Anomalous diffusive phenomena in planar hyperbolic billiards

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(joint with N. Chernov and D. Dolgopyat)

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Ergodic Theory and Dynamical Systems
Perspectives and Prospects
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Outline

Setting

- Dispersing billiards in 2D
- Intermittent billiards in 2D – cusp, stadium, ∞H

Anomalous phenomena

- Non-standard limit law
- Convergence of the second moment

Skeletons of arguments

- Non-standard limit law
- Convergence of the second moment

Further details

- Why $\frac{4+3\log 3}{4-3\log 3}$?
- Why $\sqrt{\cos\phi}$?

Summary and Outlook

Billiards

$Q = \mathbb{T}^2 \setminus \bigcup_{k=1}^{K} C_k$ strictly convex scatterers

- Billiard flow : $S^t : \mathcal{M} \to \mathcal{M}$, $(q, v) \in \mathcal{M} = Q \times \mathbb{S}^1$, $|v| = 1$ Uniform motion within Q, elastic reflection at the boundaries
- Billiard map phase space: $M = \bigcup_{k=1}^{K} M_k$
- $(r, \phi) \in M_k$, r: arclength along ∂C_k, $\phi \in [-\pi/2, \pi/2]$ outgoing velocity angle
- invariant measure $d\mu = c\, cos\phi\, dr\, d\phi$

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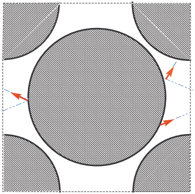

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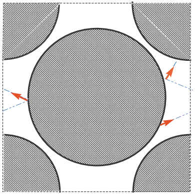

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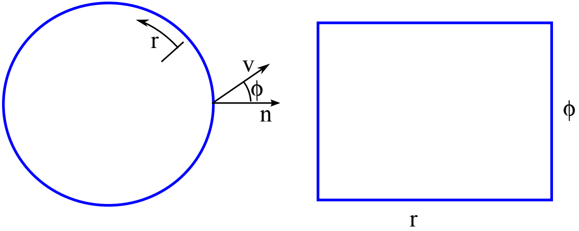

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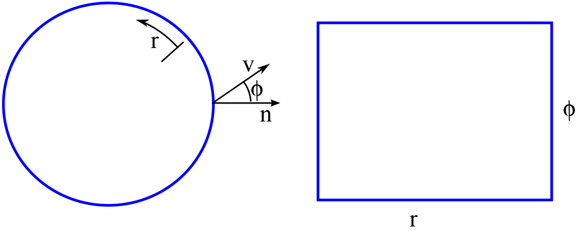

Sinai billiards

C_k are C^3 smooth and disjoint (no corner points);
finite horizon: flight length uniformly bounded from above

- Billiard map is ergodic, K-mixing (Sinai '70)
- EDC: $f, g : M \to \mathbb{R}$ Hölder continuous, $\int f d\mu = \int g d\mu = 0$ let $C_n(f,g) = \mu(f \cdot g \circ T^n)$, then $|C_n(f,g)| \le C\alpha^n$ for suitable $C > 0$ and $\alpha < 1$
  - Young '98 – tower construction with exponential tails,
  - Chernov & Dolgopyat '06 – standard pairs
- CLT: let $S_n f = f + f \circ T + ... + f \circ T^{n-1}$, then $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ where $\sigma^2 = \int f^2 d\mu + 2 \sum_{n=1}^{\infty} C_n(f,f)$.
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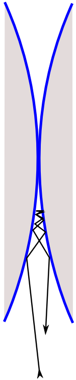

cusp: long series

stadium: bouncing orbits

∞ H: arbitrary long flights

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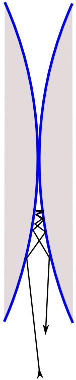

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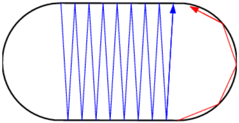

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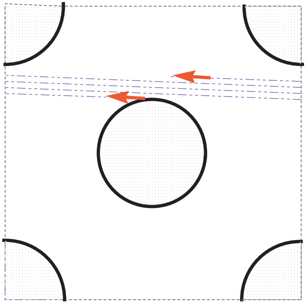

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Decay of correlations

- Cusp map:
  - Reháček ’95 ergodicity
  - Machta ’83 numerics and heuristic reasoning for $C_n(f, g) \asymp 1/n$
  - Chernov & Markarian ’07: $C_n(f, g) \leq C \frac{\log^2 n}{n}$
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- Stadium map: Markarian ’05, Chernov & Zhang ’06 $C_n(f, g) \leq C \frac{\log^2 n}{n}$
- ∞H flow: Melbourne ’09 $C_t(F, G) \leq C \frac{1}{t}$ (essentially)
- Cusp flow: Melbourne & B. ’08 $C_t(F, G)$ decays faster than any polynomial
- Not summable $\Rightarrow$ non-standard limit law?

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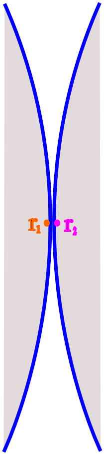
Superdiffusion in dispersing billiards with cusps



Theorem (Chernov, Dolgopyat & B. 2011)

- *Denote by $r_1 \in C_1$ and $r_2 \in C_2$ the two points that make the cusp.*
- *Let* $I_f = \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi))\rho(\phi) d\phi$

  with $\rho(\phi) = \frac{\sqrt{\cos\phi}}{\int_{-\pi/2}^{\pi/2} \sqrt{\cos\phi} d\phi}$
- *if* $I_f \neq 0$ *then* $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$

  where $D_f = c^* I_f^2$ *and* c^* *is some numerical constant.*
- *if* $I_f = 0$ *then* $S_n f$ *satisfies standard CLT.*

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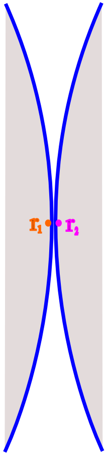

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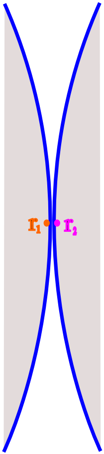

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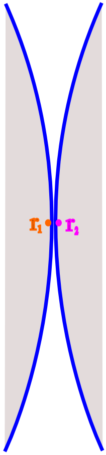

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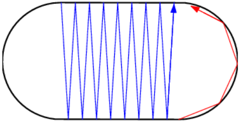

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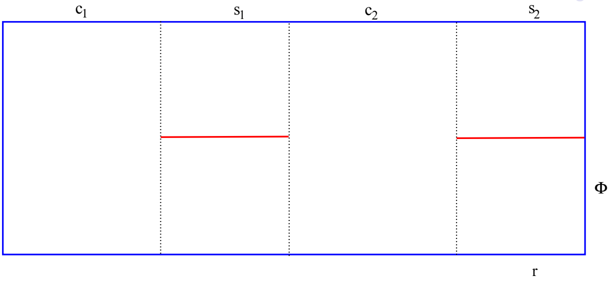
- Let $I_f = \int_{s_1 \cup s_2} f(r, \frac{\pi}{2})dr$.
- if $I_f \neq 0$ then $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ where $D_f = \frac{4+3\log 3}{4-3\log 3} c^* I_f^2$



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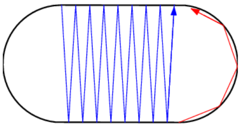

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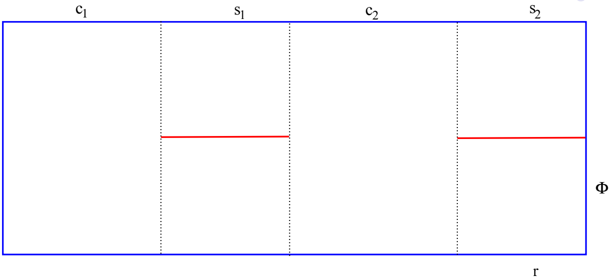

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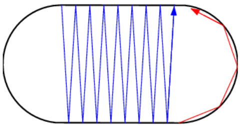

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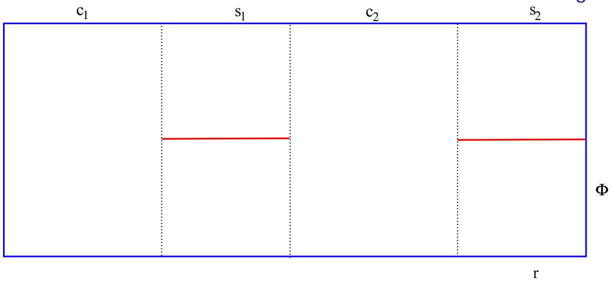
- if $I_f \neq 0$ then $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ where $D_f = \frac{4+3\log 3}{4-3\log 3} c^* I_f^2$



Superdiffusion in the ∞H Lorentz gas

- Collision map: EDC for Hölder – Chernov 1999
- an observable of particular interest: $\mathbf{L}(x)$ free flight (vector) – neither Hölder, nor in L^2
- $\frac{S_n \mathbf{L}}{\sqrt{n \log n}} \stackrel{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, D_{\mathbf{L}})$ – Szász & Varjú 2006
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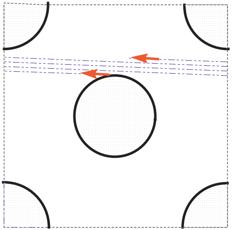

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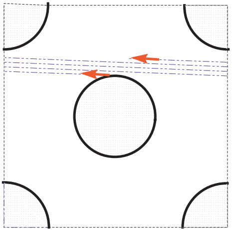

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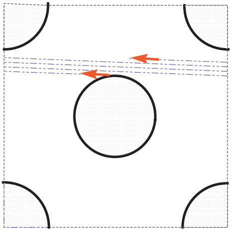

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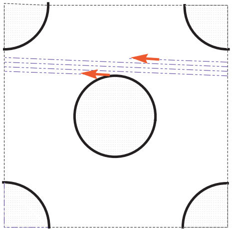

- $x \in M$ w.r.t μ
- $S_t(x)$ unfold trajectory to $\mathbb{R}^2$
- $\frac{S_t}{\sqrt{t \log t}} \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}(0, D)$,
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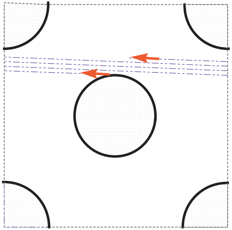

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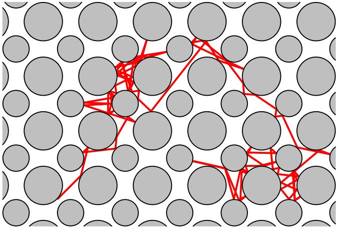

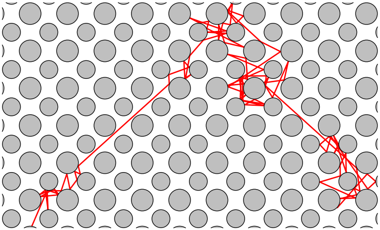

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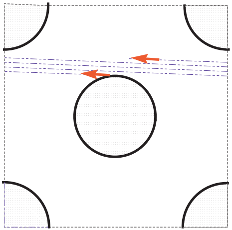

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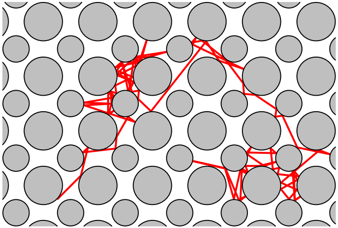

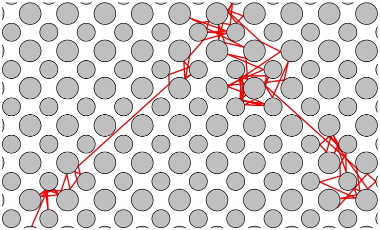

Convergence in distribution vs. moments

Let m_q denote the qth abs. moment of the standard Gaussian.

Consider (M, T, μ) ergodic and $f : M \to \mathbb{R}$ integrable such that $\frac{S_n f}{a_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$,

(say $a_n = \sqrt{D_f n}$ or $a_n = \sqrt{D_f n \log n}$).

Does that mean $\mu\left(\left|\frac{S_n f}{a_n}\right|^q\right) \to m_q$?

Melbourne & Török, 2011 yes, if

- (M, T, μ) can be modeled by a Young tower with $n^{-(\beta+1)}$ tails, $\beta > 1$ (standard CLT case)
- f Hölder, $q < 2\beta$.
- for $q > 2\beta$: $\exists C$ s.t. $C^{-1} n^{q-\beta} \le \mu(|S_n f|^q) \le C n^{q-\beta}$,
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Cusps: $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$.

Theorem (Chernov, Dolgopyat & B. 2012)

$\mu((S_n f)^2) = 2D_f n \log n(1 + o(1))$.

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numerics and heuristics; Dettmann 2012

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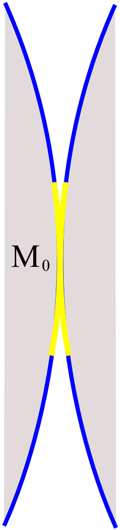
The first return map



Let $\hat{M} = M \setminus M_0$ where M_0 is a fixed small nbd. of the cusp.

- $\hat{T} : \hat{M} \to \hat{M}$ first return map
- $R : \hat{M} \to \mathbb{N}$ unbounded return time
- $\hat{f}(x) = \sum_{k=0}^{R(x)-1} f(T^k x)$ induced observable

limit law for $\hat{S}_n \hat{f}$ implies limit law for $S_n f$ (Melbourne & Török '04, Gouëzel '04)

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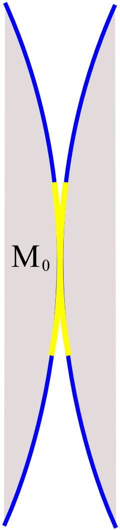

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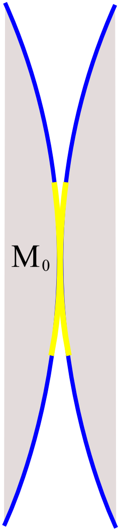

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Lemma (C1)

The map $\hat{T} : \hat{M} \to \hat{M}$ is uniformly hyperbolic and it satisfies the Growth Lemma ("Expansion prevails fractioning")

so that (via e.g Young tower or standard pairs)
EDC for Hölder observables

Lemma (C2)

$|\hat{\mu}(\hat{f} \cdot \hat{f} \circ \hat{T}^n)| \leq Ce^{-\alpha n}$ *with* $C > 0, \alpha < 1$ *for* $n \geq 1$
Not for $n = 0$ *as* $\hat{f}$ *is not Hölder and not in* L^2

Summarizing: the sequence $\hat{f} \circ \hat{T}^n$ behaves almost like an i.i.d. sequence

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- $M_n = \{x \in \hat{M} | R(x) = n\}$ n-cell
- $L_n = \bigcup_{j \le n} M_j$ low cells, $H_n = \bigcup_{j > n} M_j$ high cells

Lemma (C3)

- $\hat{f}|_{M_n} = nI(1 + o(1))$

  (recall $I = c_1 \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi))\sqrt{\cos(\phi)} d\phi$*)*
- $\hat{\mu}(H_n) = \frac{c_2}{n^2}(1 + o(1))$ *(here c_1, c_2 are numerical constants)*
- *hence* $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}|_{L_n}) = 2 \log n D_{\hat{f}} (1 + o(1))$

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Estimation of the characteristic function

Szász & Varjú; Gouëzel & B. 2006 : Young towers

- Truncation   Let $\bar{f} = \hat{f} \cdot \mathbf{1}|_{L_{\sqrt{n}\log\log n}}$

$$\mathbb{P}(\exists j \leq n : \hat{f} \circ \hat{T}^j \neq \bar{f} \circ \hat{T}^j) \leq Cn \left(\frac{1}{\sqrt{n}\log\log n} \right)^2 \to 0$$

- Decorrelation: Bernstein's big-small block technique.
- need to treat $\mathbb{E}\left(\exp(\frac{it\hat{S}_n\bar{f}}{\sqrt{n\log n}}) \right)$:
  - Taylor approximation
  - moment estimates
  - Cauchy-Schwartz inequalities

Recall: $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}|_{L_n}) = 2\log n D_{\hat{f}}(1 + o(1))$

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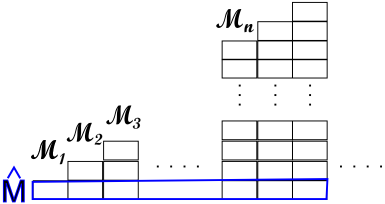

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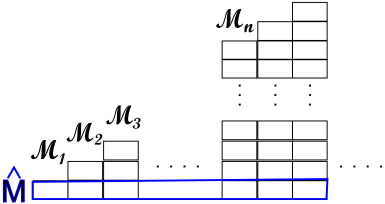

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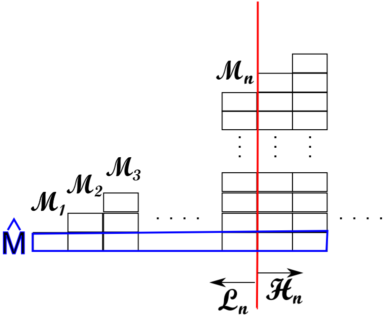

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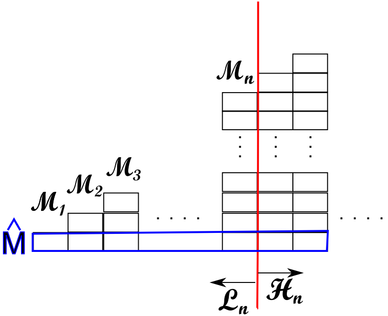

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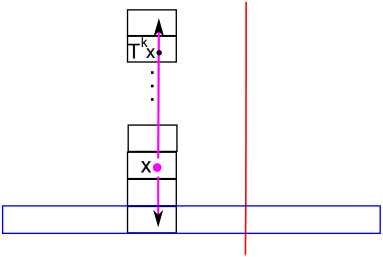

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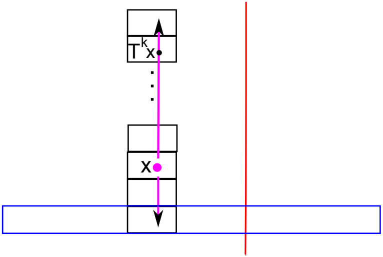

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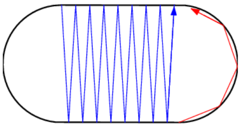
Limit law: $\frac{S(T)}{\sqrt{T\log T}} \stackrel{\mathcal{D}}{\Longrightarrow} \mathcal{N}\left(0, \frac{b}{m}\right)$

Second moment: $\mathbb{E}\left(S^2(T)\right) = 2\iint_{0<s<t<T} \mathbb{E}(\xi(s)\xi(t))\,ds\,dt$.

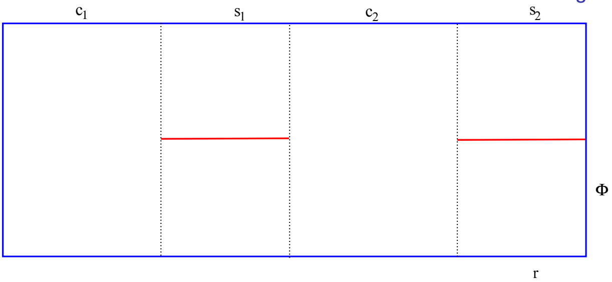
$\mathbb{E}(\xi(s)\xi(t)) = \mathbb{P}\big(H(s) > t - s\big) = \mathbb{P}(T_0 > t - s)$.

$\mathbb{E}\left(S^2(T)\right) \sim 2\int_{0<s<T} \frac{b}{m}\log(T-s)\,ds \sim \frac{2b}{m} T\log T$.

Superdiffusion in the straight stadium

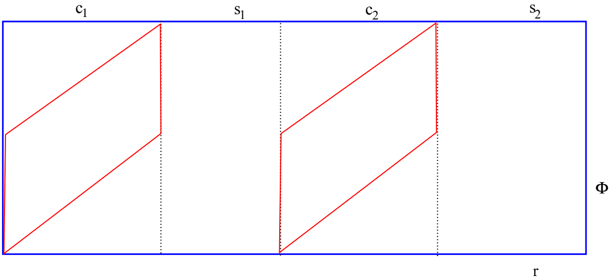


- Gouëzel & B. 2006. $f : M \to \mathbb{R}$, $\mu(f) = 0$.
- Let $I_f = \int_{S_1 \cup S_2} f(r, \frac{\pi}{2}) dr$.
- if $I_f \neq 0$ then $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ where $D_f = \frac{4+3\log 3}{4-3\log 3} c^* I_f^2$



Why $\frac{4+3\log 3}{4-3\log 3}$?

- $\hat{M}$: leaving one of the semicircular arcs.
- in cusp or infinite horizon: $E(R(Tx)|R(x)=K) = c\sqrt{K}(1+o(1))$
- in stadium: $E(R(Tx)|R(x)=K) = \alpha K(1+o(1))$ for some $\alpha < 1$, computable $\Longrightarrow$ i.i.d. clusters



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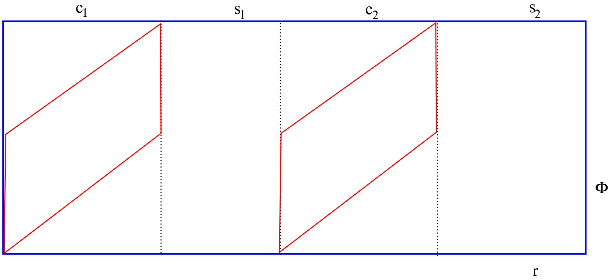

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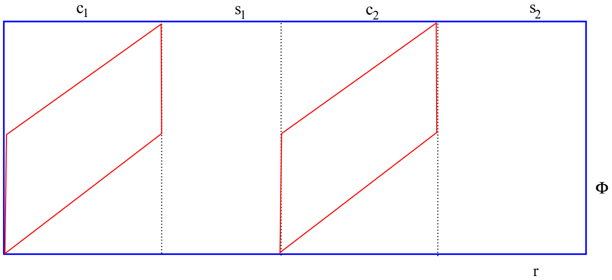

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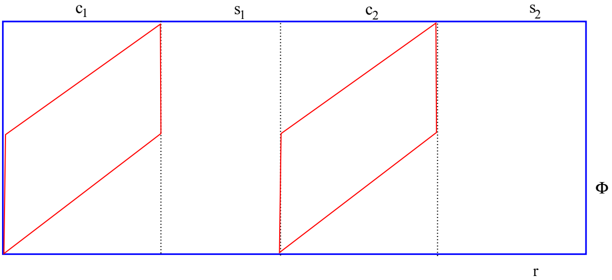
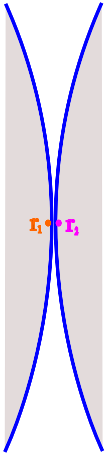


Superdiffusion in dispersing billiards with cusps



Theorem (Chernov, Dolgopyat & B. 2011)

- *Denote by* $r_1 \in C_1$ *and* $r_2 \in C_2$ *the two points that make the cusp.*
- *Let* $I_f = \int_{-\pi/2}^{\pi/2} (f(r_1,\phi) + f(r_2,\phi))\rho(\phi)d\phi$
  with $\rho(\phi) = \frac{\sqrt{\cos\phi}}{\int_{-\pi/2}^{\pi/2} \sqrt{\cos\phi}d\phi}$
- *if* $I_f \neq 0$ *then* $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$
  where $D_f = c^* I_f^2$ *and* c^* *is some numerical constant.*
- *if* $I_f = 0$ *then* $S_n f$ *satisfies standard CLT.*

Corner series

For simplicity assume that C_1 and C_2 are circles of radius 1.

Coordinates: α distance from cusp, $\gamma = \frac{\pi}{2} - \phi$

- while going down the cusp: α decreases, $\gamma : 0 \longrightarrow \frac{\pi}{2}$
- while coming out of the cusp: α increases, $\gamma : \frac{\pi}{2} \longrightarrow \pi$

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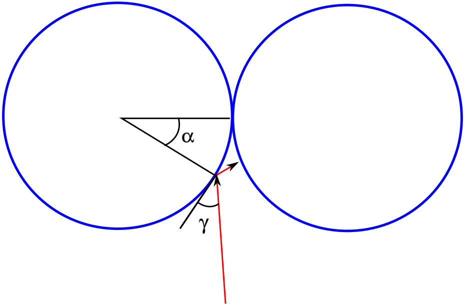

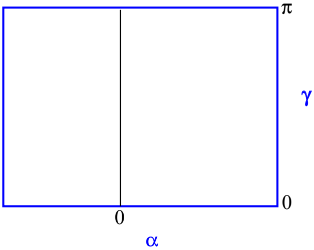

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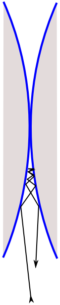

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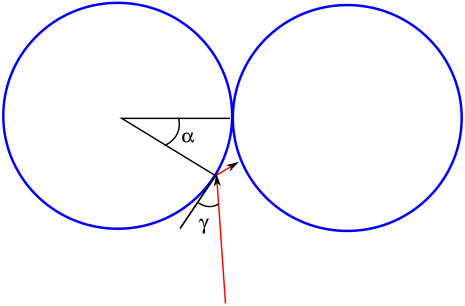
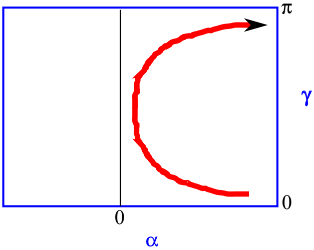
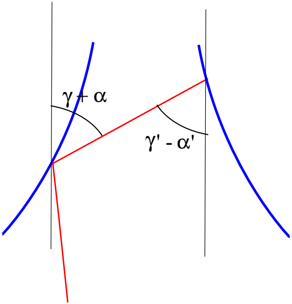




Equations of motion



$$\gamma' - \alpha' = \alpha + \gamma$$

$b = \sin\alpha - \sin\alpha'$;
$a = 2 - \cos\alpha - \cos\alpha'$
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$a = b\tan(\alpha + \gamma)$

$\sin\alpha' - \sin\alpha = -\frac{2 - \cos\alpha' - \cos\alpha}{\tan(\alpha+\gamma)}$

- Throughout the corner series: $\alpha \ll 1$, $\alpha < \gamma$;
- in a "large part" of the corner series: $\alpha \ll \gamma$.

$$\gamma' - \gamma \approx 2\alpha; \qquad \alpha' - \alpha \approx -\frac{\alpha^2}{\tan(\gamma)}.$$

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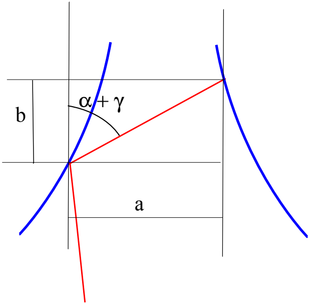

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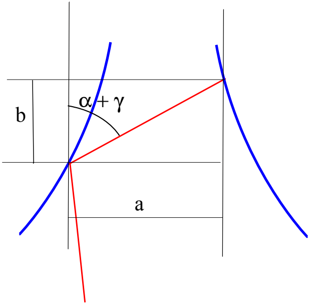

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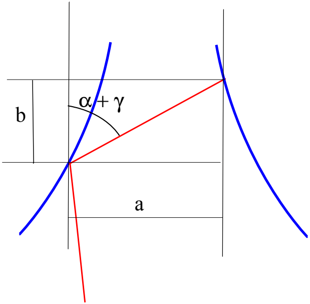

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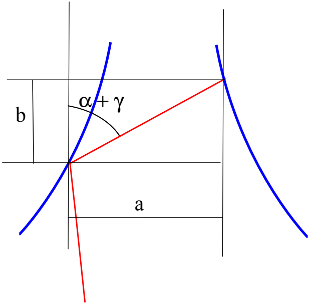

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$\dot\gamma = 2\alpha; \qquad \dot\alpha = -\frac{\alpha^2}{\tan(\gamma)}.$

$J = \alpha^2 \sin\gamma$ is first integral, so $\dot\gamma = 2\sqrt{\frac{J}{\sin\gamma}}$, $dt = \frac{2\sqrt{\sin\gamma}}{\sqrt{J}} d\gamma$

proportion of time between γ_1 and γ_2 $\asymp \int_{\gamma_1}^{\gamma_2} \sqrt{\sin\gamma}\, d\gamma$.

Recall $I_f = c \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi)) \sqrt{\cos(\phi)}\, d\phi$.

length of the excursion $R = cJ^{-\frac{1}{2}} \int_0^{\pi} \sqrt{\sin\gamma}\, d\gamma = cJ^{-\frac{1}{2}}$

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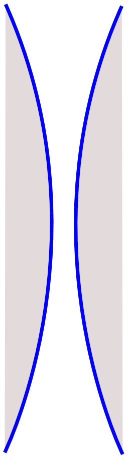
Dispersing billiards with tunnels



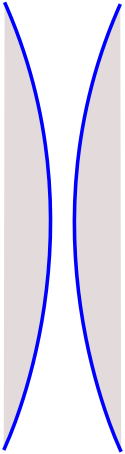
Work in progress

Denote be $T_\varepsilon : M \to M$ the billiard map

same phase space, same $f : M \to \mathbb{R}$

- *for fixed $\varepsilon > 0$ this is a Sinai billiard, hence CLT:*
- $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_{f,\varepsilon})$ *with*
- $D_{f,\varepsilon} = D_f |\log \varepsilon|(1 + o(1))$

Dispersing billiards with tunnels



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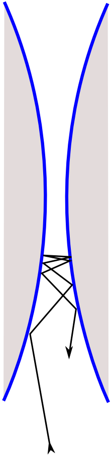

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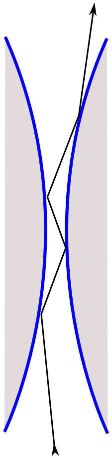

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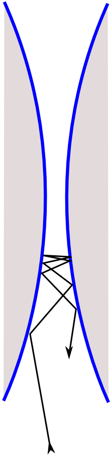

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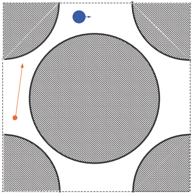
Motivation

1. Brownian Brownian motion – Chernov & Dolgopyat '09



$m \ll M$ (separation of time scales)

SDE for large particle:

$dV = \sigma_Q(f)dW$

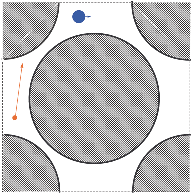
collisions of the heavy particle with the wall?

2. Triangular lattice with small opening

How does the planar diffusion depend on ε?

Motivation

1. Brownian Brownian motion – Chernov & Dolgopyat '09



$m \ll M$ (separation of time scales)

SDE for large particle:

$dV = \sigma_Q(f)dW$

collisions of the heavy particle with the wall?

2. Triangular lattice with small opening

How does the planar diffusion depend on ε?

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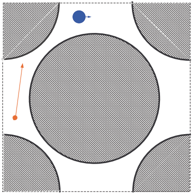

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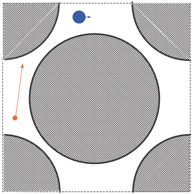

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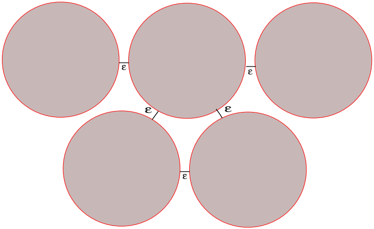
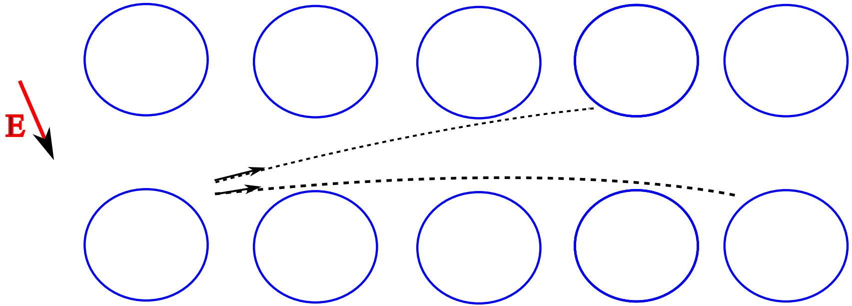


How does the planar diffusion depend on ε?

Infinite horizon with field I

- Add field $\mathbf{E}$ transversal to corridors, $|\mathbf{E}| = \varepsilon \ll 1$
- $+$ thermostating: Gaussian $\dot{\mathbf{v}} = \mathbf{E} - \langle \mathbf{E}, \mathbf{v} \rangle \mathbf{v}$
- free flight $\mathbf{L}_\varepsilon \leq \frac{C}{\sqrt{\varepsilon}}$ is bounded, but depends on ε .



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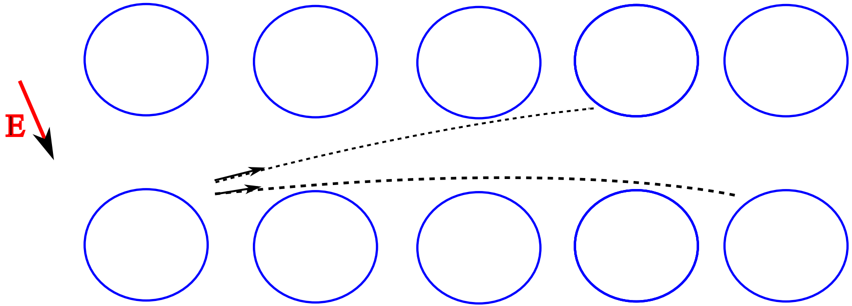

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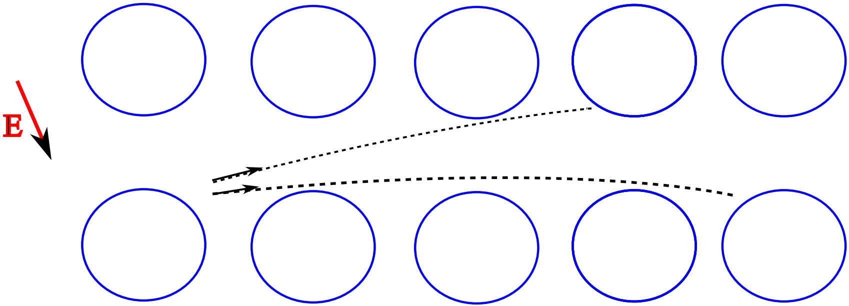

Infinite horizon with field II

Chernov-Dolgopyat 2009:

- SRB measure (non-equilibrium steady state) μ_ε
- current $\mathbf{J} = \mu_\varepsilon(L_\varepsilon) = \frac{1}{2}|\log\varepsilon|\mathbf{D_L E} + \mathcal{O}(\varepsilon)$
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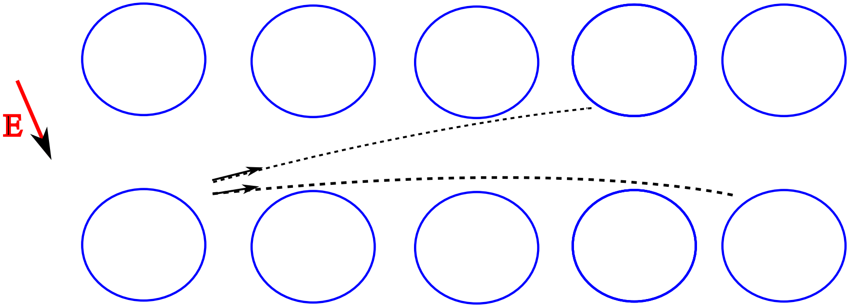

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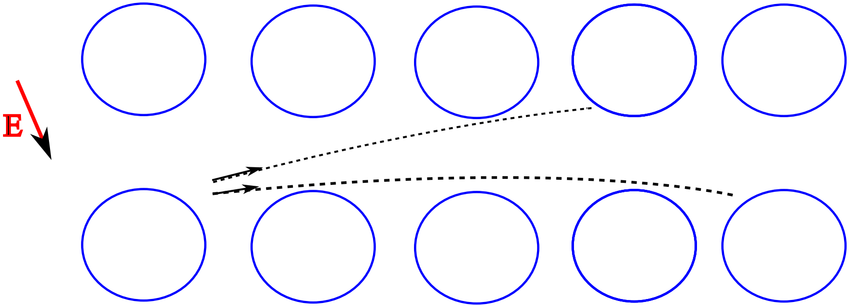

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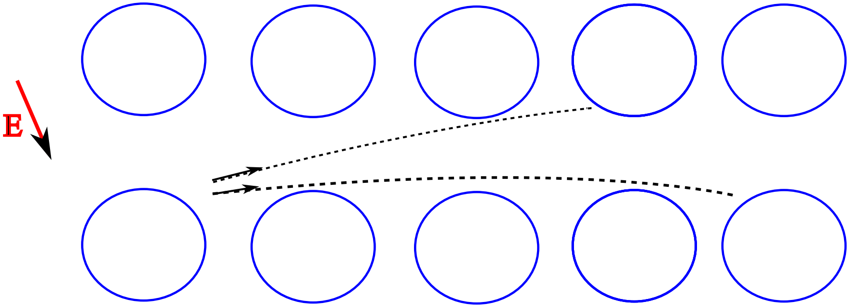
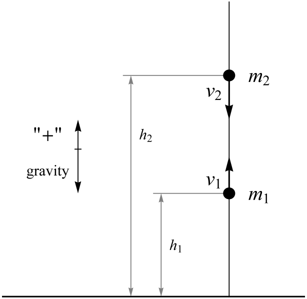


The system of two falling balls



Wojtkowski 1990, Wojtkowski & Liverani 1995

- $m_1 = m_2$ integrable
- $m_1 < m_2$ elliptic periodic orbit
- $m_1 > m_2$ hyperbolic, ergodic

Collision map: lower ball on the floor

Theorem (Borbély, Némedy Varga & B.)

For an a open set of mass ratios $\frac{m_1}{m_2} > 1$:

- *PDC* $C_n(f,g) \leq C \frac{(\log n)^3}{n^2}$,
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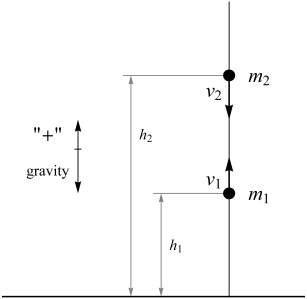

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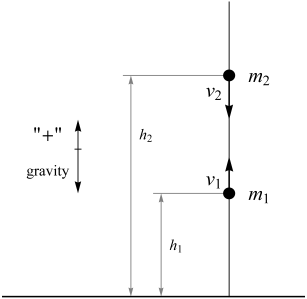

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Dispersing billiards with cusps:

- $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ with explicit D_f;
- $\mu((S_n f)^2) \sim 2D_f$.

analogous systems: stadia, ∞H Lorentz gas.

Questions (work in progress):

- extend second moment result,
- dispersing billiards with tunnels,
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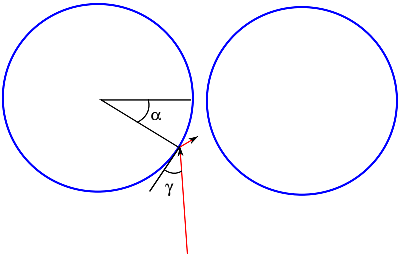
Corner series for tunnel

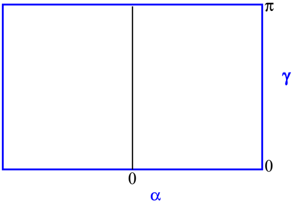
Coordinates: α, γ as for cusp

$$\gamma' - \alpha' = \alpha + \gamma$$

$$a = 2 - \cos\alpha - \cos\alpha' + \varepsilon$$

$$\sin\alpha' - \sin\alpha = -\frac{2 - \cos\alpha' - \cos\alpha + \varepsilon}{\tan(\alpha + \gamma)}$$





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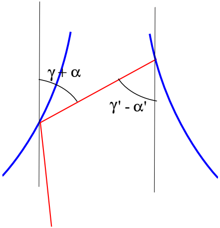

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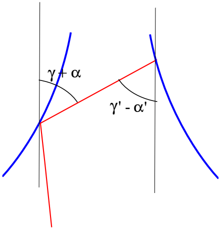

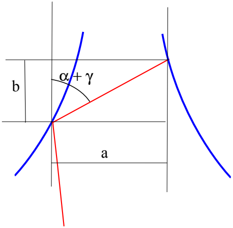

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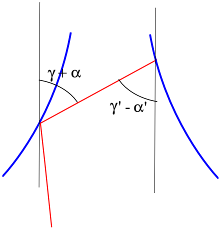

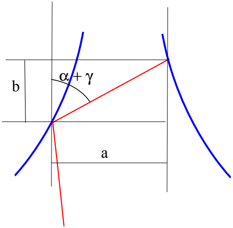

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Flow approximation for tunnel

$$\dot{\gamma} = 2\alpha; \qquad \dot{\alpha} = -\frac{\alpha^2 + \varepsilon}{\tan(\gamma)}.$$

$J = (\alpha^2 + \varepsilon)\sin\gamma$ is first integral, so $\dot{\gamma} = 2\alpha = \pm 2\sqrt{\frac{J}{\sin\gamma} - \varepsilon}$.

Fix some small δ_0. We distinguish three cases:

$$J > \varepsilon/\delta_0, \qquad J < \delta_0\varepsilon \quad \text{and} \quad J/\varepsilon \approx 1.$$

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- $\alpha > 0$ and $\alpha^2 \gg \varepsilon$ throughout the excursion
- cusp estimates apply, however $R = CJ^{-1/2} \leq \frac{C}{\sqrt{\varepsilon}}$

Contribution to the variance: $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}_{L_{\frac{C}{\sqrt{\varepsilon}}}}) = D_{\hat{f}}|\log\varepsilon|$

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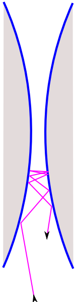

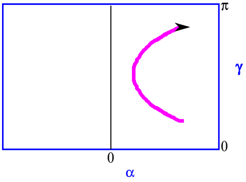

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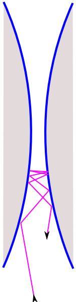

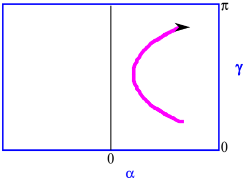

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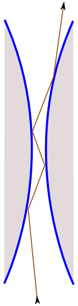

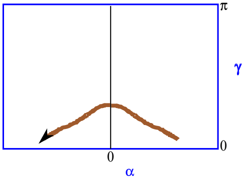

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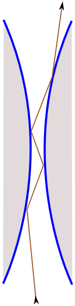

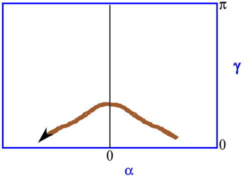

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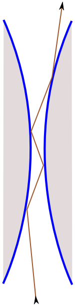

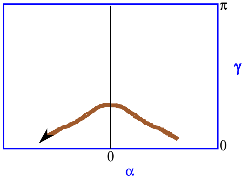

The third case

What is in between?

$\alpha = 0, \gamma = \pi/2$ is a hyperbolic fixed point (period two orbit)

Saddle case: if $J \approx \varepsilon$, R can be arbitrary large, however, it is dominated by the hyperbolic periodic orbit

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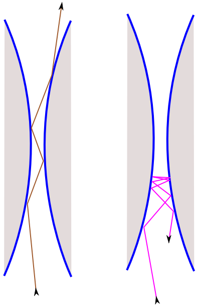

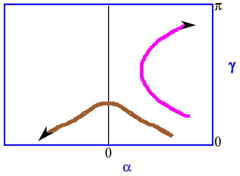

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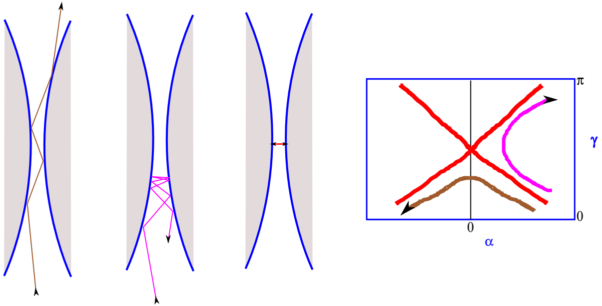

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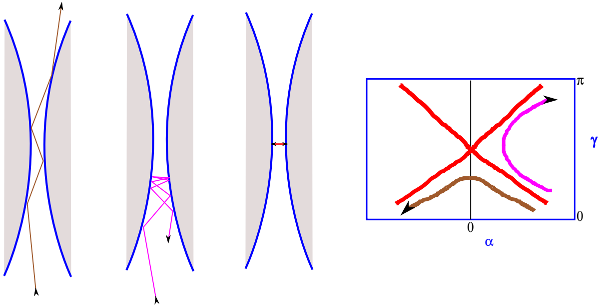

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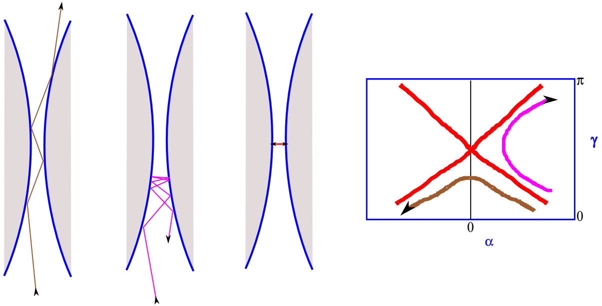
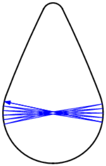


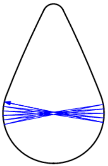
Stadium – what is ε?



skewed stadia: similar, bouncing $\Longrightarrow$ diametrical

Numerics and heuristic reasoning: Ergodicity for large enough finite c (Halász, Sanders, Tahuilán, B. 2011)

Stadium – what is ε?



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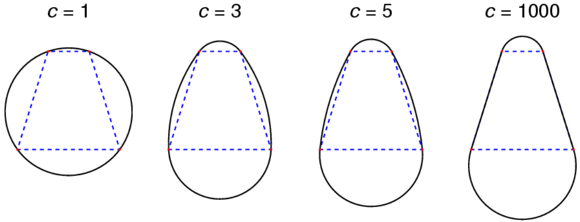

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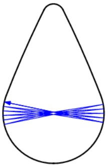

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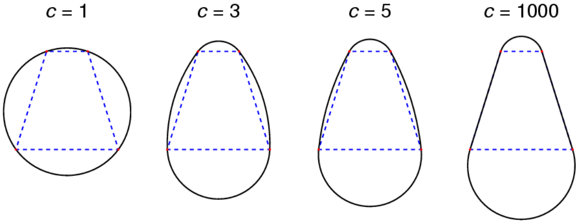

Numerics and heuristic reasoning: Ergodicity for large enough finite c (Halász, Sanders, Tahuilán, B. 2011)