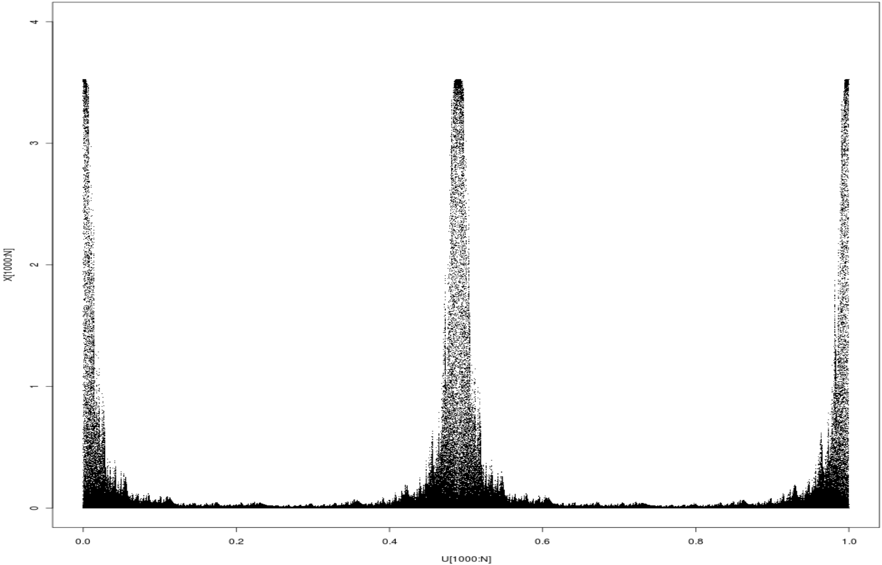

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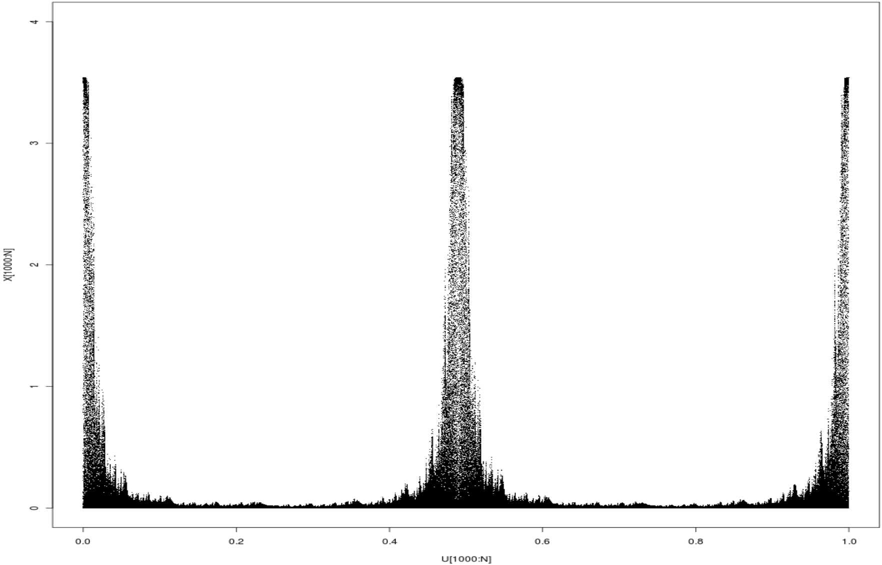

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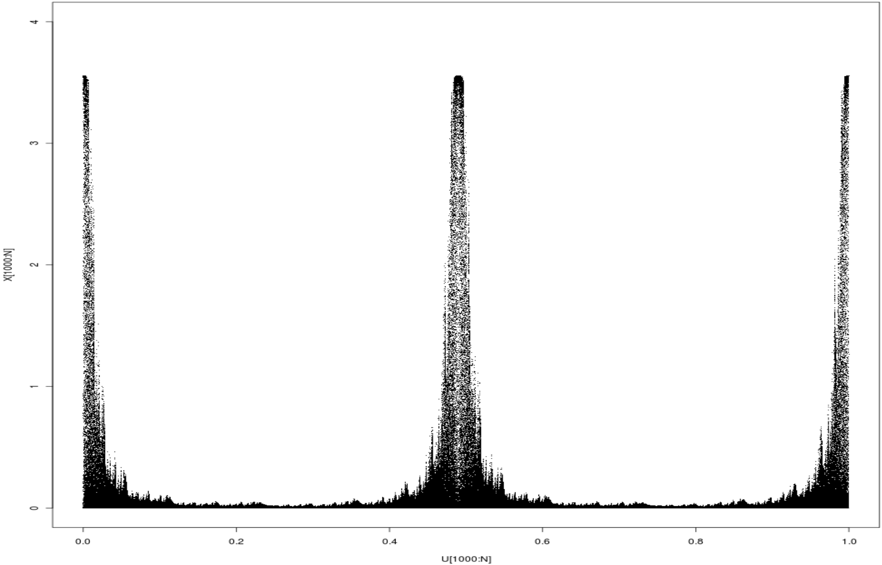

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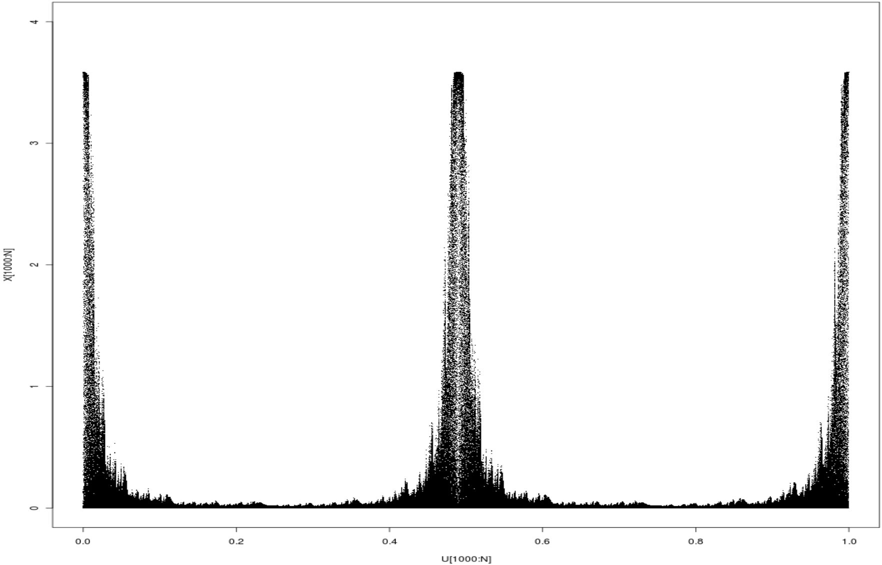

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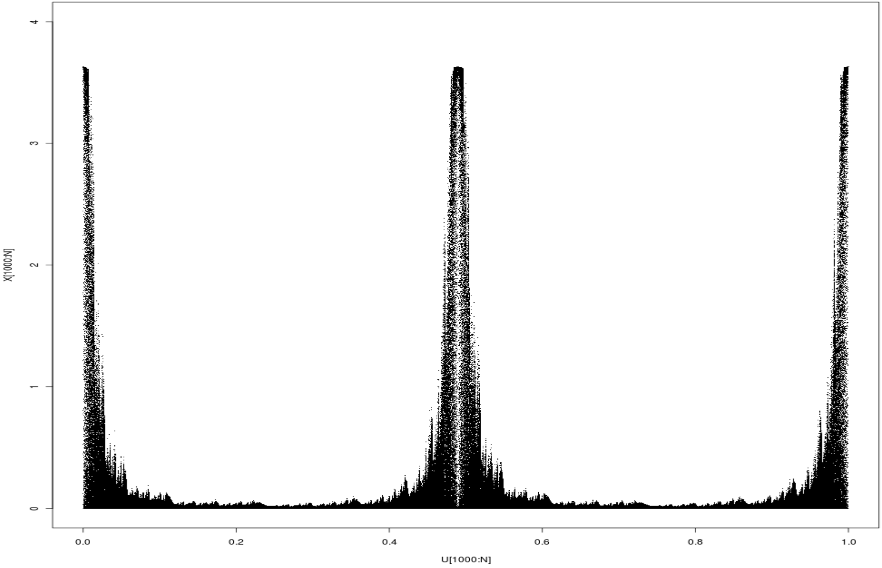

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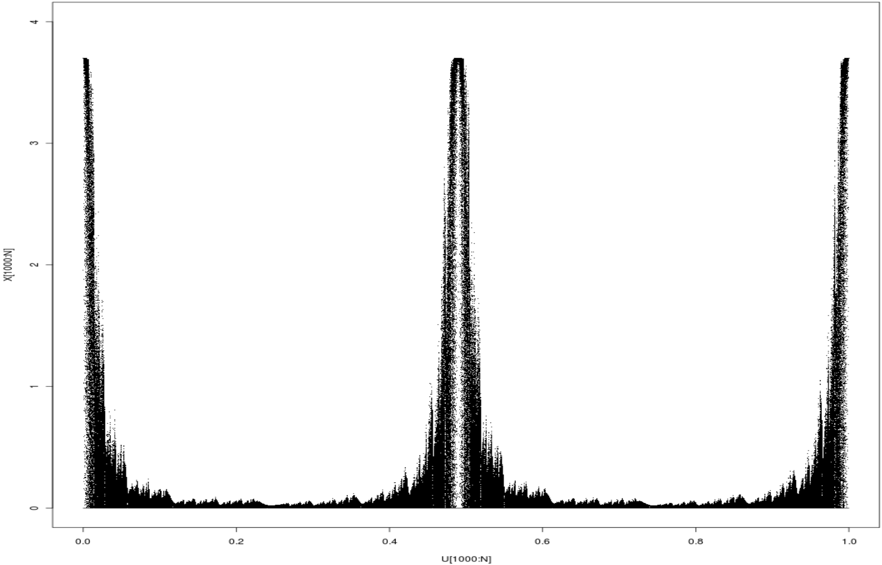

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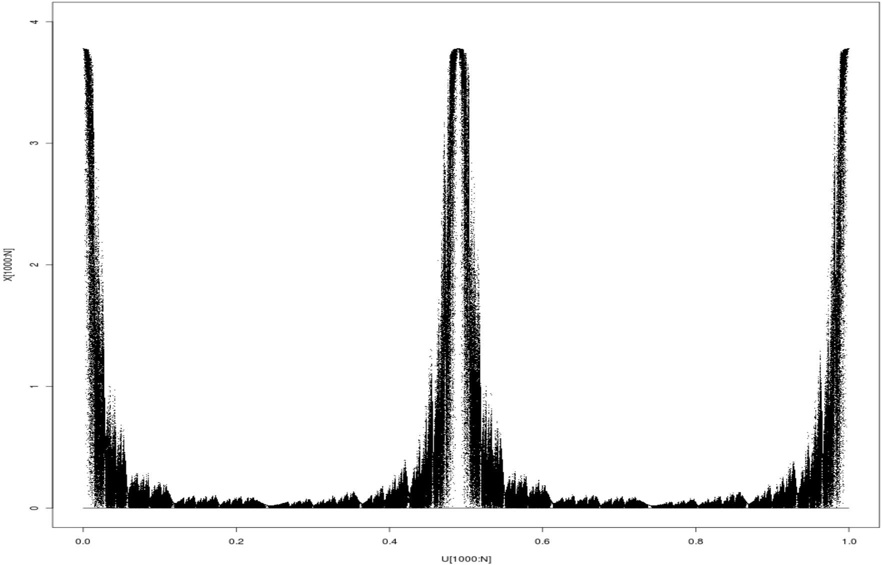

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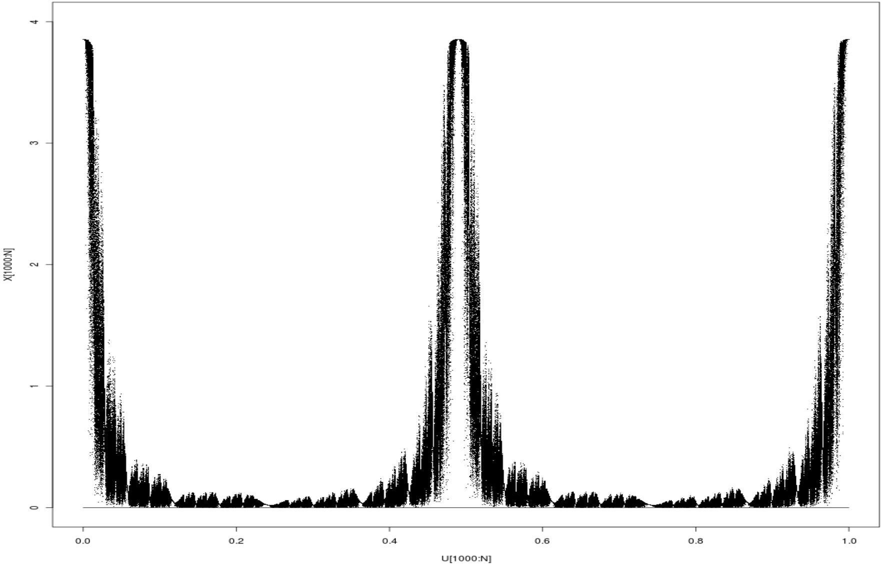

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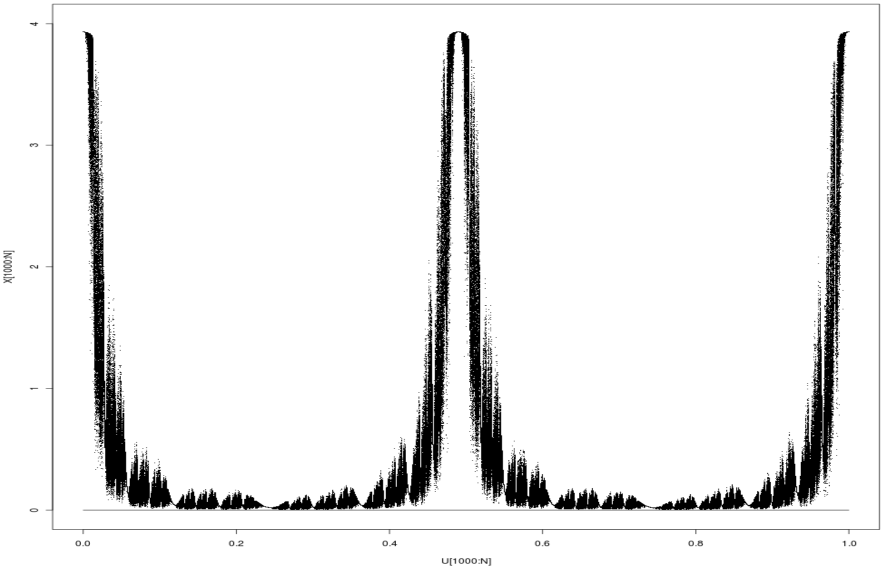

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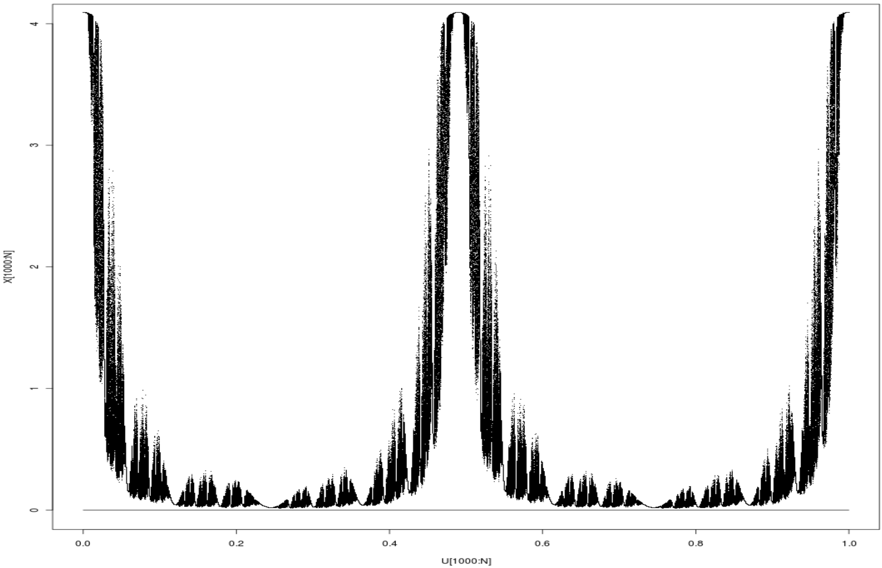
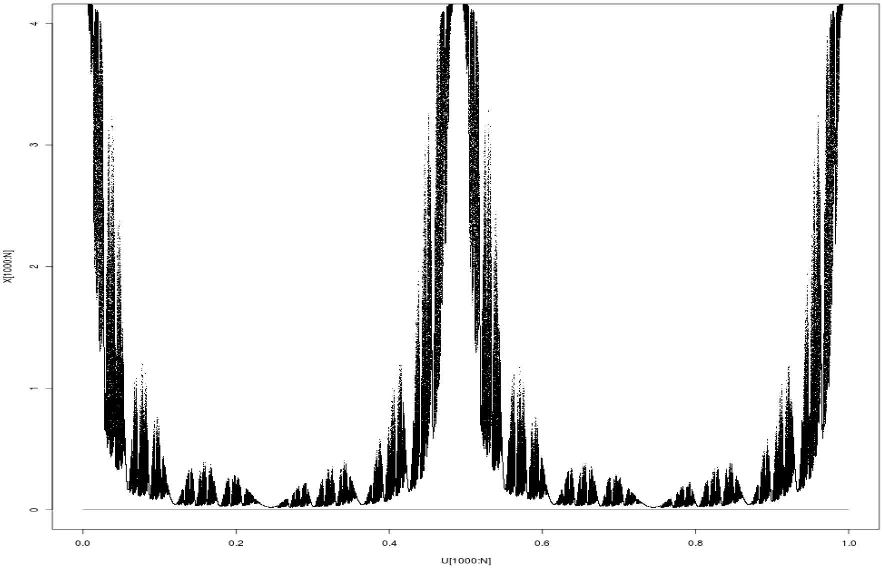


Pictures ($A = 0.49$, $\epsilon = 0.01$, critical t)



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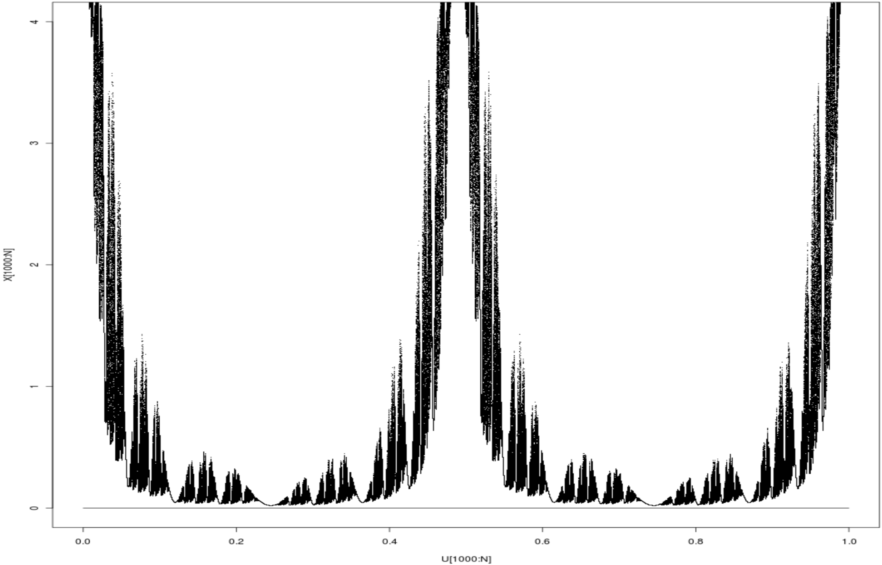

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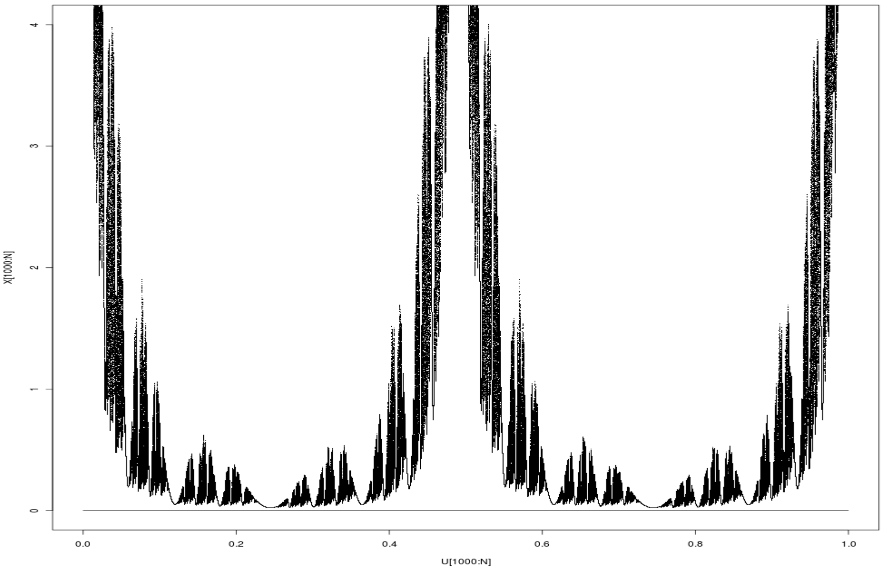

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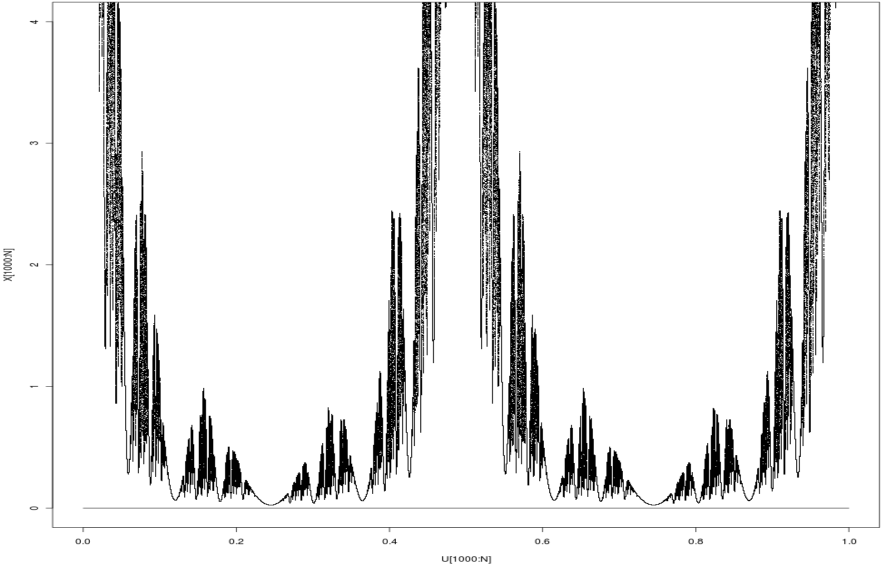

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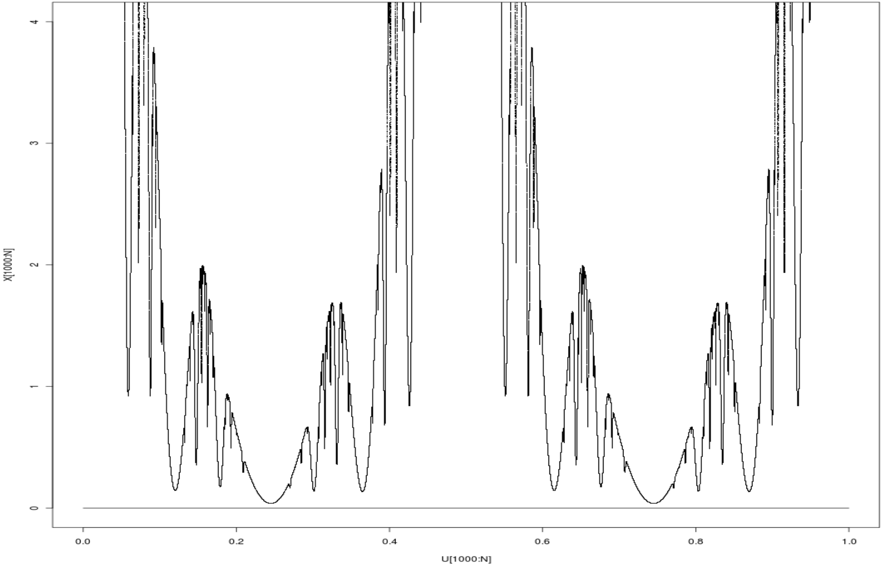

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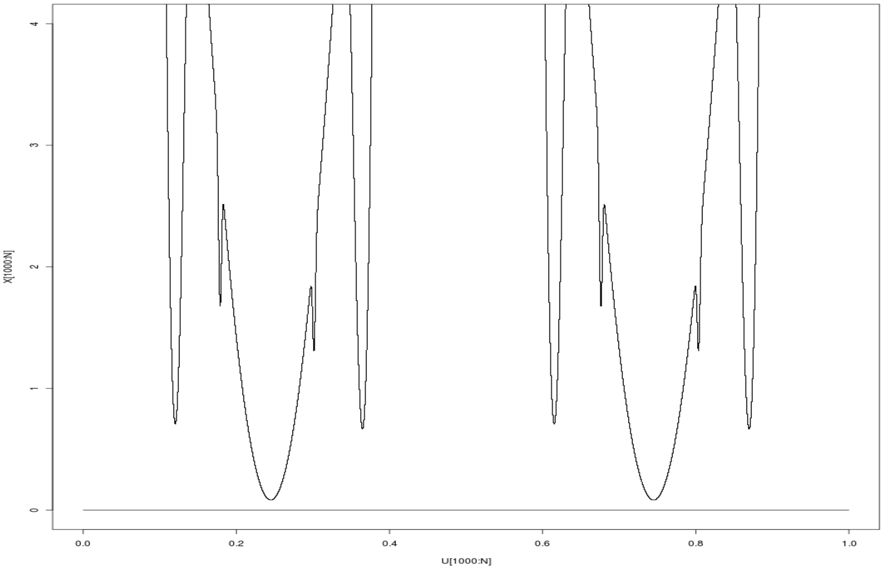

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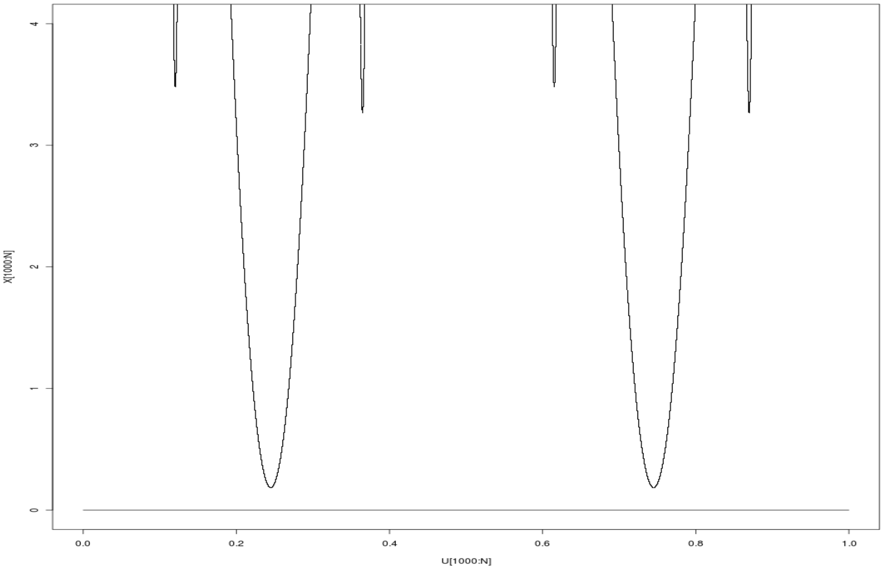

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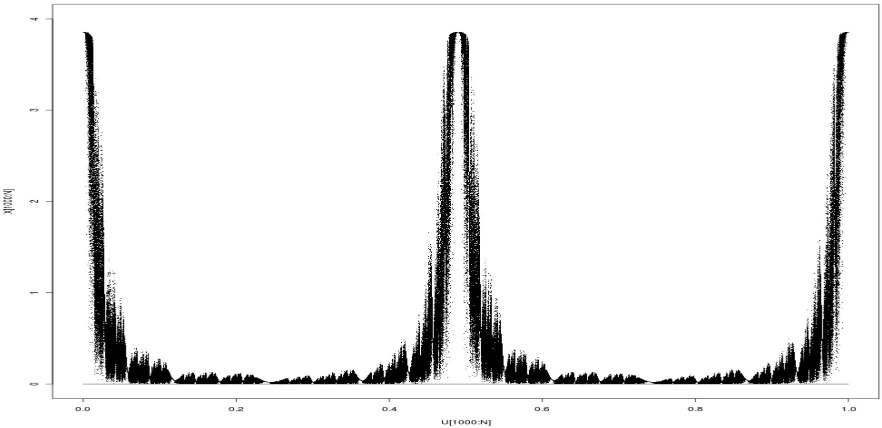

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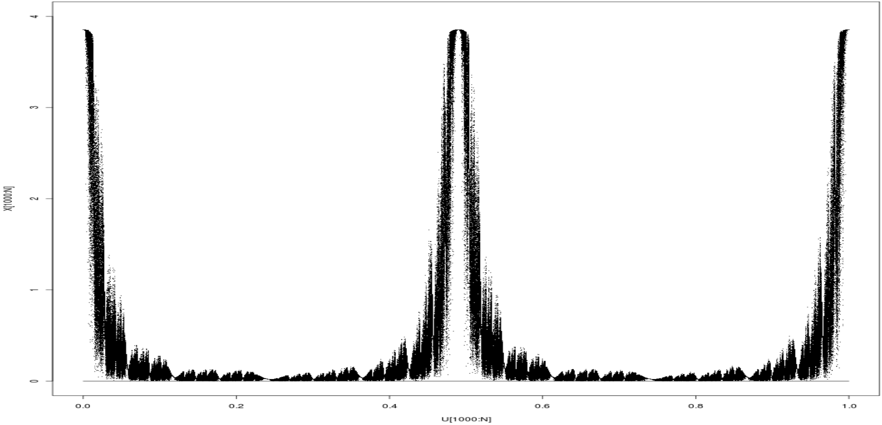

- "Size" of $N_t := \{\theta \in \Theta : \varphi_t(\theta) = 0\}$ as function of t?
- Scaling behaviour of φ_t close to 0-axis (global and local)?

Characterization of N_t [with Atsuya Otani, Erlangen]

- $N_t := \{\theta : \varphi_t(\theta) = 0\}$, hence $s < t \Rightarrow N_s \supseteq N_t$.
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Observe:

- N_t consists of unstable fibres of S due to pullback construction of φ_t.

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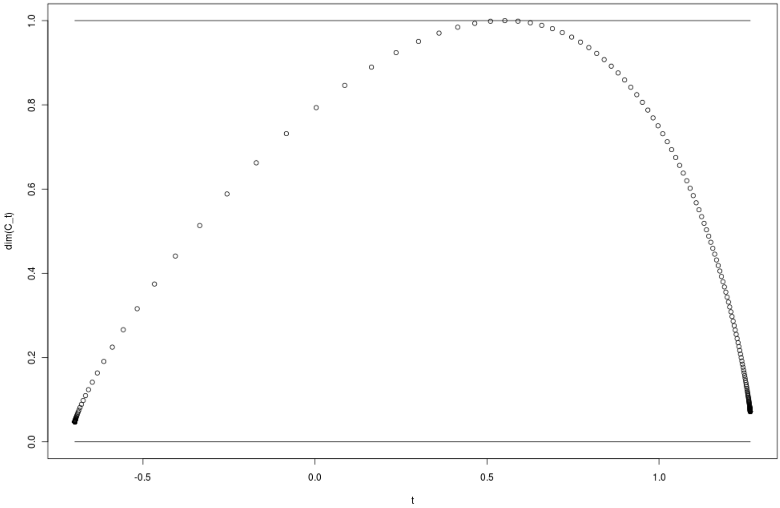
Theorem 1 (for Anosov surface diffeo, baker)

$$\dim_H(C_t) = D(t) + 1$$

where $D : (t_{min}, t_{max}) \to [0,1]$ real analytic s.th.

$$D(t_c) = 1, \quad D'(t) \begin{cases} > 0 & (t_{min} < t < t_c) \\ < 0 & (t_c < t < t_{max}) \end{cases}, \quad D''(t_c) < 0.$$

$D(t)$ for baker map and $g(v) = 1 + \epsilon + \cos(2\pi v)$



$a = 0.45$, $\epsilon = 0.01$ [with H. Jafri and R. Ramaswamy, New Delhi]

The stability index [→ Podvigina/Ashwin 2011]

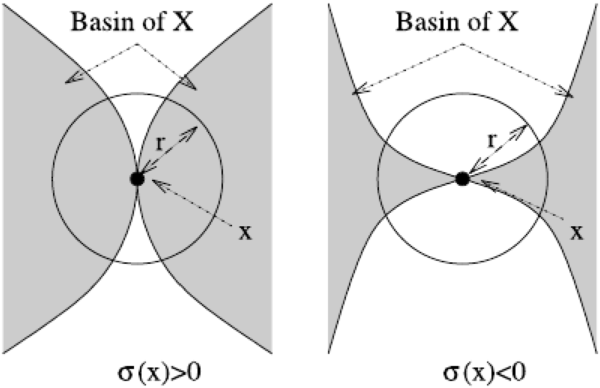
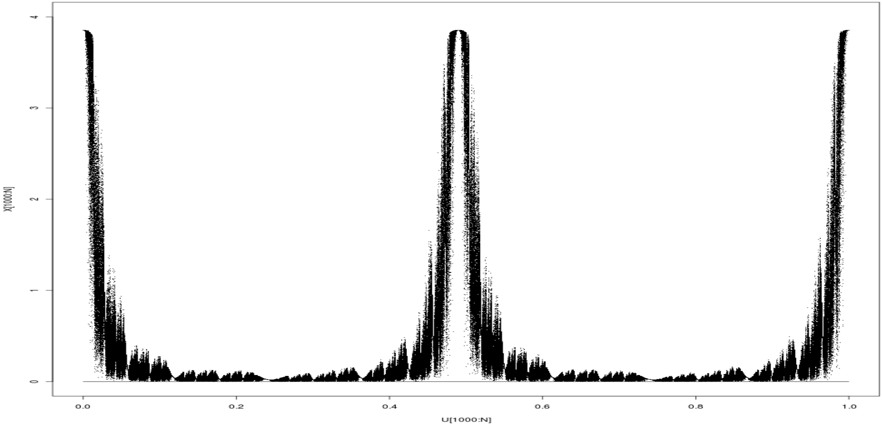


Figure 1. Schematic diagram illustrating how the stability index $\sigma(x)$ of a point $x \in X$ relates to the local geometry of the basin of attraction of X (shaded region). For $\sigma(x) > 0$, the measure of points in a ball of radius r that are in the complement of the basin goes to zero, relative the measure of the ball, as $r^{|\sigma(x)|}$. For $\sigma(x) < 0$, this estimate applies to the basin itself.

The stability index for $t > t_c$



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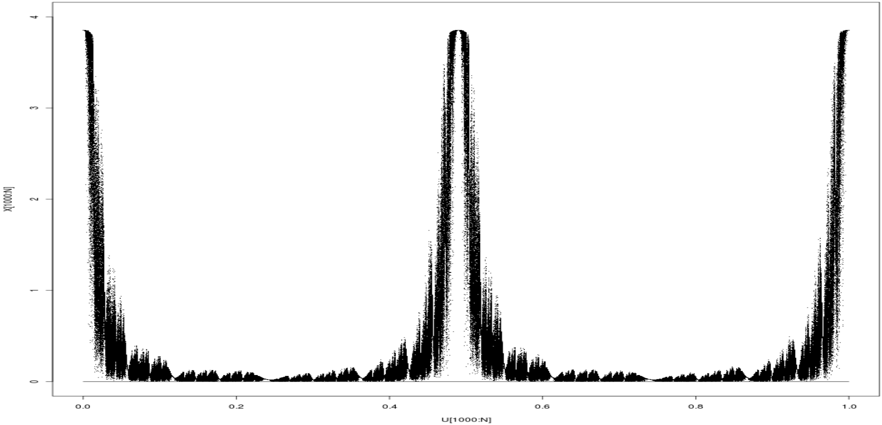

1. Global scaling close to 0-line
2. Local scaling close to 0-line

Global scaling: Anosov surface diffeo, baker

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2. Let $\Xi_\epsilon(t) := \frac{1}{\epsilon}\int_\Theta \min\{\varphi(\theta),\epsilon\}\, dm^2$. Then
$$\lim_{\epsilon\to 0} \frac{\log(1-\Xi_\epsilon(t))}{\log \epsilon} = s_*(t), \quad \lim_{\epsilon\to 0} \frac{\log \Xi_\epsilon(t)}{\log \epsilon} = 0$$

Global scaling: Anosov surface diffeo, baker

- $p_t(s) :=$topological pressure$\big(\log|D_u S^{-1}| - s(\log g + t)\big) = p_0(s) - st$
- $p_t(0) = 0$, $p_t'(0) = -\mu^-_{SRB}(\log g + t) < 0$
- $\exists!\ s_*(t) > 0 : p_t(s_*(t)) = 0$

Theorem 2

1. $$\lim_{\epsilon\to 0} \frac{\log m^2\{\theta : \varphi_t(\theta) < \epsilon\}}{\log \epsilon} = s_*(t)$$

2. Let $\Xi_\epsilon(t) := \frac{1}{\epsilon}\int_\Theta \min\{\varphi(\theta), \epsilon\}\, dm^2$. Then
$$\lim_{\epsilon\to 0} \frac{\log(1 - \Xi_\epsilon(t))}{\log \epsilon} = s_*(t), \quad \lim_{\epsilon\to 0} \frac{\log \Xi_\epsilon(t)}{\log \epsilon} = 0$$

Proof: Large deviations, motivated by papers on Loyne's exponent in queuing theory.

Local scaling: Anosov surface diffeo, baker

$$\Sigma_\epsilon(\theta, t) := \frac{1}{\epsilon \cdot m^2(U_\epsilon(\theta))} \int_{U_\epsilon(\theta)} \min\{\varphi_t, \epsilon\} \, dm^2$$

Local scaling: Anosov surface diffeo, baker

$$\Sigma_\epsilon(\theta, t) := \frac{1}{\epsilon \cdot m^2(U_\epsilon(\theta))} \int_{U_\epsilon(\theta)} \min\{\varphi_t, \epsilon\}\, dm^2$$

$$\sigma_+(\theta, t) := \lim_{\epsilon \to 0} \frac{\log(1 - \Sigma_\epsilon(\theta, t))}{\log \epsilon}, \quad \sigma_-(\theta, t) := \lim_{\epsilon \to 0} \frac{\log \Sigma_\epsilon(\theta, t)}{\log \epsilon}$$

Local scaling: Anosov surface diffeo, baker

$$\Sigma_\epsilon(\theta,t) := \frac{1}{\epsilon\cdot m^2(U_\epsilon(\theta))}\int_{U_\epsilon(\theta)} \min\{\varphi_t,\epsilon\}\,dm^2$$

$$\sigma_+(\theta,t) := \lim_{\epsilon\to 0}\frac{\log(1-\Sigma_\epsilon(\theta,t))}{\log\epsilon},\quad \sigma_-(\theta,t) := \lim_{\epsilon\to 0}\frac{\log\Sigma_\epsilon(\theta,t)}{\log\epsilon}$$

$$\Gamma(\theta) := \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log g(S^{-k}\theta),\quad \Lambda(\theta) := \lim_{n\to\infty}\frac{1}{n}\log|D_u S^{-n}(\theta)|$$

Local scaling: Anosov surface diffeo, baker

$$\Sigma_\epsilon(\theta, t) := \frac{1}{\epsilon \cdot m^2(U_\epsilon(\theta))} \int_{U_\epsilon(\theta)} \min\{\varphi_t, \epsilon\}\, dm^2$$

$$\sigma_+(\theta, t) := \lim_{\epsilon\to 0} \frac{\log(1 - \Sigma_\epsilon(\theta, t))}{\log \epsilon}, \quad \sigma_-(\theta, t) := \lim_{\epsilon\to 0} \frac{\log \Sigma_\epsilon(\theta, t)}{\log \epsilon}$$

$$\Gamma(\theta) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log g(S^{-k}\theta), \quad \Lambda(\theta) := \lim_{n\to\infty} \frac{1}{n} \log |D_u S^{-n}(\theta)|$$

Theorem 3 Let $\theta \in \Theta$. Suppose $\Gamma(\theta)$ and $\Lambda(\theta)$ exist.

1. If $\Lambda(\theta) + \Gamma(\theta) + t \geqslant 0$, then

$$\sigma_+(\theta, t) = \frac{\Lambda(\theta) + \Gamma(\theta) + t}{\Lambda(\theta)} \cdot s_*(t), \quad \sigma_-(\theta, t) = 0$$

Local scaling: Anosov surface diffeo, baker

$$\Sigma_\epsilon(\theta,t) := \frac{1}{\epsilon \cdot m^2(U_\epsilon(\theta))} \int_{U_\epsilon(\theta)} \min\{\varphi_t,\epsilon\}\, dm^2$$

$$\sigma_+(\theta,t) := \lim_{\epsilon\to 0} \frac{\log\left(1-\Sigma_\epsilon(\theta,t)\right)}{\log \epsilon}, \quad \sigma_-(\theta,t) := \lim_{\epsilon\to 0} \frac{\log \Sigma_\epsilon(\theta,t)}{\log \epsilon}$$

$$\Gamma(\theta) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log g(S^{-k}\theta), \quad \Lambda(\theta) := \lim_{n\to\infty} \frac{1}{n} \log |D_u S^{-n}(\theta)|$$

Theorem 3 Let $\theta \in \Theta$. Suppose $\Gamma(\theta)$ and $\Lambda(\theta)$ exist.

1. If $\Lambda(\theta)+\Gamma(\theta)+t \geqslant 0$, then

$$\sigma_+(\theta,t) = \frac{\Lambda(\theta)+\Gamma(\theta)+t}{\Lambda(\theta)} \cdot s_*(t), \quad \sigma_-(\theta,t) = 0$$

2. If $\Lambda(\theta)+\Gamma(\theta)+t \leqslant 0$, then

$$\sigma_-(\theta,t) = -\frac{\Lambda(\theta)+\Gamma(\theta)+t}{\Lambda(\theta)}, \quad \sigma_+(\theta,t) = 0$$