Two special invariant measures for the random β-transformation

Karma Dajani

April 12, 2011

1. Definition of K_β, the random β-transformation

2. Unique measure of maximal entropy

3. A K_β-invariant measure of product type

4. A special Pisot case

Terminology

Let $\beta > 1$ be a non-integer. By a β-expansion we mean an expression of the form

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i},$$

with $b_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$.

Definition of K_β

Roughly, K_β is obtained by randomizing the greedy map, and the lazy map.

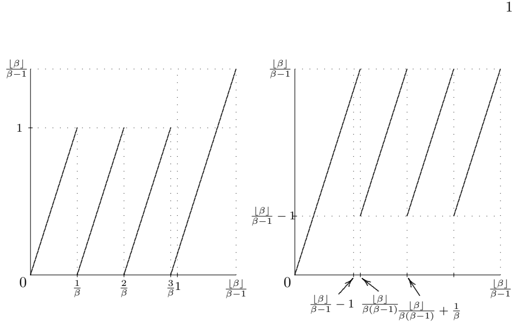

Figure 1: The greedy map T_β (left), and lazy map L_β (right). Here $\beta = \pi$.

Definition of K_β

If we take the common refinement, or simply superimpose the two maps, we get the following picture on $[0, \lfloor\beta\rfloor/(\beta - 1)]$.

1

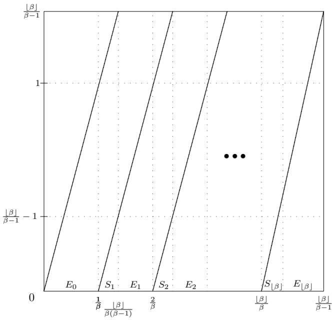

Figure 1: The greedy and lazy maps, and their switch regions.

Definition of K_β: a special partition

We get a partition of the interval $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ into $\lfloor\beta\rfloor$ switch regions, $S_1, \ldots, S_{\lfloor\beta\rfloor}$, and $\lfloor\beta\rfloor + 1$ uniqueness regions, $E_0, \ldots, E_{\lfloor\beta\rfloor}$, where

$$E_0 = \left[0, \frac{1}{\beta}\right), \quad E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor - 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right],$$

$$E_k = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta}\right), \quad k = 1, \ldots, \lfloor\beta\rfloor - 1,$$

$$S_k = \left[\frac{k}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}\right], \quad k = 1, \ldots, \lfloor\beta\rfloor.$$

Definition of K_β: a special partition

- On S_k, the greedy map assigns the digit k, while the lazy map assigns the digit $k-1$. On E_k both maps assign the same digit k.
- We use a random rule by flipping a coin every time the orbit is in the switch region.

Definition of K_β: a special partition

- On S_k, the greedy map assigns the digit k, while the lazy map assigns the digit $k-1$. On E_k both maps assign the same digit k.
- We use a random rule by flipping a coin every time the orbit is in the switch region.

The map K_β

Consider $\Omega = \{0,1\}^{\mathbb{N}}$ with product σ-algebra. Let $\sigma : \Omega \to \Omega$ be the left shift. Define $K = K_\beta : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$K(\omega, x) = \begin{cases} (\omega, \beta x - \ell) & x \in E_\ell,\ \ell = 0, 1, \ldots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - \ell) & x \in S_\ell \text{ and } \omega_1 = 1,\ \ell = 1, \ldots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - \ell + 1) & x \in S_\ell \text{ and } \omega_1 = 0,\ \ell = 1, \ldots, \lfloor\beta\rfloor. \end{cases}$$

Random Digits

Let

$$d_1 = d_1(\omega, x) = \begin{cases} \ell & \text{if } x \in E_\ell,\ \ell = 0, 1, \ldots, \lfloor\beta\rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_\ell,\ \ell = 1, 2, \ldots, \lfloor\beta\rfloor, \\ \ell - 1 & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_\ell,\ \ell = 1, 2, \ldots, \lfloor\beta\rfloor, \end{cases}$$

then

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - d_1) & \text{if } x \in E, \\ (\sigma(\omega), \beta x - d_1) & \text{if } x \in S. \end{cases}$$

Set $d_n = d_n(\omega, x) = d_1\left(K_\beta^{n-1}(\omega, x)\right)$, and let $\pi_2 : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$ be the canonical projection onto the second coordinate. Then

$$\pi_2\left(K_\beta^n(\omega, x)\right) = \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

rewriting gives

$$x = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_2\left(K_\beta^n(\omega, x)\right)}{\beta^n}.$$

Since $\pi_2\left(K_\beta^n(\omega, x)\right) \in [0, \lfloor\beta\rfloor/(\beta-1)]$, it follows that

$$\left| x - \sum_{i=1}^{n} \frac{d_i}{\beta^i} \right| = \frac{\pi_2\left(K_\beta^n(\omega, x)\right)}{\beta^n} \to 0 \qquad \text{as} \quad n \to \infty.$$

Random β-expansions

Theorem

(D. + de Vries) Suppose $x \in [0, \lfloor\beta\rfloor/(\beta-1)]$ *can be written as*

$$x = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_n}{\beta^n} + \cdots,$$

with $b_i \in \{0, 1, \cdots, \lfloor\beta\rfloor\}$*. Then, there exists* $\omega \in \Omega$ *such that* $b_n = d_n(\omega, x) = d_1\left(K_\beta^{n-1}(\omega, x)\right)$ *for all* $n \geq 1$*.*

Random β-expansions

Theorem

(D. + de Vries) Suppose $x \in [0, \lfloor\beta\rfloor/(\beta-1)]$ *can be written as*

$$x = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_n}{\beta^n} + \cdots ,$$

with $b_i \in \{0, 1, \cdots, \lfloor\beta\rfloor\}$. *Then, there exists* $\omega \in \Omega$ *such that* $b_n = d_n(\omega, x) = d_1\left(K_\beta^{n-1}(\omega, x)\right)$ *for all* $n \geq 1$.

The proof relies on the behavior of the sequence $\{x_n = \sum_{i=1}^{\infty} \frac{b_{n-1+i}}{\beta^i} : n \geq 1\}$. If the set $N(x) = \{n : x_n \in S\}$ is infinite, then there is a unique $\omega \in \Omega$ such that $b_n = d_n(\omega, x)$. If $N(x)$ is finite, then there are uncountably many $\omega \in \Omega$ such that $b_n = d_n(\omega, x)$.

Unique measure of maximal entropy

The measure of maximal entropy is basically obtained by identifying K_β with the full shift on $(\lfloor \beta \rfloor + 1)$ symbols with the uniform product measure. It is easy to see that the full shift is a *factor* (no measure yet) of K_β

Unique measure of maximal entropy

- Consider the Bernoulli shift $(D, \mathcal{F}, \mathbb{P}, \sigma')$, where $D = \{0, 1, \cdots, \lfloor\beta\rfloor\}^{\mathbb{N}}$, $\mathcal{F}$ the product σ-algebra, $\mathbb{P}$ the uniform product measure, and σ' the left shift.
- Define $\phi : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$\phi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \cdots,) .$$

- Then, ϕ is (i) measurable, (ii) surjective and (iii) $\phi \circ K_\beta = \sigma' \circ \phi$.
- ϕ is **not** invertible.

Unique measure of maximal entropy

- Consider the Bernoulli shift $(D, \mathcal{F}, \mathbb{P}, \sigma')$, where $D = \{0, 1, \cdots, \lfloor \beta \rfloor\}^{\mathbb{N}}$, $\mathcal{F}$ the product σ-algebra, $\mathbb{P}$ the uniform product measure, and σ' the left shift.
- Define $\phi : \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)] \to D$ by

$$\phi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \cdots,) .$$

- Then, ϕ is (i) measurable, (ii) surjective and (iii) $\phi \circ K_\beta = \sigma' \circ \phi$.
- ϕ is **not** invertible.

Unique measure of maximal entropy

- Consider the Bernoulli shift $(D, \mathcal{F}, \mathbb{P}, \sigma')$, where $D = \{0, 1, \cdots, \lfloor\beta\rfloor\}^{\mathbb{N}}$, $\mathcal{F}$ the product σ-algebra, $\mathbb{P}$ the uniform product measure, and σ' the left shift.
- Define $\phi : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$\phi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \cdots,) .$$

- Then, ϕ is (i) measurable, (ii) surjective and (iii) $\phi \circ K_\beta = \sigma' \circ \phi$.
- ϕ is **not** invertible.

Unique measure of maximal entropy

- Consider the Bernoulli shift $(D, \mathcal{F}, \mathbb{P}, \sigma')$, where $D = \{0, 1, \cdots, \lfloor\beta\rfloor\}^{\mathbb{N}}$, $\mathcal{F}$ the product σ-algebra, $\mathbb{P}$ the uniform product measure, and σ' the left shift.
- Define $\phi : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$\phi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \cdots,).$$

- Then, ϕ is (i) measurable, (ii) surjective and (iii) $\phi \circ K_\beta = \sigma' \circ \phi$.
- ϕ is **not** invertible.

- Let

$$\begin{aligned} Z &= \{(\omega, x) \in \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)] : \pi(K_\beta^n(\omega, x)) \in S \text{ i.o.}\}, \\ D' &= \{(b_1, b_2, \ldots) \in D : \sum_{i=1}^{\infty} \frac{b_{j+i-1}}{\beta^i} \in S \text{ for infinitely many } j\text{'s}\}. \end{aligned}$$

- Then, (i) $\phi(Z) = D'$, (ii) $K_\beta^{-1}(Z) = Z$, and (iii) $(\sigma')^{-1}(D') = D'$.

Unique measure of maximal entropy

- Let

$$\begin{aligned} Z &= \{(\omega,x)\in\Omega\times[0,\lfloor\beta\rfloor/(\beta-1)] : \pi(K_\beta^n(\omega,x))\in S \text{ i.o.}\}, \\ D' &= \{(b_1,b_2,\dots)\in D : \sum_{i=1}^{\infty}\frac{b_{j+i-1}}{\beta^i}\in S \text{ for infinitely many } j\text{'s}\}. \end{aligned}$$

- Then, (i) $\phi(Z)=D'$, (ii) $K_\beta^{-1}(Z)=Z$, and (iii) $(\sigma')^{-1}(D')=D'$.

Unique measure of maximal entropy

Lemma

(D. de Vries) Let ϕ' be the restriction of ϕ to Z, then $\phi' : Z \to D'$ is a measurable bijection, and $\mathbb{P}(D') = 1$.

Unique measure of maximal entropy

Lemma

(D. de Vries) Let ϕ' be the restriction of ϕ to Z, then $\phi' : Z \to D'$ is a measurable bijection, and $\mathbb{P}(D') = 1$.

Define the K_β-invariant measure ν_β by $\nu_\beta(A) = \mathbb{P}(\phi'(Z \cap A))$.

Unique measure of maximal entropy

Lemma

(D. de Vries) Let ϕ' be the restriction of ϕ to Z, then $\phi' : Z \to D'$ is a measurable bijection, and $\mathbb{P}(D') = 1$.

Define the K-invariant measure ν_β by $\nu({}_\beta A) = \mathbb{P}(\phi'(Z \cap A))$. Then,

Theorem

(D.+ de Vries) Let $\beta > 1$ be a non-integer. Then the dynamical systems $(\Omega \times [0, \lfloor\beta\rfloor/(\beta-1)], \nu_\beta, K_\beta)$ and $(D, \mathbb{P}, \sigma')$ are measurably isomorphic.

Unique measure of maximal entropy

A consequence of the above theorem is that among all the K_β-invariant measures with support Z, ν_β has the largest entropy, namely $\log(1 + \lfloor\beta\rfloor)$. It is the only one with support Z, and this value of the entropy.

Unique measure of maximal entropy

A consequence of the above theorem is that among all the K_β-invariant measures with support Z, ν_β has the largest entropy, namely $\log(1+\lfloor\beta\rfloor)$. It is the only one with support Z, and this value of the entropy.
In fact we have more,

Lemma

(D.+ de Vries) Let μ be a K_β-invariant measure such that $\mu(Z^c)>0$, then $h_\mu(K_\beta)<\log(1+\lfloor\beta\rfloor)$.

Unique measure of maximal entropy

A consequence of the above theorem is that among all the K_β-invariant measures with support Z, ν_β has the largest entropy, namely $\log(1 + \lfloor \beta \rfloor)$. It is the only one with support Z, and this value of the entropy.
In fact we have more,

Lemma

(D.+ de Vries) Let μ be a K_β-invariant measure such that $\mu(Z^c) > 0$, then $h_\mu(K_\beta) < \log(1 + \lfloor \beta \rfloor)$.

This leads to

Theorem

(D.+ de Vries) ν_β is the unique K_β-invariant measure of maximal entropy.

The marginals

The projection of ν_β in the second coordinate is the Erdős measure:

$$\begin{array}{ccc} \Omega \times [0, \lfloor \beta \rfloor / (\beta - 1)] & \xrightarrow{\pi_2} & [0, \lfloor \beta \rfloor / (\beta - 1)] \\ & \searrow \phi & \Big\uparrow h \\ & & \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}} \end{array}$$

,
where $h(b_1, b_2, \ldots) = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$, i.e. $\nu_\beta \circ \pi_2^{-1}$ gives the distribution of the random variable h.

The marginals

Projection in the first coordinate need not be product measure, only in special cases, namely if the greedy expansion of 1 has the form

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots + \frac{a_n}{\beta^n},$$

with $a_1, \ldots, a_n > 0$. In this case the dynamics can be identified with (a symmetric) Markov chain (more later).

The marginals

Projection in the first coordinate need not be product measure, only in special cases, namely if the greedy expansion of 1 has the form

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots + \frac{a_n}{\beta^n},$$

with $a_1, \ldots, a_n > 0$. In this case the dynamics can be identified with (a symmetric) Markov chain (more later).

However, the projection is *symmetric*:

$$\nu_\beta \circ \pi_1^{-1}(\{\omega_1 = i_1, \ldots, \omega_n = i_n\}) = \nu_\beta \circ \pi_1^{-1}(\{\omega_1 = 1 - i_1, \ldots, \omega_n = 1 - i_n\}).$$

The marginals

Projection in the first coordinate need not be product measure, only in special cases, namely if the greedy expansion of 1 has the form

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots + \frac{a_n}{\beta^n},$$

with $a_1, \ldots, a_n > 0$. In this case the dynamics can be identified with (a symmetric) Markov chain (more later).

However, the projection is *symmetric*:

$$\nu_\beta \circ \pi_1^{-1}(\{\omega_1 = i_1, \ldots, \omega_n = i_n\}) = \nu_\beta \circ \pi_1^{-1}(\{\omega_1 = 1 - i_1, \ldots, \omega_n = 1 - i_n\}).$$

As a consequence,

$$\nu_\beta \circ \pi_1^{-1}(\{\omega_1 = 1\}) = \nu_\beta \circ \pi_1^{-1}(\{\omega_1 = 0\}) = 1/2.$$

The invariant measure $m_p \times \mu_{\beta,p}$

For $0 < p < 1$, let m_p be the $(p, 1-p)$ product measure on Ω.

The invariant measure $m_p \times \mu_{\beta,p}$

For $0 < p < 1$, let m_p be the $(p, 1-p)$ product measure on Ω.

Theorem

(D.+de Vries) There exists a probability measure $\mu_{\beta,p}$ on $[0, \lfloor\beta\rfloor/(\beta-1)]$ equivalent with Lebesgue measure λ such that

(i) *the measure $m_p \times \mu_{\beta,p}$ is K_β-invariant, ergodic and is equivalent to $m_p \times \lambda$.*

(ii) *$\mu_{\beta,p}$ has density bounded away from 0.*

(iii) *$\mu_{\beta,p}$ satisfies*

$$\mu_{\beta,p} = p\mu_{\beta,p} \circ T_\beta^{-1} + (1-p)\mu_{\beta,p} \circ L_\beta^{-1}.$$

(iv) *Let ν_β be the measure of maximal entropy, then $\nu_\beta \neq m_p \times \mu_{\beta,p}$. As a consequence the two measures are mutually singular.*

The invariant measure $m_p \times \mu_{\beta,p}$

For $0 < p < 1$, let m_p be the $(p, 1-p)$ product measure on Ω.

Theorem

(D.+de Vries) There exists a probability measure $\mu_{\beta,p}$ on $[0, \lfloor\beta\rfloor/(\beta-1)]$ equivalent with Lebesgue measure λ such that

(i) *the measure $m_p \times \mu_{\beta,p}$ is K_β-invariant, ergodic and is equivalent to $m_p \times \lambda$.*

(ii) *$\mu_{\beta,p}$ has density bounded away from 0.*

(iii) *$\mu_{\beta,p}$ satisfies*

$$\mu_{\beta,p} = p\mu_{\beta,p} \circ T_\beta^{-1} + (1-p)\mu_{\beta,p} \circ L_\beta^{-1}.$$

(iv) *Let ν_β be the measure of maximal entropy, then $\nu_\beta \neq m_p \times \mu_{\beta,p}$. As a consequence the two measures are mutually singular.*

The invariant measure $m_p \times \mu_{\beta,p}$

For $0 < p < 1$, let m_p be the $(p, 1-p)$ product measure on Ω.

Theorem

(D.+de Vries) There exists a probability measure $\mu_{\beta,p}$ on $[0, \lfloor\beta\rfloor/(\beta-1)]$ equivalent with Lebesgue measure λ such that

(i) *the measure $m_p \times \mu_{\beta,p}$ is K_β-invariant, ergodic and is equivalent to $m_p \times \lambda$.*

(ii) *$\mu_{\beta,p}$ has density bounded away from 0.*

(iii) *$\mu_{\beta,p}$ satisfies*

$$\mu_{\beta,p} = p\mu_{\beta,p} \circ T_\beta^{-1} + (1-p)\mu_{\beta,p} \circ L_\beta^{-1}.$$

(iv) *Let ν_β be the measure of maximal entropy, then $\nu_\beta \neq m_p \times \mu_{\beta,p}$. As a consequence the two measures are mutually singular.*

The invariant measure $m_p \times \mu_{\beta,p}$

For $0 < p < 1$, let m_p be the $(p, 1-p)$ product measure on Ω.

Theorem

(D.+de Vries) There exists a probability measure $\mu_{\beta,p}$ on $[0, \lfloor\beta\rfloor/(\beta-1)]$ equivalent with Lebesgue measure λ such that

(i) *the measure $m_p \times \mu_{\beta,p}$ is K_β-invariant, ergodic and is equivalent to $m_p \times \lambda$.*

(ii) *$\mu_{\beta,p}$ has density bounded away from 0.*

(iii) *$\mu_{\beta,p}$ satisfies*

$$\mu_{\beta,p} = p\mu_{\beta,p} \circ T_\beta^{-1} + (1-p)\mu_{\beta,p} \circ L_\beta^{-1}.$$

(iv) *Let ν_β be the measure of maximal entropy, then $\nu_\beta \neq m_p \times \mu_{\beta,p}$. As a consequence the two measures are mutually singular.*

For $0 < p < 1$, let m_p be the $(p, 1-p)$ product measure on Ω.

Theorem

(D.+de Vries) There exists a probability measure $\mu_{\beta,p}$ on $[0, \lfloor\beta\rfloor/(\beta-1)]$ equivalent with Lebesgue measure λ such that

(i) *the measure $m_p \times \mu_{\beta,p}$ is K_β-invariant, ergodic and is equivalent to $m_p \times \lambda$.*

(ii) *$\mu_{\beta,p}$ has density bounded away from 0.*

(iii) *$\mu_{\beta,p}$ satisfies*

$$\mu_{\beta,p} = p\mu_{\beta,p} \circ T_\beta^{-1} + (1-p)\mu_{\beta,p} \circ L_\beta^{-1}.$$

(iv) *Let ν_β be the measure of maximal entropy, then $\nu_\beta \neq m_p \times \mu_{\beta,p}$. As a consequence the two measures are mutually singular.*

Comments on the construction

To construct $m_p \times \mu_{\beta,p}$, we use an intermediate transformation, namely the genuine skew product

$$R_\beta(\omega, x) = (\sigma\omega, T_{\omega_1}x),$$

where $T_0 = L_\beta$ and $T_1 = T_\beta$.

Some observations

Let μ be any Borel probability measure on $[0, \lfloor\beta\rfloor/(\beta - 1)]$. The following are equivalent:

- The measure $m_p \times \mu$ is R_β-invariant
- μ satisfies $\mu = p\,\mu \circ T_\beta^{-1} + (1-p)\,\mu \circ L_\beta^{-1}$
- The measure $m_p \times \mu$ is K_β-invariant

Some observations

Let μ be any Borel probability measure on $[0, \lfloor\beta\rfloor/(\beta-1)]$. The following are equivalent:

- The measure $m_p \times \mu$ is R_β-invariant
- μ satisfies $\mu = p\,\mu \circ T_\beta^{-1} + (1-p)\,\mu \circ L_\beta^{-1}$
- The measure $m_p \times \mu$ is K_β-invariant

Some observations

Let μ be any Borel probability measure on $[0, \lfloor\beta\rfloor/(\beta - 1)]$. The following are equivalent:

- The measure $m_p \times \mu$ is R_β-invariant
- μ satisfies $\mu = p\,\mu \circ T_\beta^{-1} + (1 - p)\,\mu \circ L_\beta^{-1}$
- The measure $m_p \times \mu$ is K_β-invariant

Some observations

Let μ be any Borel probability measure on $[0, \lfloor\beta\rfloor/(\beta - 1)]$. The following are equivalent:

- The measure $m_p \times \mu$ is R_β-invariant
- μ satisfies $\mu = p\,\mu \circ T_\beta^{-1} + (1-p)\,\mu \circ L_\beta^{-1}$
- The measure $m_p \times \mu$ is K_β-invariant

An R_β-invariant measure of product type

We consider a randomized version of the Perron-Frobenius operator defined for probability density functions:

$$Pf = p\,P_{T_\beta} + (1-p)\,P_{L_\beta},$$

where P_{T_β}, and P_{L_β} are the Perron-Frobenius operator of T_β and L_β respectively.

Theorem

(Pelikan) For any probability density f, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*$$

exists in L^1, and $Pf^* = f^*$.

An R_β-invariant measure of product type

Theorem

(Pelikan) For any probability density f, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*$$

exists in L^1,and $Pf^* = f^*$.

In particular, if we take $f = 1$, then $P1^* = 1^*$, and the measure defined by

$$\mu_{\beta,p}(A) = \int_A 1^* \, d\lambda$$

satisfies

$$\mu_{\beta,p} = p\,\mu_{\beta,p} \circ T_\beta^{-1} + (1-p)\,\mu_{\beta,p} \circ L_\beta^{-1}$$

i.e. $m_p \times \mu_{\beta,p}$ is R_β-invariant, and ergodic (follows from the fact that T_β and L_β are ergodic w.r.t. an absolutely continuous probability measure).

Ergodicity w.r.t. K_β

The measure $m_p \times \mu$ is K_β-invariant. Ergodicity follows from the following.

- Define $F : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to D$ by

  $$F(\omega, x) = \left(d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \ldots, \right).$$

  Then, $\sigma' \circ F = F \circ R_\beta$. Hence the measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is σ'-invariant and ergodic.
- The measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is concentrated on $\phi(Z) = D'$.
- Therefore, the measure ρ defined by $\rho(A) = m_p \times \mu_{\beta,p} \circ F^{-1}(\phi(A \cap Z))$ is K_β-invariant and ergodic.
- $\rho = m_p \times \mu_{\beta,p}$.

Ergodicity w.r.t. K_β

The measure $m_p \times \mu$ is K_β-invariant. Ergodicity follows from the following.

- Define $F : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$F(\omega, x) = \left(d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \ldots, \right).$$

  Then, $\sigma' \circ F = F \circ R_\beta$. Hence the measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is σ'-invariant and ergodic.

- The measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is concentrated on $\phi(Z) = D'$.
- Therefore, the measure ρ defined by $\rho(A) = m_p \times \mu_{\beta,p} \circ F^{-1}(\phi(A \cap Z))$ is K_β-invariant and ergodic.
- $\rho = m_p \times \mu_{\beta,p}$.

Ergodicity w.r.t. K_β

The measure $m_p \times \mu$ is K_β-invariant. Ergodicity follows from the following.

- Define $F : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$F(\omega, x) = \left(d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \ldots, \right).$$

  Then, $\sigma' \circ F = F \circ R_\beta$. Hence the measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is σ'-invariant and ergodic.

- The measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is concentrated on $\phi(Z) = D'$.
- Therefore, the measure ρ defined by $\rho(A) = m_p \times \mu_{\beta,p} \circ F^{-1}(\phi(A \cap Z))$ is K_β-invariant and ergodic.
- $\rho = m_p \times \mu_{\beta,p}$.

Ergodicity w.r.t. K_β

The measure $m_p \times \mu$ is K_β-invariant. Ergodicity follows from the following.

- Define $F : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$F(\omega, x) = \left(d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \ldots, \right).$$

  Then, $\sigma' \circ F = F \circ R_\beta$. Hence the measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is σ'-invariant and ergodic.
- The measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is concentrated on $\phi(Z) = D'$.
- Therefore, the measure ρ defined by $\rho(A) = m_p \times \mu_{\beta,p} \circ F^{-1}(\phi(A \cap Z))$ is K_β-invariant and ergodic.
- $\rho = m_p \times \mu_{\beta,p}$.

Ergodicity w.r.t. K_β

The measure $m_p \times \mu$ is K_β-invariant. Ergodicity follows from the following.

- Define $F : \Omega \times [0, \lfloor \beta \rfloor/(\beta - 1)] \to D$ by

$$F(\omega, x) = \left(d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \dots, \right).$$

  Then, $\sigma' \circ F = F \circ R_\beta$. Hence the measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is σ'-invariant and ergodic.
- The measure $m_p \times \mu_{\beta,p} \circ F^{-1}$ is concentrated on $\phi(Z) = D'$.
- Therefore, the measure ρ defined by $\rho(A) = m_p \times \mu_{\beta,p} \circ F^{-1}(\phi(A \cap Z))$ is K_β-invariant and ergodic.
- $\rho = m_p \times \mu_{\beta,p}$.

Comments on the mutual singularity of ν_β and $m_p \times \mu_{\beta,p}$

Since K_β is ergodic w.r.t. ν_β and $m_p \times \mu_{\beta,p}$, we only need to show that $\nu_\beta \neq m_p \times \mu_{\beta,p}$.

Comments on the mutual singularity of ν_β and $m_p \times \mu_{\beta,p}$

Since K_β is ergodic w.r.t. ν_β and $m_p \times \mu_{\beta,p}$, we only need to show that $\nu_\beta \neq m_p \times \mu_{\beta,p}$.

The answer is clear for $p \neq 1/2$ since

$$\nu_\beta(\{\omega_1 = 1\} \times [0, \lfloor\beta\rfloor/(\beta - 1)]) = 1/2,$$

while

$$m_p \times \mu_{\beta,p}(\{\omega_1 = 1\} \times [0, \lfloor\beta\rfloor/(\beta - 1)]) = p.$$

Comments on the mutual singularity of ν_β and $m_p \times \mu_{\beta,p}$

Assume $p = 1/2$. Choose n large enough so that $[1/\beta, 1/\beta + 1/\beta^n]) \subset S_1$. By symmetry of the measure ν_β we have

$$\nu_\beta(\{\omega_1 = 1\} \times [0, \lfloor\beta\rfloor/(\beta - 1)] | \Omega \times [1/\beta, 1/\beta + 1/\beta^n)) = 1/2.$$

Comments on the mutual singularity of ν_β and $m_p \times \mu_{\beta,p}$

Assume $p = 1/2$. Choose n large enough so that $[1/\beta, 1/\beta + 1/\beta^n]) \subset S_1$. By symmetry of the measure ν_β we have

$$\nu_\beta(\{\omega_1 = 1\} \times [0, \lfloor\beta\rfloor/(\beta-1)] | \Omega \times [1/\beta, 1/\beta + 1/\beta^n)) = 1/2.$$

On the other hand, if $\nu_\beta = m_p \times \mu_{\beta,p}$, then using the fact that ν_β is the uniform Bernoulli measure on the (random) digits, and that $\mu_{\beta,p}$ is bounded away from 0, we get

$$\nu_\beta(\{\omega_1 = 1\} \times [0, \lfloor\beta\rfloor/(\beta-1)] | \Omega \times [1/\beta, 1/\beta + 1/\beta^n)) \leq C \frac{\beta^n}{(1 + \lfloor\beta\rfloor)^n}$$

which tends to 0 as $n \to \infty$, leading to a contradiction.

Some consequences: uncountably many expansions

Ergodicity of $m_p \times \mu_{\beta,p}$ gives

$$m_p \times \mu_{\beta,p}(\{(\omega, x) : K_\beta^i(\omega, x) \in \Omega \times S \text{ i.o. }\}) = 1.$$

Ergodicity of $m_p \times \mu_{\beta,p}$ gives

$$m_p \times \mu_{\beta,p}(\{(\omega, x) : K^i_\beta(\omega, x) \in \Omega \times S \text{ i.o. }\}) = 1.$$

Thus, the set $G = \{x : x \text{ has a unique } \beta - \text{expansion}\}$ has Lebesgue measure 0.

Some consequences: uncountably many expansions

Ergodicity of $m_p \times \mu_{\beta,p}$ gives

$$m_p \times \mu_{\beta,p}(\{(\omega, x) : K_\beta^i(\omega, x) \in \Omega \times S \text{ i.o. }\}) = 1.$$

Thus, the set $G = \{x : x \text{ has a unique } \beta - \text{expansion}\}$ has Lebesgue measure 0. By non-singularity of the greedy and lazy maps, we get that the set

$$F = \bigcup_{n=1}^{\infty} \{x : T_{u_0} \circ T_{u_1} \circ \ldots \circ T_{u_n} \in G \text{ for some } u_1, \ldots, u_n\}$$

has Lebesgue measure zero, where $T_0 = L_\beta$ and $T_1 = T_\beta$.
For $x \notin F$ different elements of Ω lead to different expansions. Hence a.e. x has uncountably many β-expansions.

Some consequences: universal expansions

An expansion of x,

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i},$$

with digits in $A = \{0, 1, \cdots, \lfloor \beta \rfloor\}$ is called universal if every possible block of digits in A appears somewhere in the the above expansion of x.

Erdős and Komornik (1998) proved that there exists a $\beta_0 \in (1, 2)$ such that for each $\beta \in (1, \beta_0)$, every x has a universal expansion in base β.

Some consequences: universal expansions

An expansion of x,

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i},$$

with digits in $A = \{0, 1, \cdots, \lfloor \beta \rfloor\}$ is called universal if every possible block of digits in A appears somewhere in the the above expansion of x.

Erdős and Komornik (1998) proved that there exists a $\beta_0 \in (1, 2)$ such that for each $\beta \in (1, \beta_0)$, every x has a universal expansion in base β.

Sidorov (2003) showed that for each $\beta \in (1, 2)$, Lebesgue a.e. point has a universal expansion.

Some consequences: universal expansions

Using the ergodicity of the map K_β w.r.t the measure $m_p \times \mu_{\beta,p}$, together with the equivalence of the measure $\mu_{\beta,p}$ w.r.t. Lebesgue measure λ, one can show (using the Ergodic Theorem and Fubini) the following result.

Some consequences: universal expansions

Using the ergodicity of the map K_β w.r.t the measure $m_{\boldsymbol{p}} \times \mu_{\beta,\boldsymbol{p}}$, together with the equivalence of the measure $\mu_{\beta,\boldsymbol{p}}$ w.r.t. Lebesgue measure λ, one can show (using the Ergodic Theorem and Fubini) the following result.

Theorem

(D. de Vries) For any non-integer $\beta > 1$, and for λ a.e. $x \in [0, \lfloor \beta \rfloor / (\beta - 1)]$, there exists a continuum of universal expansions of x in base β.

Underlying Markov partition

For the rest of this talk, we assume that the greedy expansion of 1 has the form

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots + \frac{a_n}{\beta^n},$$

with $a_i > 0$ for $i = 1, \ldots, n$.

Underlying Markov partition

For the rest of this talk, we assume that the greedy expansion of 1 has the form

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \ldots + \frac{a_n}{\beta^n},$$

with $a_i > 0$ for $i = 1, \ldots, n$.

The underlying dynamics of K_β is given by a simple Markov chain.

Underlying Markov partition

Start with the partition $\{E_0, S_1, \ldots, S_{\lfloor\beta\rfloor}, E_{\lfloor\beta\rfloor}\}$.

Underlying Markov partition

Start with the partition $\{E_0, S_1, \ldots, S_{\lfloor\beta\rfloor}, E_{\lfloor\beta\rfloor}\}$.
Refine using the orbit of 1 and $\frac{\lfloor\beta\rfloor}{\beta-1} - 1$. The refinement gives the desired Markov partition

$$\{C_0, C_1, \ldots, C_L\},$$

where C_i is either S_j for some j, or is a subset of E_k for some k.

We consider the associated topological Markov chain and its corresponding adjacency matrix. We use the Parry recipe to find the (Markov) measure Q of maximal entropy.

An easy calculation shows that

$$Q([j_1,\ldots,j_\ell]) = \frac{v_{j_\ell}}{(1+\lfloor\beta\rfloor)^{\ell-1}},$$

where the probability vector $v=(v_1,\ldots,v_L)$ is a right Perron eigenvalue.

Measure of maximal entropy

We consider the associated topological Markov chain and its corresponding adjacency matrix. We use the Parry recipe to find the (Markov) measure Q of maximal entropy.

An easy calculation shows that

$$Q([j_1, \dots, j_\ell]) = \frac{v_{j_\ell}}{(1 + \lfloor\beta\rfloor)^{\ell-1}},$$

where the probability vector $v = (v_1, \dots, v_L)$ is a right Perron eigenvalue. When viewed as a measure on $\Omega \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}]$, one can show that the projection in the first coordinate is the uniform Bernoulli measure (the proof uses the strong Markov property, and the fact the elements of Ω depend on the times the Markov chain is in the S-region)

The measure $m_p \times \mu_{\beta,p}$

To identify the measure $\mu_{\beta,p}$, we consider the transition matrix $P = (p_{i,j})$, given by

$$p_{i,j} = \begin{cases} \lambda(C_i \cap T_\beta^{-1} C_j)/\lambda(C_i) & \text{if } C_i \subseteq E_k \text{ for some } k, \\ p & \text{if } C_i \subseteq S_k \text{ for some } k \text{ and } j = 0, \\ 1 - p & \text{if } C_i \subseteq S_k \text{ for some } k \text{ and } j = L. \end{cases}$$

Denote by $\pi = (\pi_1, \ldots, \pi_L)$ the stationary distribution of P.

The measure $m_p \times \mu_{\beta,p}$

An easy calculation shows that

$$\mu_{\beta,p}(B) = \sum_{j=0}^{L} \frac{\lambda(B \cap C_j)}{\lambda(C_j)} \cdot \pi(j) \qquad [B \in \mathcal{B}],$$

and $\mu_{\beta,p}$ has density

$$\mathbf{1}^* = \sum_{i=0}^{L} \frac{\pi_i}{\lambda(C_i)} \mathbb{I}_{C_i}.$$